RIPS COMPLEXES OF PLANAR POINT SETS

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ABSTRACT. Fix a finite set of points in Euclidean n-space $\mathbb{E}^n$, thought of as a point-cloud sampling of a certain domain $D \subset \mathbb{E}^n$. The Rips complex is a combinatorial simplicial complex based on proximity of neighbors that serves as an easily-computed but high-dimensional approximation to the homotopy type of D. There is a natural "shadow" projection map from the Rips complex to $\mathbb{E}^n$ that has as its image a more accurate n-dimensional approximation to the homotopy type of D.

We demonstrate that this projection map is 1-connected for the planar case $n = 2$. That is, for planar domains, the Rips complex accurately captures connectivity and fundamental group data. This implies that the fundamental group of a Rips complex for a planar point set is a free group. We show that, in contrast, introducing even a small amount of uncertainty in proximity detection leads to 'quasi'-Rips complexes with nearly arbitrary fundamental groups. This topological noise can be mitigated by examining a pair of quasi-Rips complexes and using ideas from persistent topology. Finally, we show that the projection map does not preserve higher-order topological data for planar sets, nor does it preserve fundamental group data for point sets in dimension larger than three.

1. INTRODUCTION

Given a set X of points in Euclidean space $\mathbb{E}^n$, the **Vietoris-Rips complex** $\mathcal{R}_\epsilon(X)$ is the abstract simplicial complex whose k-simplices are determined by subsets of $k+1$ points in X with diameter at most ϵ. For simplicity, we set $\epsilon = 1$ and write $\mathcal{R} := \mathcal{R}_1(X)$ for the remainder of the paper, with the exception of §4. For brevity (and to conform to typical usage), we refer to $\mathcal{R}$ as the **Rips complex**. The Rips complex is an example of a **flag complex** — the maximal simplicial complex with a given 1-skeleton.

The Rips complex was used by Vietoris [17] in the early days of homology theory, as a means of creating finite simplicial models of metric spaces. Within the past two decades, the Rips complex has been utilized frequently in geometric group theory [13] as a means of building simplicial models for group actions. Most recently, Rips complexes have been used heavily in computational topology, as a simplicial model for point-cloud data

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[3, 4, 5, 7], and as simplicial completions of communication links in sensor networks [8, 9, 16].

The utility of Rips complexes in computational topology stems from the ability of a Rips complex to approximate the topology of a cloud of points. We make this notion more specific. To a collection of points, one can assign a different simplicial model called the Čech complex that accurately captures the homotopy type of the cover of these points by balls. Formally, given a set X of points in some Euclidean space $\mathbb{E}^n$, the **Čech complex** $\mathcal{C}_\epsilon(X)$ is the abstract simplicial complex where a subset of $k+1$ points in X determines a k-simplex if and only if they lie in a ball of radius $\epsilon/2$. The Čech complex is equivalently the nerve of the set of closed balls of radius $\epsilon/2$ centered at points in X. The **Čech theorem** (or **Nerve lemma**, see, *e.g.*, [2]) states that $\mathcal{C}_\epsilon(X)$ has the homotopy type of the union of these balls. Thus, the Čech complex is an appropriate simplicial model for the topology of the point cloud (where the parameter ϵ is a variable).

There is a price for the high topological fidelity of a Čech complex. Given the point set, it is nontrivial to compute and store the simplices of the Čech complex. The virtue of a Rips complex is that it is determined completely by its 1-skeleton — the proximity graph of the points. (This is particularly useful in the setting of *ad hoc* wireless networks, where the hardware establishes communication links based, ideally, on proximity of nodes.) The penalty for this simplicity is that it is not immediately clear what is encoded in the homotopy type of $\mathcal{R}$. Like the Čech complex, it is not generally a subcomplex of its host Euclidean space $\mathbb{E}^n$, and, unlike the Čech complex it need not behave like an n-dimensional space at all: $\mathcal{R}$ may have nontrivial topological invariants (homotopy or homology groups) of dimension n and above.

The disadvantage of both Čech and Rips complexes are in their rigid cut-offs as a function of distance between points. Arbitrarily small perturbations in the locations of the points can have dramatic effects on the topology of the associated simplicial complexes. Researchers in sensor networks are acutely aware of this limitation, given the amount of uncertainty and fluctuation in wireless networks. To account for this, several researchers in sensor networks have used a notion of a distance-based communication graph with a region of uncertain edges [1, 15]. This motivates the following construction.

Fix an open **uncertainty interval** (ϵ, ϵ') which encodes connection errors as a function of distance. For all nodes of distance $\leq \epsilon$, there is an edge, and for all nodes of distance $\geq \epsilon'$, no edge exists. For nodes of distance within (ϵ, ϵ'), a communication link may or may not exist. A **quasi-Rips complex** with uncertainty interval (ϵ, ϵ') is the simplicial flag complex of such a graph. We note that this does not model *temporal* uncertainty, merely spatial.

A completely different model of simplicial complexes associated to a point cloud comes from considering shadows. Any abstract simplicial complex with vertices indexed by geometric points in $\mathbb{E}^n$ (*e.g.*, a Rips, Čech, or quasi-Rips complex) has a canonical **shadow** in $\mathbb{E}^n$, which strikes a balance between computability and topological faithfulness. For, say, a Rips complex, the canonical **projection** $p\colon \mathcal{R} \to \mathbb{E}^n$ is the well-defined function that maps each simplex in $\mathcal{R}$ affinely onto the convex hull of its vertices in $\mathbb{E}^n$. This projection map is continuous and piecewise-linear. The shadow $\mathcal{S}$ is the image $p(\mathcal{R})$ of this projection map.

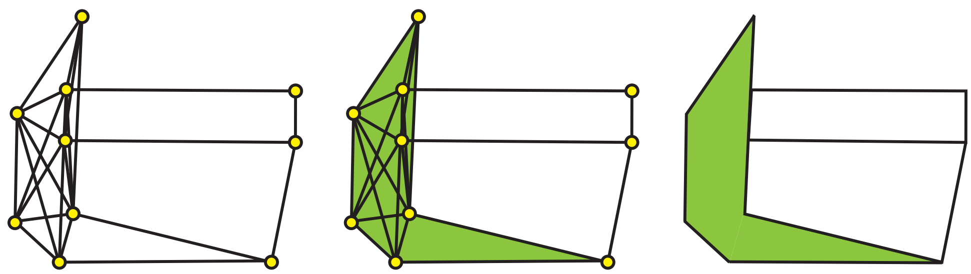

FIGURE 1. A connectivity graph in the plane [left] determines a 5-dimensional (Vietoris-) Rips complex [center] and its 2-dimensional projected shadow [right].

This paper studies the topological faithfulness of the projection map p (see Figure 1). Specifically, we look at the connectivity of p. Recall that a topological map $f : X \to Y$ is k**-connected** if the induced homomorphisms on homotopy groups $p_* : \pi_i(X) \to \pi_i(Y)$ are isomorphisms for all $0 \leq i \leq k$: *e.g.*, a 1-connected map preserves path-connectivity and fundamental group data.

We can now list the principal results of the paper, ordered as they appear in the following sections.

(1) For any set of points in $\mathbb{E}^2$, $\pi_1(p)\colon \pi_1(\mathcal{R}) \to \pi_1(\mathcal{S})$ is an isomorphism.
(2) The fundamental group of any planar Rips complex is free.
(3) Given any finitely presented group G, there exists a quasi-Rips complex $\mathcal{R}_Q$ with arbitrarily small uncertainty interval such that $\pi_1(\mathcal{R}_Q)$ is a free extension of G.
(4) Given a pair of quasi-Rips complexes $\mathcal{R}_Q$, $\mathcal{R}_{Q'}$ with disjoint uncertainty intervals, the image of $\iota_* : \pi_1(\mathcal{R}_Q) \to \pi_1(\mathcal{R}_{Q'})$ is free.
(5) The projection map p on $\mathbb{R}^n$ is always k-connected for $k = 0$ or $n = 1$. For all other cases except $(k, n) = (1, 2)$ and, perhaps, $(1, 3)$, k-connectivity fails on $\mathbb{R}^n$ (see Figure 8).

2. PLANAR RIPS COMPLEXES AND THEIR SHADOWS

In this section, we restrict attention to the 2-dimensional case.

2.1. **The shadow complex.** The shadow $\mathcal{S}$ is a polyhedral subset of the plane. By Carathéodory's theorem [10], $\mathcal{S}$ is the projection of the 2-skeleton of $\mathcal{R}$. Since the vertices of $\mathcal{R}$ are distinct points in the plane, it follows that distinct edges of $\mathcal{R}$ have distinct images under p, and these are nondegenerate. Informally we will identify vertices and edges of $\mathcal{R}$ with their images under p. On the other hand, p may be degenerate on 2-simplices.

We can canonically decompose $\mathcal{S}$ into a 2-dimensional **shadow complex** as follows:

- A **shadow vertex** is either a vertex of $\mathcal{R}$ or a point of transverse intersection of two edges of $\mathcal{R}$. We write $\mathcal{S}^{(0)}$ for the set of shadow vertices.
- A **shadow edge** is the closure of any component of $p(\mathcal{R}^{(1)}) \setminus \mathcal{S}^{(0)}$. Each shadow edge is a maximal line segment contained in a Rips edge, with no shadow vertices in its interior. We write $\mathcal{S}^{(1)}$ for the union of all shadow vertices and shadow edges.
- Finally, a **shadow face** is the closure of any bounded component of $\mathbb{E}^2 \setminus \mathcal{S}^{(1)}$.

The fundamental group $\pi_1(\mathcal{S})$ may now be described in terms of combinatorial paths of shadow edges modulo homotopy across shadow faces, whereas $\pi_1(\mathcal{R})$ may be described in terms of combinatorial paths of Rips edges modulo homotopy across Rips faces. This description opens the door to combinatorial methods in the proof that $\pi_1(p)$ is an isomorphism.

2.2. **Technical Lemmas.** Theorem 3.1 will follow from reduction to three special cases. We prove these cases in this subsection. We use the following notation. Simplices of a Rips complex will be specified by square braces, e.g., $[ABC]$. Images in the shadow complex will be denoted without adornment, e.g., ABC. The Euclidean length of an edge AB will be denoted $|AB|$. Braces $\langle\cdot\rangle$ will be used to denote the span in $\mathcal{R}$: the smallest subcomplex containing a given set of vertices, e.g., $\langle ABCD\rangle$.

The following propositions address the three special cases of Theorem 3.1 which are used to prove the theorem. Certain induced subcomplexes of $\mathcal{R}$ are shown to be simply connected. In the first two cases, it is helpful to establish the stronger conclusion that these subcomplexes are **cones**: all maximal simplices share a common vertex, called the **apex**. The first of these cases is trivial and well-known (*viz.*, [8, 12]).

Proposition 2.1. *Let $\mathcal{R} = \langle ABYZ\rangle$ be a Rips complex containing simplices $[AB]$ and $[YZ]$ whose images in $\mathcal{S}$ intersect. Then $\mathcal{R}$ is a cone.*

Proof. Let x be the common point of AB and YZ. Each edge is split at x into two pieces, at most one of which can have length more than one-half. The triangle inequality implies that the shortest of these four half-edges must have its endpoint within unit distance of both endpoints of the traversing edge, thus yielding a 2-simplex in $\mathcal{R}$. $\square$

Proposition 2.2. *Let $\mathcal{R} = \langle ABXYZ \rangle$ be a Rips complex containing simplices $[AB]$ and $[XYZ]$ whose images in $\mathcal{S}$ intersect. Then $\mathcal{R}$ is a cone.*

Proof. The edge AB intersects the triangle XYZ. If AB intersects only one edge of XYZ, then one vertex of AB (say, A) lies within XYZ and cones off a 3-simplex $[AXYZ]$ in $\mathcal{R}$. Therefore, without a loss of generality we may assume AB crosses ZY and ZX.

By Proposition 2.1, the subcomplexes $\langle ABXZ \rangle$ and $\langle ABYZ \rangle$ are cones. If these two cones have the same apex, then the entire Rips complex $\mathcal{R}$ is a cone with that apex. Similarly, if either apex lies inside the image triangle XYZ, then $\mathcal{R}$ is a cone with that apex. The only remaining possibility is that A is the apex of one subcomplex and B is the apex of the other; in this case, $\mathcal{R}$ is a cone over Z, since both A and B are connected to Z. □

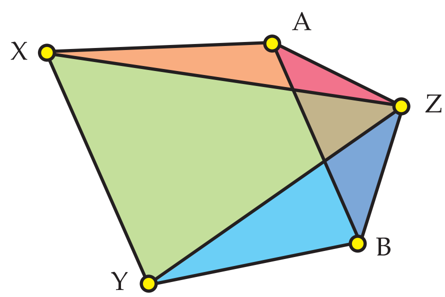

FIGURE 2. The last case of Proposition 2.2.

Proposition 2.3. *Let $\mathcal{R} = \langle ABCDXYZ \rangle$ be a Rips complex containing simplices $[AB]$, $[CD]$ and $[XYZ]$ whose images in $\mathcal{S}$ meet in a common point. Moreover, assume that none of A, B, C, D lies in the interior of XYZ. Then $\pi_1(\mathcal{R})$ is trivial.*

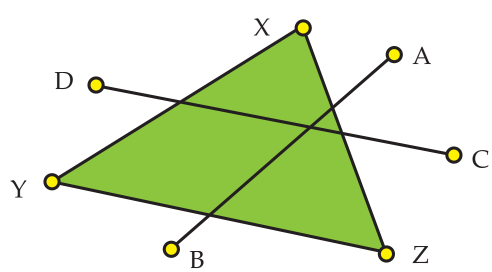

FIGURE 3. The setup for Proposition 2.3.

To prove Proposition 2.3, we use two further geometric lemmas.

Lemma 2.4. *Let $\mathcal{R} = \langle BXYZ \rangle$ be a Rips complex containing simplex $[XYZ]$. If M is a point in XYZ such that $|BM| \le \frac{1}{2}$, then $\mathcal{R}$ contains at least one of the edges $[BX]$, $[BY]$, $[BZ]$.*

Proof. If B lies in XYZ then all three edges belong to $\mathcal{R}$. Otherwise, BM meets the boundary of XYZ at a point M'. We may assume that M' lies on XY, with $|M'X| \leq |M'Y|$. Then $|BX| \leq |BM'| + |M'X| \leq \frac{1}{2} + \frac{1}{2} = 1$. □

Lemma 2.5. *Let $\mathcal{R} = \langle ABCXYZ \rangle$ be a Rips complex containing simplices $[ABC]$ and $[XYZ]$. Suppose that AB intersects XYZ but BC and AC do not. Then $\mathcal{R}$ is a cone.*

Proof. The hypotheses of the lemma imply that at least one of the points X, Y, or Z lies in the interior of ABC. $\mathcal{R}$ is a cone on this point. □

Proof of Proposition 2.3. We argue by exhaustive case analysis that $\mathcal{R}$ contains no minimal non-contractible cycle.

Suppose γ is a minimal non-contractible cycle in $\mathcal{R}$. Because $\mathcal{R}$ is a flag complex, γ must consist of at least four Rips edges. Our previous Propositions imply that this cycle intersects each simplex $[AB]$, $[CD]$, and $[XYZ]$ at least once. By minimality, γ contains at most one edge of $[XYZ]$. Thus, we may assume without loss of generality (by relabeling if necessary) that γ is of the form $A(B)C(D)X(Y)$ where $(\cdot)$ denotes an optional letter.

Claim 1: In a minimal cycle, the subwords $ABCD$, $CDXY$, $XYAB$ are impossible. Proposition 2.1 (in the first case) and Proposition 2.2 (in the last two cases) imply that the subpaths corresponding to these subwords are homotopic (relative to endpoints) within a cone subcomplex to a path with at most two edges, contradicting the minimality of γ.

Claim 1 implies that that there is at most (i.e. exactly) one optional letter. This leaves three possible minimal non-contractible cycles: $ACXY$, $ABCX$, and $ACDX$. The last two cases differ only by relabeling, so it suffices to consider only $ACXY$ and $ABCX$.

Claim 2: $ACXY$ is impossible. Suppose $ACXY$ is a cycle in $\mathcal{R}$. If AC meets XYZ then Proposition 2.2 implies that $\langle ACXYZ \rangle$ is a cone, so $ACXY$ is contractible. Thus, we can assume that AC does not meet XYZ.

By Proposition 2.1, either $[BC]$ or $[AD]$ is a Rips edge. Without loss of generality, assume $[BC]$ is a Rips edge; then $[ABC]$ is a Rips triangle. If BC does not meet XYZ, then Lemma 2.5 implies that $\langle ABCXYZ \rangle$ is a cone, and hence that $ACXY$ is contractible. Thus we can assume that BC intersects XYZ.

Proposition 2.2 now implies that both $\langle ABXYZ \rangle$ and $\langle BCXYZ \rangle$ are cones. If any of the segments $[BX]$, $[BY]$, $[BZ]$ is a Rips edge, then the cycle $ACXY$ is homotopic to the sum of two cycles, contained respectively in the cones $\langle ABXYZ \rangle$ and $\langle BCXYZ \rangle$, and hence is contractible. See Figure 4(a).

We can therefore assume that none of the segments $[BX]$, $[BY]$, $[BZ]$ is a Rips edge. In this case, the apex of $\langle ABXYZ \rangle$ must be A. In particular, the diagonal $[AX]$ of the cycle $ACXY$ belongs to $\mathcal{R}$, and so $ACXY$ is contractible. This completes the proof of Claim 2.

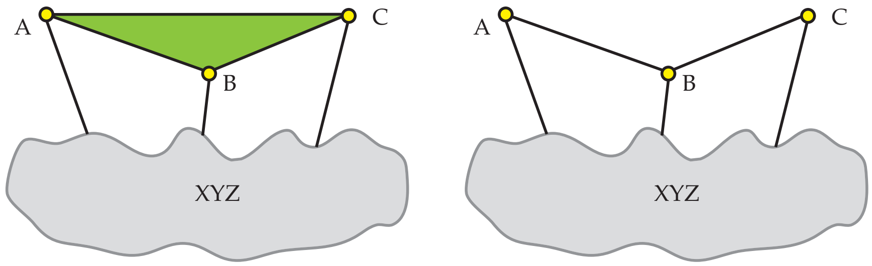

FIGURE 4. $ACXY$ (left), or $ABCX$ (right), splits into two cycles in the presence of $[BX]$, $[BY]$, or $[BZ]$.

Claim 3: $ABCX$ is impossible. Suppose $ABCX$ is a cycle in $\mathcal{R}$. If either $[AC]$ or $[BX]$ is a Rips edge, then $ABCX$ is trivially contractible. Moreover, if either $[BY]$ or $[BZ]$ is a Rips edge, then the cycle $ABCX$ reduces to the sum of two cycles, as in Figure 4(b). The left cycle is contractible by Proposition 2.2, and the right cycle is contractible by Claim 2 (suitably relabeled), so $ABCX$ is contractible in that case too. We can therefore assume that none of the segments $[AC]$, $[BX]$, $[BY]$, or $[BZ]$ is a Rips edge.

Now let M be a common point of intersection of AB, CD, and XYZ. Lemma 2.4 implies that $|BM| > \frac{1}{2}$, and so $|AM| = |AB| - |BM| \leq \frac{1}{2}$. Since $|AC| > 1$, we have $|CM| = |AC| - |AM| > \frac{1}{2}$, and so $|DM| = |CD| - |CM| \leq \frac{1}{2}$. These inequalities imply that $|AD| \leq |AM| + |DM| \leq 1$, so $[AD]$ is a Rips edge.

It follows that $\mathcal{R}$ contains the cycle $ADCX$. This cycle is homotopic to $ABCX$, since $\langle ABCD \rangle$ is a cone by Proposition 2.1. Lemma 2.4 implies that at least one of the segments $[DX]$, $[DY]$, $[DZ]$ must be a Rips edge. Arguing as before, with D in place of B, we conclude that $ADCX$, and thus $ABCX$, is contractible. This completes the proof of Claim 3. $\square$

2.3. **Lifting Paths via Chaining.** For any path α in $\mathcal{R}^{(1)}$, the projection $p(\alpha)$ is a path in $\mathcal{S}^{(1)}$, but not every shadow path is the projection of a Rips path. Every oriented shadow edge in $\mathcal{S}$ is covered by one or more oriented edges in $\mathcal{R}$. Thus to every path in $\mathcal{S}^{(1)}$ can be associated a sequence of oriented edges in $\mathcal{R}$. These edges do not necessarily form a path, but projections of consecutive Rips edges necessarily intersect at a shadow vertex.

Definition 2.6. *Let $[AB]$ and $[CD]$ be oriented Rips edges induced by consecutive edges in some shadow path. A* ***chaining sequence*** *is a path from A to D in the subcomplex $\langle ABCD \rangle$ which begins with the edge AB and ends with the edge CD.*

If we concatenate chaining sequences of shadow edges in $\mathcal{S}$ by identifying the Rips edges in the beginning and end of adjacent lifting sequences, we obtain a **lift** of the shadow path to $\mathcal{R}$. For any shadow path α in $\mathcal{S}$,

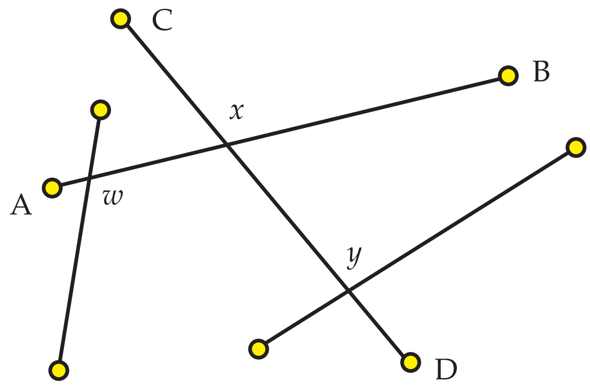

FIGURE 5. The setting for Lemma 2.8

we let $\widehat{\alpha}$ denote a lift of α to the Rips complex by means of chaining sequences. Note that the lift of a shadow path is not a true lift with respect to the projection map p — the endpoints, for example, may differ.

Lemma 2.7. *For any path α in $\mathcal{S}^{(1)}$, any two lifts of α to $\mathcal{R}$ with the same endpoints are homotopic in $\mathcal{R}$ rel endpoints.*

Proof. Let σ and τ be consecutive shadow edges in α, and let $[AB]$ and $[CD]$ be Rips edges such that $\sigma \subseteq AB$ and $\tau \subseteq CD$. Proposition 2.1 implies that all chaining sequences from A to D are homotopic rel endpoints in $\langle ABCD \rangle$, and thus in $\mathcal{R}$. If every shadow edge in α lifts to a unique Rips edge, the proof is complete.

On the other hand, suppose $\tau \subseteq CD \cap C'D'$ for some Rips edge $[C'D']$ that overlaps $[CD]$. Proposition 2.1 implies that both $[CC']$ and $[DD']$ are Rips edges. Moreover, since AB intersects $CD \cap C'D'$, any chaining sequence from A to D is homotopic rel endpoints in $\mathcal{R}$ to any chaining sequence from A to D' followed by $[D'D]$. Thus, concatenation of chaining sequences is not dependent on uniqueness of edge lifts. □

We next show that the projection of a lift of any two consecutive shadow edges is homotopic to the original edges.

Lemma 2.8. *For any two adjacent shadow edges wx and xy, where AB and CD are Rips edges with $wx \subseteq AB$ and $xy \subseteq CD$, $p(\widehat{wx \cdot xy})$ is homotopic rel endpoints to the path $ABxCD$ in $\mathcal{S}$.*

Proof. Consider the possible chaining sequences from A to D for $wx \cdot xy$. Either BC or AD must exist in $\mathcal{R}$ by Proposition 2.1.

Suppose BC exists. By Lemma 2.7, the chaining sequence is the Rips path $ABCD$ (up to homotopy rel endpoints). Either the triangle $[ABC]$ or the triangle $[BCD]$ exists in $\mathcal{R}$ by Proposition 2.1, so the triangle BCx is in shadow. This gives that $ABCD \simeq AxD \simeq ABxCD$ in $\mathcal{S}$.

If BC is not a Rips edge, then AD must be a Rips edges. By Lemma 2.7, the chaining sequence is the Rips path $ABADCD$ (up to homotopy rel endpoints). Either the triangle $[ACD]$ or the triangle $[ABD]$ exists in $\mathcal{R}$ by Proposition 2.1. Therefore, ADx lies in the shadow, so we get $ABADCD \simeq ABxCD$ in $\mathcal{S}$. □

Lemma 2.9. *For any lift $\widehat{\alpha}$ of any shadow path α with endpoints in $p(\mathcal{R}^{(0)})$, we have $p(\widehat{\alpha}) \simeq \alpha$ rel endpoints.*

Proof. For each pair of edges consecutive shadow edges wx and xy in α, where $wx \subseteq AB$, $xy \subseteq CD$, and AB and CD are Rips edges, Lemma 2.8 says that the projection of their lifting sequence deforms back to $ABxCD$. Every adjacent pair of chaining sequences can still be identified along common edges, since each ends with the first edge in the next one along α. The projection is homotopic rel endpoints to the original path α except for spikes of the form xB and xC at each shadow junction, which can be deformation retracted, giving $p(\widehat{\alpha}) \simeq \alpha$. □

3. 1-CONNECTIVITY ON $\mathbb{R}^2$

The following is the main theorem of this paper.

Theorem 3.1. *For any set of points in $\mathbb{E}^2$, $\pi_1(p)\colon \pi_1(\mathcal{R}) \to \pi_1(\mathcal{S})$ is an isomorphism.*

Proof. Assume that all π_1 computations are performed with a basepoint in $p(\mathcal{R}^{(0)})$, to remove ambiguity of endpoints in lifts of shadow paths to $\mathcal{R}$. Surjectivity of p on π_1 follows from Lemma 2.9 and the fact that any loop in $\mathcal{S}$ is homotopic to a loop of shadow edges thanks to the cell structure of $\mathcal{S}$.

To prove injectivity, note that any contractible cycle in $\mathcal{S}$ is expressible as a concatenation of boundary loops of shadow faces (conjugated to the basepoint). Thanks to Lemma 2.9, injectivity of $\pi_1(p)$ will follow by showing that the boundary of any shadow face lifts to a contractible loop in $\mathcal{R}$. Consider therefore a shadow face Ψ contained in the projection of a Rips 2-simplex $[XYZ]$, and choose $[XYZ]$ to be minimal in the partial order of such 2-simplices generated by inclusion on the projections.

Write $\partial\Psi$ as $\alpha_1 \cdot \alpha_2 \cdots \alpha_n$, where the α_i are the shadow edges, and let $[A_iB_i]$ be a sequence of directed Rips edges with $\alpha_i \subseteq [A_iB_i]$. Neither the A_i nor the B_i project to the interior of XYZ (see Figure 6); if any Rips vertex W did so, the edges $[XW]$, $[YW]$ and $[ZW]$ would exist in $\mathcal{R}$. As Ψ cannot be split by the image of any of these three edges, it must be contained in the projected image of a Rips 2-simplex, say $[XYW]$, whose image lies within that of $[XYZ]$, contradicting the minimality assumption on $[XYZ]$. The hypotheses of Proposition 2.3 thus apply to $[XYZ]$ and the consecutive edges $[A_iB_i]$, $[A_{i+1}B_{i+1}]$, and each complex $\langle A_iB_iA_{i+1}B_{i+1}XYZ\rangle$ is simply connected.

Fix the vertex X as a basepoint and fix a sequence of edge paths β_i in $\langle A_iB_iXYZ\rangle$ from X to A_i. Such paths exist and are unique up to homotopy

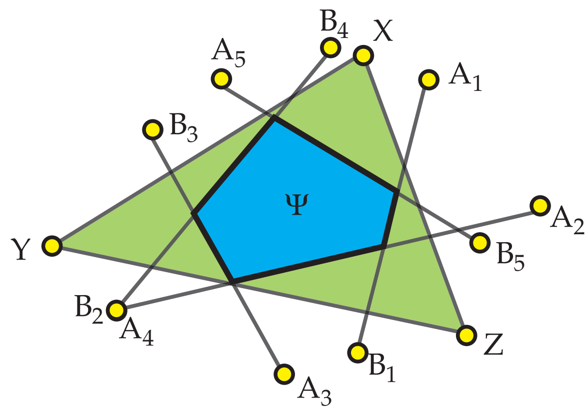

FIGURE 6. The boundary of a shadow face Ψ within XYZ is determined by Rips edges $[A_iB_i]$ whose projected endpoints lie outside XYZ.

since (by Proposition 2.2) $\langle A_iB_iXYZ\rangle$ is a cone. We decompose $\widehat{\partial\Psi}$ into loops $\gamma_1\cdots\gamma_n$, where γ_i is the loop with basepoint X given by

$$\gamma_i = \beta_i \cdot (\widehat{\alpha_i \cdot \alpha_{i+1}}) \cdot [B_{i+1}A_{i+1}] \cdot \beta_{i+1}^{-1}$$

where all indices are computed modulo n. By Proposition 2.3, each of these loops γ_i is contractible; hence, so is $\widehat{\Psi}$. □

Corollary 3.2. *The fundamental group of a Rips complex of a planar point set is free.*

4. QUASI RIPS COMPLEXES AND SHADOWS

We observe that Theorem 3.1 fails for quasi-Rips complexes, even for those with arbitrarily small uncertainty intervals. The failure of Proposition 2.1 in the quasi-Rips case makes it a simple exercise for the reader to generate examples of a quasi-Rips complexes which are simply-connected but whose shadows are not. Worse failure than this is possible.

Theorem 4.1. *Given any uncertainty interval (ϵ, ϵ') and any finitely presented group G, there exists a quasi-Rips complex $\mathcal{R}_Q$ with $\pi_1(\mathcal{R}_Q) \cong G * F$, where F is a free group.*

Proof. It is well-known that any finitely presented group G can be realized as the fundamental group of a 2-dimensional cell complex whose 1-skeleton is a wedge of circles over the generators and whose 2-cells correspond to relations. Such a complex can be triangulated, and, after a barycentric subdivision, can be assumed to be 3-colored: that is, there are no edges between vertices of the same color. Call this vertex 3-colored 2-d simplicial complex K.

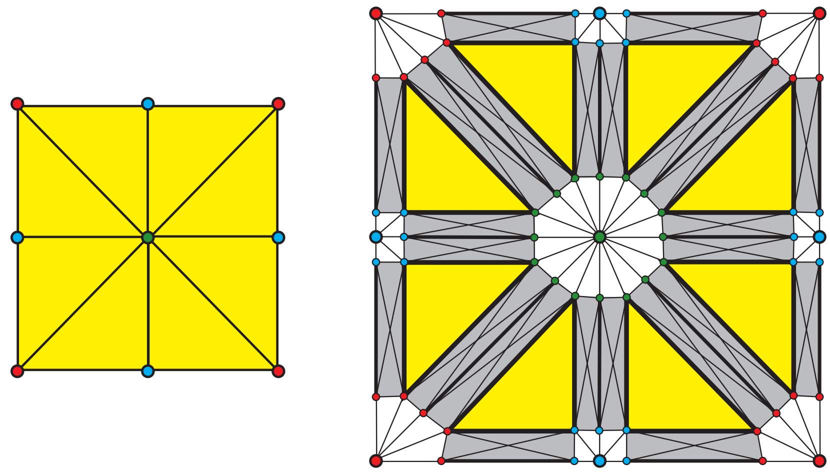

FIGURE 7. A 3-colored simplicial complex K and its blowup $\tilde{K}$, whose flag completion is homotopy equivalent to K. Opposite edges of K (and thus $\tilde{K}$) can be identified to yield a torus, projective plane, or Klein bottle.

We perform a 'blowup' of the complex K to a 3-d simplicial complex $\tilde{K}$ as follows (see Figure 7 for an example). Recall, the geometric realization of K can be expressed as the disjoint union of closed i-simplices with faces glued via simplicial gluing maps (the Δ-complex [14]). To form $\tilde{K}$, take the disjoint union of closed i-simplices of K and instead of simplicial gluing maps, use the **join** to connect all faces. The 3-coloring of K is inherited by $\tilde{K}$ via the blowup process.

There is a natural collapsing map $c : \tilde{K} \to K$ which collapses the joins to simplicial identification maps. The inverse image of any point in an open 2-simplex (1-simplex, resp.) of K is a closed 0-simplex (2-simplex resp.) of $\tilde{K}$. The inverse image of a vertex $v \in K$ consists of the 1-skeleton of the link of v in K. If we fill in $\tilde{K}$ by taking the flag completion, then $c^{-1}(v)$ is a copy of the star of v in K. Thus, upon taking the flag complex of $\tilde{K}$, the fiber of c for each point in K is contractible, which shows that the flag complex of $\tilde{K}$ is homotopic to K and thus preserves π_1.

We now embed $\tilde{K}$ in a quasi-Rips complex $\mathcal{R}_Q$. Define the vertices of $\mathcal{R}_Q$ in $\mathbb{R}^2$ as follows. Fix an equilateral triangle of side length $(\epsilon + \epsilon')/2$ in $\mathbb{R}^2$. Embed the vertices of $\tilde{K}$ arbitrarily in sufficiently small open balls (of radii no larger than $(\epsilon' - \epsilon)/4$) centered at the vertices of this triangle, respecting the 3-coloring. For this vertex set in $\mathbb{R}^2$, we define $\mathcal{R}_Q$ by placing an edge between vertices according to the edges of $\tilde{K}$, using the fact that any two vertices not of the same color are separated by a distance within the uncertainty interval. Of course, we must also add a complete connected graph on all vertices with a given color, since these lie within the small balls.

The quasi-Rips complex $\mathcal{R}_Q$ is the flag complex of this graph. It contains the flag complex of $\tilde{K}$, along with three high-dimensional simplices, one for each color.

We claim that any 2-simplex of $\mathcal{R}_Q$ which is not also a 2-simplex of $\tilde{K}$ has all vertices of the same color. Proof: Consider a 2-simplex $\sigma \in \mathcal{R}_Q$ spanning more than one color. Since the only edges added to form $\mathcal{R}_Q$ from $\tilde{K}$ have both ends with identical colors, it must be that $\sigma \cap \tilde{K}$ contains two edges which share a vertex. Any two edges in $\tilde{K}$ which share a vertex are sent by the collapsing map c to either (1) two edges of a 2-simplex in K; or (2) a single 1-simplex of K; or (3) a single vertex of K. In either case, the entire 2-simplex σ exists in the flag complex of $\tilde{K}$.

We end by showing that $\pi_1(\mathcal{R}_Q)$ is a free extension of G. Each of the three large colored simplices added to form $\mathcal{R}_Q$ from $\tilde{K}$ is homotopy equivalent to adding an abstract colored vertex (the apex of the cone) and an edge from this apex to the blowup of each 0-simplex of K in $\tilde{K}$. This is homotopy equivalent to taking a wedge with (many) circles and thus yields a free extension of the fundamental group of the flag complex of $\tilde{K}$, G. $\square$

We note that the construction above may be modified so that the lower-bound Rips complex $\mathcal{R}_\epsilon$ is connected. If necessary, the complex can be so constructed that the inclusion map $\mathcal{R}_\epsilon \hookrightarrow \mathcal{R}_{\epsilon'}$ induces an isomorphism on π_1 (which factors through $\pi_1(\mathcal{R}_Q)$).

Theorem 4.1 would appear to be a cause for despair, especially for applications to sensor networks, in which the rigid unit-disc graph assumption is unrealistic. The following result shows that Theorem 3.1 is not without utility, even when only quasi-Rips complexes are available.

Corollary 4.2. *Let $\mathcal{R}_Q$ and $\mathcal{R}_{Q'}$ denote two quasi-Rips complexes whose uncertainty intervals are disjoint. Then the image of $\pi_1(\mathcal{R}_Q)$ in $\pi_1(\mathcal{R}_{Q'})$ is a free subgroup of $\mathcal{S}_{\epsilon'}$ for any ϵ' in between the uncertainty intervals of the quasi-Rips complexes.*

Roughly speaking, this result says that a *pair* of quasi-Rips complexes, graded according to sufficiently distinct strong and weak signal links, suffices to induce information about a shadow complex.

Proof. The inclusions $\mathcal{R}_Q \subset \mathcal{R}_{\epsilon'} \subset \mathcal{R}_{Q'}$ imply that the induced homomorphism $\pi_1(\mathcal{R}_Q) \to \pi_1(\mathcal{R}_{Q'})$ factors through $\pi_1(\mathcal{R}_{\epsilon'})$. Thus, the image of $\pi_1(\mathcal{R}_Q)$ in $\pi_1(\mathcal{R}_{Q'})$ is a subgroup of $\pi_1(\mathcal{R}_{\epsilon'}) \cong \pi_1(\mathcal{S}_{\epsilon'})$, a free group. Any subgroup of a free group is free. $\square$

This is another example of the principle of **topological persistence**: there is more information in the inclusion map between two spaces than in the two spaces themselves. Knowing two 'noisy' quasi-Rips complexes and the inclusion relating them yields true information about the shadow.

5. k-CONNECTIVITY IN $\mathbb{R}^n$

Theorem 3.1 points to the broader question of whether higher-order topological data are preserved by the shadow projection map. Recall that a topological space is k-**connected** if the homotopy groups π_i vanish for all $0 \leq i \leq k$. A map between topological spaces is k-connected if the induced homomorphisms on π_i are isomorphisms for all $0 \leq i \leq k$.

We summarize the results of this section in Figure 8.

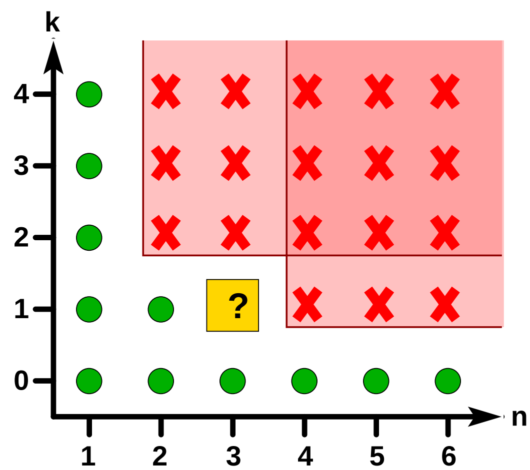

FIGURE 8. For which (n, k) is the Rips projection map in $\mathbb{E}^n$ k-connected? The only unresolved case is $(3, 2)$.

Throughout this paper, we have ignored basepoint considerations in the description and computation of π_1. The following proposition excuses our laziness.

Proposition 5.1. *For any set of points in $\mathbb{E}^n$, the map $p\colon \mathcal{R} \to \mathcal{S}$ is 0-connected.*

Proof. Certainly $\pi_0(p)$ is surjective, since p is surjective. The injectivity of $\pi_0(p)$ is a consequence of the following claim: If two Rips simplices σ and τ have intersecting shadows, then σ and τ belong to the same connected component of $\mathcal{R}$.

To prove the claim, suppose that $p(\sigma)$ and $p(\tau)$ intersect. By translation, we can suppose that $0 \in p(\sigma) \cap p(\tau)$. If $\{x_i\}$ and $\{y_j\}$ respectively denote the vertices of σ and τ, then

$$\sum_i \lambda_i x_i = 0 = \sum_j \mu_j y_j$$

for suitable convex coefficients $\{\lambda_i\}$ and $\{\mu_j\}$. Then

$$\begin{aligned}\sum_{i,j}\lambda_i\mu_j|x_i-y_j|^2 &= \sum_{i,j}\lambda_i\mu_j|x_i|^2 - 2\sum_{i,j}\lambda_i\mu_j(x_i\cdot y_j) + \sum_{i,j}\lambda_i\mu_j|y_j|^2 \\ &= \sum_i \lambda_i|x_i|^2 - 2\sum_i\lambda_i x_i\cdot\sum_j\mu_j y_j + \sum_j\mu_j|y_j|^2 \\ &= \sum_i\lambda_i|x_i|^2 + \sum_j\mu_j|y_j|^2,\end{aligned}$$

and similarly

$$\begin{aligned}\sum_{i,i'}\lambda_i\lambda_{i'}|x_i-x_{i'}|^2 &= 2\sum_i\lambda_i|x_i|^2, \\ \sum_{j,j'}\mu_j\mu_{j'}|y_j-y_{j'}|^2 &= 2\sum_j\mu_j|y_j|^2.\end{aligned}$$

Since every edge $x_ix_{i'}$ and $y_jy_{j'}$ has length at most 1, the left-hand sides of these last equations have value at most 1. Thus $\sum_i\lambda_i|x_i|^2 \leq 1/2$ and $\sum_j\mu_j|y_j|^2 \leq 1/2$. It follows that $\sum_{i,j}\lambda_i\mu_j|x_i-y_j|^2 \leq (1/2)+(1/2) = 1$ and so at least one edge x_iy_j has length at most 1.

Thus the simplices σ, τ are connected by an edge, as required. □

Proposition 5.2. *For any set of points in $\mathbb{E}^1$, the map $p\colon \mathcal{R} \to \mathcal{S}$ is a homotopy equivalence.*

Proof. Both $\mathcal{R}$ and $\mathcal{S}$ are homotopy equivalent to finite unions of closed intervals in $\mathbb{E}^1$, and therefore to finite sets of points. This is clear for $\mathcal{S}$. For $\mathcal{R}$, we note that $\mathcal{R}_1$ is equal to the Čech complex $\mathcal{C}_1$ in $\mathbb{E}^1$. Certainly the two complexes have the same 1-skeleton. Moreover, Helly's theorem implies that Čech complexes are flag complexes in 1D: a collection of convex balls has nonempty intersection if all pairwise intersections are nonempty. Thus $\mathcal{R}_1 = \mathcal{C}_1$. By the nerve theorem, this complex has the homotopy type of a union of closed intervals in $\mathbb{E}^1$.

Since a 0-connected map between finite point sets is a homotopy equivalence, the same conclusion now holds for the 0-connected map $p\colon \mathcal{R} \to \mathcal{S}$. □

Proposition 5.3. *There exists a configuration of points in $\mathbb{E}^2$ for which p is not 2-connected.*

Proof. Consider the vertices $rx_1, rx_2, rx_3, rx_4, rx_5, rx_6$ of a regular hexagon of radius r centered at the origin. If $1/2 < r \leq 1/\sqrt{3}$ then only the three main diagonals are missing from $\mathcal{R}$. Thus $\mathcal{R}$ has the structure of a regular octahedron, and therefore the homotopy type of a 2-sphere. On the other hand $\mathcal{S}$ is just the hexagon itself (including interior), and is contractible. □

The example of Proposition 5.3 extends to higher homotopy groups by constructing cross-polytopes, as in [8].

Proposition 5.4. *There exists a configuration of points in $\mathbb{E}^4$ for which p is not 1-connected.*

Proof. Consider the six points

$$(rx_1, \epsilon x_1), \quad (rx_2, 0), \quad (rx_3, \epsilon x_3), \quad (rx_4, 0), \quad (rx_5, \epsilon x_5), \quad (rx_6, 0)$$

in $\mathbb{E}^4$, in the notation of the previous proposition. Then $\mathcal{R}$ has the structure of a regular octahedron, but the map $p\colon \mathcal{R} \to \mathcal{S}$ identifies one pair of antipodal points (specifically, the centers of the two large triangles, 135 and 246). Thus $\mathcal{R}$ is simply-connected, whereas $\pi_1(\mathcal{S}) = \mathbb{Z}$. □

We note that these counterexamples may be embedded in higher dimensions and perturbed to lie in general position.

6. Conclusion

The relationship between a Rips complex and its projected shadow is extremely delicate, as evidenced by the universality result for quasi-Rips complexes (Theorem 4.1) and the lack od general k-connectivity in $\mathbb{R}^n$ (§5). These results act as a foil to Theorem 3.1: it is by no means *a priori* evident that a planar Rips complex should so faithfully capture its shadow.

We close with a few remarks and open questions.

(1) Are the cross-polytopes of Proposition 5.3 the only significant examples of higher homology in a (planar) Rips complex? If all generators of the homology $H_k(\mathcal{R})$ for $k > 1$ could be classified into a few such 'local' types, then, after a local surgery on $\mathcal{R}$ to eliminate higher homology, one could use the Euler characteristic combined with Theorem 3.1 as a means of quickly computing the number of holes in the shadow of a planar Rips complex. This method would have the advantage of being local and thus distributable.

(2) Does the projection map preserve π_1 for a Rips complex of points in $\mathbb{R}^3$? Our proofs for the 2-d case rest on some technical lemmas whose extensions to 3-d would be neither easy to write nor enjoyable to read. A more principled approach would be desirable, but is perhaps not likely given the 1-connectivity on $\mathbb{R}^3$ is a borderline case.

(3) What are the computational and algorithmic issues associated with determining the shadow of a (planar) Rips complex? See [6] for recent progress, including algorithms for test contractibility of cycles in a planar Rips complex and a positive lower bound on the diameter of a hole in the shadow.

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