"Chaotic" knots and "wild" dynamics[1]

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Abstract

The delicate interplay between knot theory and dynamical systems is surveyed. Numerous bridges between these fields allow us to apply dynamical perspectives (entropy, "chaotic") to knot theory, as well as knot-theoretic perspectives (cablings, "wild") to dynamical phenomena and bifurcations thereof. This intricate relationship has opened new doors in the study of ODE models of physical systems, while conversely yielding interesting topological objects from dynamical flows.

1 The topology and dynamics of flows

The concept of *flow* is central to several fields of inquiry: fluid dynamicists consider the motion of liquids and gasses, plasma physicists work with trajectories of particles in magnetic fields, chemical engineers are concerned with the mixing of media through stirred agents. Mathematicians, pure and applied, are also deeply involved in examining flows which arise as solutions to differential equations. Apart from these physical applications, the topological and dynamical properties of flows play important roles in the solution to deep mathematical problems: *e.g.*, S. Smale's classical solution to the high-dimensional Poincaré Conjecture, or, on a more accessible level, the invariance of the Euler characteristic on surfaces.

Any understanding of flows requires a consideration of instantaneous spatial properties (topology/geometry) as well as temporal behaviour (dynamics). There is a beautiful history of the interplay between topological and dynamical features of flows, dating back to Poincaré [36, 37], continuing with Birkhoff [5], and resurfacing intermittently in the works of several authors before the resurgence of interest inspired, in part, by the Smale school of dynamics [42] (see any text on dynamical systems for a more thorough history).

It remained until the late sixties for knot theory *per se* to enter the dynamical picture (to the extent of this author's knowledge). At the Seminar on Turbulence at Berkeley in the Fall of 1976, the principal theme was one of trying to understand complex behavior in simple models of systems relevant to fluid dynamics. Bob Williams introduced the notion of examining the dynamical complexity of a particular system (the Lorenz attractor, to be explained shortly) via examining the *topological* complexity, on the level of the *knotting and linking* of periodic orbits within the flow. This perspective grew into an elaborate theory and theme, both of which are still evolving.

For more information on dynamical systems, see, [13, 24, 38] (a brief and biased sampling of the many excellent texts). For a complete introduction to the theory of knots and links, we refer the reader to [1, 39, 10].

First, recall that a *knot* is a simple closed curve in a three-manifold, typically $\mathbb{R}^3$ or S^3, the unit sphere in $\mathbb{R}^4$. A *link* is a disjoint collection of knots. Two knots K_0 and K_1 are said to be *isotopic* if there is a smoothly varying one-parameter family of knots K_t joining K_0 to K_1 — in other words, two knots are considered equivalent if one can deform one to the other without cutting.

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1.1 Three examples

Let us consider three examples of flows on S^3 having interesting topological/dynamical properties:

Example 1.1 (integrable Hamiltonian oscillators) The following example arises as a two degree-of-freedom Hamiltonian mechanical system (a pair of uncoupled simple harmonic oscillators: see [2]). Let $H : \mathbb{R}^4 \to \mathbb{R}$ be given by $H(p_1, q_1, p_2, q_2) = \frac{1}{2}m(p_1^2 + q_1^2) + \frac{1}{2}n(p_2^2 + q_2^2)$, where $m, n \in \mathbb{Z}^+$. The level set $H^{-1}(\frac{1}{2})$ is a three-dimensional ellipsoid diffeomorphic to the unit 3-sphere $S^3 \subset \mathbb{R}^4$. The reader may verify that the solution to Hamilton's equations of motion,

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i} \qquad\qquad \frac{dp_i}{dt} = \Leftrightarrow \frac{\partial H}{\partial q_i},$$

is given as

$$\begin{array}{ll} q_1(t) = P_1 \sin(mt) + Q_1 \cos(mt) & q_2(t) = P_2 \sin(nt) + Q_2 \cos(nt) \\ p_1(t) = P_1 \cos(mt) \Leftrightarrow Q_1 \sin(mt) & p_2(t) = P_2 \cos(nt) \Leftrightarrow Q_2 \sin(nt) \end{array}, \qquad (1)$$

where $\sum(P_i^2 + Q_i^2) = \frac{1}{2}$. These solution curves fill up $H^{-1}(\frac{1}{2}) \cong S^3$ with closed orbits. There are two special orbits given by $Q_i, P_i = 0$ for $i = 1, 2$: these are simple unknotted orbits in S^3 which link each other in a *Hopf link*. The components of this Hopf link form the "cores" for a 1-parameter family of invariant tori that fill up the complement. Each of these tori is filled with closed orbits that travel m times around one angular direction and n times about the other before closing: see Figure 1.

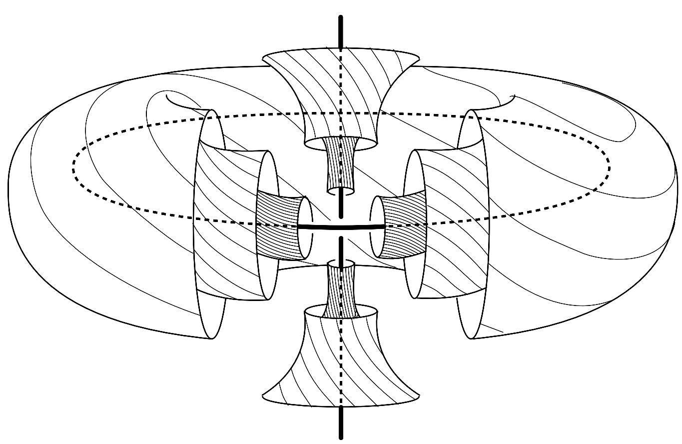


Figure 1: The flow of a pair of integrable Hamiltonian oscillators, represented in $\mathbb{R}^3$. The "vertical" orbit is a closed curve passing through the point at infinity.

Definition 1.2 A simple closed curve which lies on a (standardly embedded) torus is a (m, n) *torus knot*, where m and n are given as the integral winding numbers in each of the angular directions.

The flow of the integrable Hamiltonian oscillator example contains two unknots, and an uncountable family of (m, n) torus knots. Note also that the dynamics are particularly simple: all orbits are periodic.

Example 1.3 (north - south flow) Consider in the above example the function $F = \frac{1}{2}m(p_1^2 + q_1^2) \Leftrightarrow \frac{1}{2}n(p_2^2 + q_2^2)$. This function is an *integral* for the system: *i.e.*, F is constant along solutions. In fact, F is constant on the invariant tori of Figure 1, attaining its maximum and minimum on the two unknotted orbits which form a Hopf link. Now consider the flow $\dot{x} = \Leftrightarrow\nabla_x F$ along the gradient of F. This flow has two circles worth of fixed points, along the Hopf link, with all other orbits flowing from a fixed point on the first component of the link to a fixed point on the second component (see Figure 2).

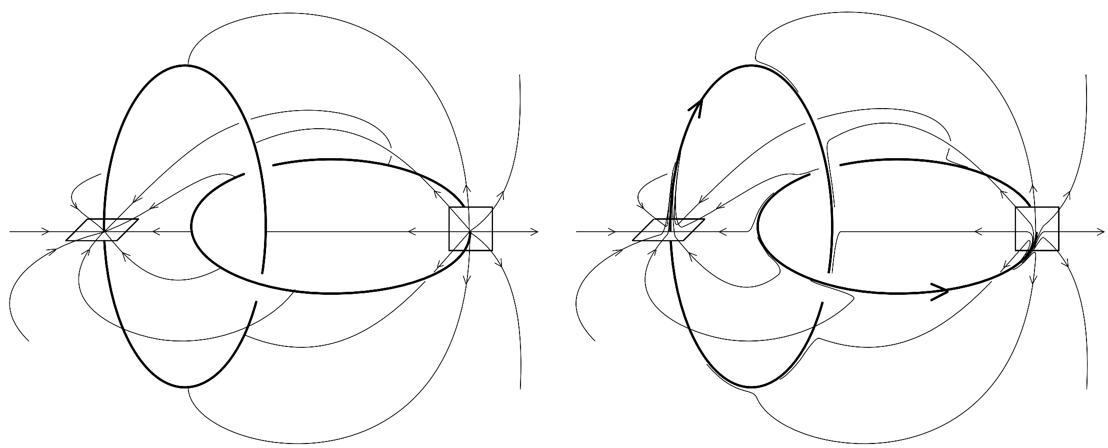


Figure 2: The gradient-flow (left) is perturbed into a flow with one attractor and one repellor (right) arranged in a Hopf link.

If we perturb the flow slightly in a neighborhood of the fixed point sets by adding a component of the vector field in the tangent direction of the fixed point curve, we can turn the two fixed point circles into closed orbits which are, respectively, attracting and repelling. Hence, the flow envisaged is one with two unknotted closed orbits, with all other orbits emanating from the repellor and limiting onto the attractor. This special flow will be called the *north-south flow*, since it is the higher-dimensional analogue of the gradient flow on the 2-sphere which flows from north to south poles.

Example 1.4 (Lorenz system) The solution to the well-known *Lorenz equations* yields a flow on $\mathbb{R}^3$ which has inspired countless analyses [30, 25]:

$$\begin{array}{rcl} \dot{x} & = & 10(y \Leftrightarrow x) \\ \dot{y} & = & 28x \Leftrightarrow y \Leftrightarrow xz \ , \\ \dot{z} & = & \Leftrightarrow \frac{8}{3}z + xy \end{array} \qquad (2)$$

Unlike the previous two examples, this flow has singularities, or fixed points. Also, as one can clearly see via numerical experiments, every orbit apart from these fixed points immediately collapses onto a complicated attractor which has decorated untold numbers of books and papers, including this one: see Figure 3. This example has been described as the canonical "chaotic" system arising from a differential equation. While actually proving that this system satisfies certain properties which comprise a sensible definition of the term "chaotic" is difficult [33], it is nonetheless clear to the observer that this three-dimensional flow exhibits what no two-dimensional flow could ever attain: a genuinely complicated behaviour.

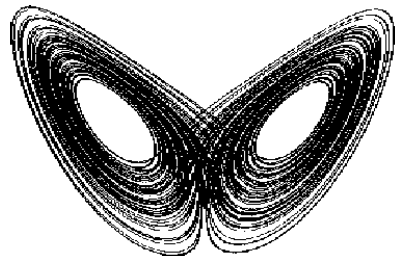


Figure 3: The Lorenz attractor.

The topology of this example is more difficult to grasp, particularly the (unstable) periodic orbits. As observed by Lorenz [30], the attractor resembles a surface which is *branched* in the sense that two strips overlap and fuse together along an interval. Hence, any periodic orbits which might live within the Lorenz attractor must reside within this branched surface. This will provide us with the model of a *template*, or branched surface, to be examined more closely in the next section. In the sequel, we will see that such surfaces actually contain an infinite collection of periodic orbits; however, the way in which these are knotted and packed together will be found to be rather surprising. Needless to say, we are not finished with this example.

1.2 The link of closed orbits

Let us begin with a more formal definition of the object of study in the preceding examples:

Definition 1.5 Given a flow ϕ_t on a three-manifold M, the *link of closed orbits*, L_ϕ, is defined to be the set of closed (periodic) orbits of ϕ_t, considered as a (perhaps infinite) link in M.

Several basic questions present themselves. How "small" can this link be? How "large?" Does the existence of one type of knot or link force the coexistence of another? In Examples 1.1 and 1.3, we note very simple types of knots in the periodic orbit links — along with dynamics which are relatively simple. It remains to see what the periodic orbit link of the Lorenz attractor is.

If we try to answer the question of how small a link of closed orbits can be, we run into a sort of degeneracy. Given any finite link, we can realize it as the set of periodic orbits for a flow — simply construct a vector field which points along the tangent vectors to the link, then quickly tapers off to zero within a small neighborhood of the link. This is not very satisfying. So consider the problem of trying to realize a finite link as the set of closed orbits for a *nonsingular* (fixed point free) flow (Example 1.1 as opposed to Example 1.3).

The classic *Seifert conjecture* states that it is impossible to have a nonsingular flow on S^3 having an *empty* link of closed orbits. This conjecture is squarely false, as first indicated by P. Schweitzer in 1974 [41]; however, the concluding blow has recently been dealt by K. Kuperburg [28], who constructs examples of *smooth* nonsingular flows with empty periodic orbit link. From her "plug" techniques, it follows that given a finite periodic orbit link for a nonsingular flow on S^3, there is a new nonsingular flow which has one less component in the closed orbit link.

The converse question of how "big" a periodic orbit link may be will be treated in §3. Until then, we challenge the reader with the task of imagining a smooth flow containing infinite collections of different knots (or even all!).

2 Tame vs. wild dynamics

The first stop on our tour of the intersection between knot theory and dynamics is an exploration of the following Theme:

simple dynamics	$\Leftrightarrow$	**simple knots**
complicated dynamics	$\Leftrightarrow$	**complicated knots**

The concept certainly seems plausible, and with the proper formulations we will find it to ring true.

One of the (few) measures of complexity for a dynamical system is the concept of *topological entropy*. This measures the growth rate of recurrence among orbits of increasing length: see [8, 38] for a proper (if not transparent) definition. In the context of flows, entropy is either zero or positive, with zero-entropy flows having (relatively) simple dynamics and positive-entropy flows having very complicated dynamics. In Examples 1.1 and 1.3, the topological entropy is zero; the topological entropy of the flow in Example 1.4 is positive.[3]

2.1 Zero-entropy flows

Note that the orbit structures in the zero-entropy examples we present are relatively tame, from a knot-theoretic point of view. This is a general principal which has been discovered several times in several different forms by independent researchers [35, 44, 27, 29, 19, 40, 16, 11]. While the details of this body of work are manifold, the general principal is straightforward and encapsulated within two operations on knots: *cabling* and *connected sums* (see Figure 4).

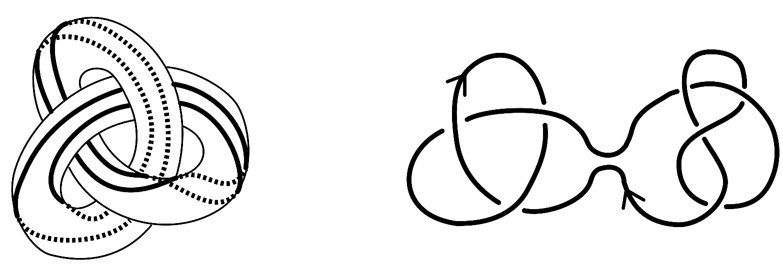


Figure 4: A 2-cable of a trefoil knot (left) and the connected sum of a trefoil and a figure-eight (right).

Definition 2.1 Given a knot K with tubular neighborhood N_K, a *cable* of K is a (nontrivial) knot which lives on the boundary of N_K.

Definition 2.2 Given two oriented knots K and K', their *connected sum*, denoted $K\#K'$, is the knot formed by deleting a small portion of each knot and identifying ends via orientation-preserving maps.

[3] This is a bit misleading — we really cannot compute the entropy exactly for this system. But, for an idealized model [25] it is certainly positive.

A cable can be thought of as a more general type of torus knot. In fact, the *iterated torus knots* are defined to be the set of knots obtained from the unknot by repeated cablings. Similarly, we define the set of *zero-entropy knots* to be the class of all knots generated from unknots via the operations of cabling and connected sums. These knots form, from the topological point of view, an extremely narrow class of "simple" knots.

There are several examples of zero-entropy flows which permit us to justify the terminology *zero-entropy knots*: we briefly mention these, leave the (many) details to the literature as referenced.

Example 2.3 A *nonsingular Morse-Smale flow*, or NMS flow, is a flow without fixed points having finite periodic orbit link such that each component is either a hyperbolic attractor, repellor, or saddle, with no other recurrence, as in Example 1.3. Such flows have entropy zero. It was shown by Wada [44], following earlier work by Morgan [35], that every knot in a NMS flow on S^3 must be a zero-entropy knot.

Example 2.4 An *integrable Hamiltonian flow* on S^3 is a flow derived from a Hamiltonian system having S^3 as an energy surface with an additional integral conserved by the solutions (recall Example 1.1 along with the integral F). Such systems appear frequently in models of oscillators, or in the motion of rigid bodies [2] and have zero entropy. Fomenko and his coworkers have carried out a program to classify the global topology of such systems on arbitrary three-manifolds [16]. The results on S^3 have as a corollary the statement that every closed orbit for an integrable Hamiltonian flow on S^3 forms a zero-entropy knot.

Example 2.5 Plane fields on three-manifolds are extremely interesting from the geometric and topological points of view, as evidenced by the current interest in *contact structures* [14, 15, 2]. Recently, John Etnyre and the author have shown that, given a plane field on a 3-manifold and a vector field contained within the plane field, then, generically, the fixed points of the vector field fill simple closed curves. In particular, given a *gradient-like* flow on S^3 within a plane field (the dynamically simplest kind of flow), the fixed point sets must form zero-entropy knots. An example of such a flow is the gradient flow of Example 1.3 before the perturbation turning it into an NMS flow. The fixed point set is a Hopf link.

The types of flows considered above are radically different in form and function: Morse-Smale flows are completely hyperbolic, whereas integrable Hamiltonian flows may be thought of as completely anti-hyperbolic. Yet, the common thread of zero-entropy knots runs through each class of flows.

Remark 2.6 Extensions of this work to zero-entropy *links* has been done by Wada (NMS flows) [44] and Cassasayas *et al.* (integrable Hamiltonian flows) [11].

2.2 Positive-entropy flows

On the other hand, a three-dimensional flow which has positive entropy must have a large number of periodic orbits, since entropy measures a growth rate of periodic orbits under increasing resolution. Hence, the periodic orbit links are expected to be numerically large. The pleasant feature of interest to this section is that the periodic orbit links must be large in a knot-theoretic sense:

Theorem 2.7 (Franks & Williams, [17] 1985) *If a sufficiently smooth[4] flow on S^3 (or $\mathbb{R}^3$) has positive topological entropy, then the link of periodic orbits is infinite and contains an infinite number of distinct knot types as components.*

[4]For example, C^2 smoothness is sufficient.

In particular, it follows from the proof that any such flow must contain infinitely many knots which are not in the class of zero-entropy knots. Additionally, there must be knotted orbits of arbitrarily long length (under any fixed metric); hence, there are “asymptotically wild” knots within the flow. Orbits of increasing length within a compact manifold (like S^3) must accumulate on themselves in perhaps strange ways (see [21] for an account of topological restrictions on the accumulations).

So given a positive-entropy flow, such as appears in Example 1.4, it remains to see if there is some classification of the knot types contained in the system. Does such a flow support all knot types? It would certainly seem not to be the case. However, a complete description of the topology of the periodic orbit link is by no means answered by Theorem 2.7. In the next section, we will describe a useful tool for examining such “chaotic” links.

Remark 2.8 Jerôme Los has considered the implications of entropy in knot theory. In [31], he uses the entropy of a certain map to define an invariant for knots. This is a nice twist on the theme of applying knot theory to dynamics.

3 Templates and knot-theoretic chaos

In Theorem 2.7 we alluded to the fact that “chaotic” flows in three-dimensions must have a link of closed orbits which is very complicated topologically. A systematic study of such orbit links is made possible through an idea of Bob Williams — the *template* (also known as *knotholder* or *branched 2-manifold*). As an example, consider the well-known *Lorenz attractor*, pictured in Figure 3. The attractor certainly appears to have the structure of a 2-manifold with a branch line, where the left and right “strips” merge, as noted by Lorenz [30].

Definition 3.1 A *template* is a compact branched two-manifold with boundary along with a smooth expanding semiflow. Templates are built from a finite number of *joining* and *splitting charts*, as pictured in Figure 5.

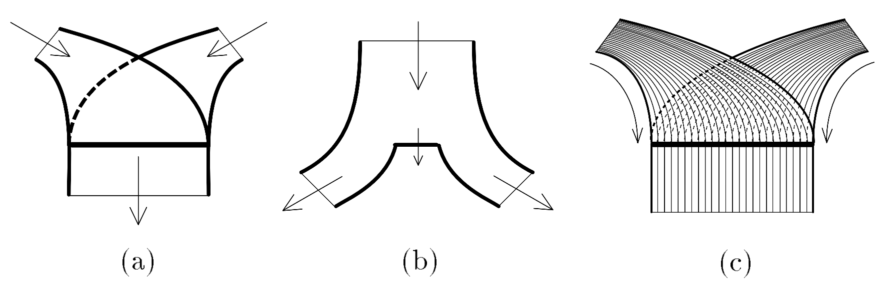
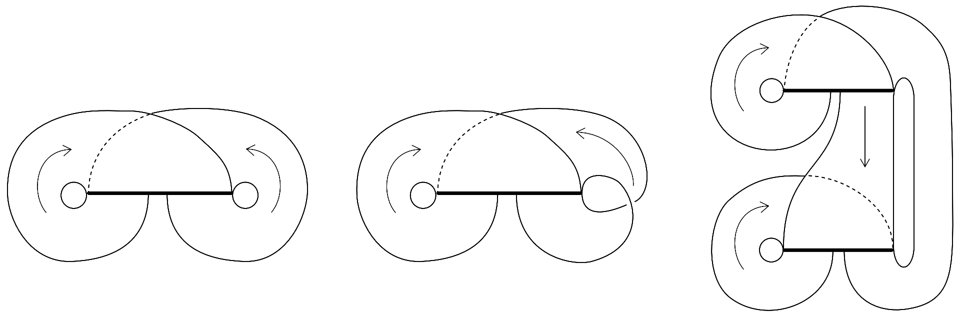


Figure 5: (a) Joining and (b) splitting charts for templates come with an expanding semiflow (c).

Simply put, a template is what you get when you take a finite set of the joining and splitting charts, and put them together so that the ends match up. Each chart has a semiflow (a one-way flow) on it, so you have to match up the directions of the flow. Several examples appear in Figure 6.

One may note that the semiflow on a template is “overflowing” on the splitting chart — orbits terminate at gaps. Given a template, we are interested only in those orbits which remain on the



(a) (b) (c)

Figure 6: Examples of templates: (a) the Lorenz template, (b) the horseshoe template, and (c) the universal template $\mathcal{V}$.

template for all time (*e.g.*, the periodic orbits); hence, we usually redraw our templates to eliminate the gaps by "pushing them backwards" until they reach the nearest branch line: see Figure 6.

The use of templates in the study of knot-theoretic dynamics is established by the Template Theorem of Joan Birman and Bob Williams:

Theorem 3.2 (Birman & Williams, [7] 1983) *Given a flow ϕ_t on $\mathbb{R}^3$ which has a nontrivial hyperbolic invariant set Λ, the link of periodic orbits of Λ are in $1:1$ correspondence with the link of periodic orbits on some embedded template $\mathcal{T} \subset \mathbb{R}^3$.*

The hyperbolicity condition means that there exists a continuously varying frame of directions orthogonal to the flow, along which the flow is expanding and contracting respectively. To produce the template, one simply "crushes" the flow along the contracting directions. This procedure, which identifies orbits having the same asymptotic behaviour, collapses a three-dimensional neighborhood of the invariant set down to a branched 2-manifold.

Combining Theorems 2.7 and 3.2, we see that a template must contain an infinite number of disjoint knotted orbits. This seemingly surprising result is actually quite simple to demonstrate: consider the Lorenz template $\mathcal{L}$. If we label the left and right strips x_1 and x_2 respectively, then we may represent any orbit of the semiflow on the template by an *itinerary*, or infinite sequence of symbols. Because of the nature of the semiflow on the template (it is *expanding*), there is in fact a $1:1$ correspondence between itineraries and orbits on $\mathcal{L}$ (this follows from some basic *symbolic dynamics* [8, 38]). The periodic orbits correspond precisely to itineraries which have periodically repeating words. For example, the two orbits on the boundary of $\mathcal{L}$ are $x_1x_1x_1\ldots = (x_1)^\infty$ and $x_2x_2x_2\ldots = (x_2)^\infty$. The periodic orbit $(x_1(x_1x_2)^2)^\infty$ is given in Figure 7 — it is a (2,3) torus knot or *trefoil*.

The Lorenz template is the simplest possible template. There are but two strips, and these are joined together without additional twisting, linking, or knotting. It is thus not surprising that those knots which live on $\mathcal{L}$ share several nice properties. For example, every knot on $\mathcal{L}$ is *fibred* — its exterior in S^3 looks like a "stacked" family of spanning surfaces parametrized by S^1. In addition, many "simple" families of knots reside on $\mathcal{L}$: *e.g.*, every kind of torus knot lives on $\mathcal{L}$. However, certain simple knots, such as the figure-eight knot, cannot live on $\mathcal{L}$ at all. It is worthwhile to consider how all these knots fit together disjointly to fill up the template.

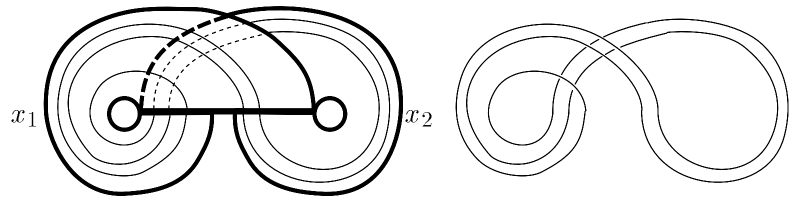


Figure 7: The periodic orbit $(x_1(x_1x_2)^2)^\infty$ on the Lorenz template is a trefoil knot.

The link of closed orbits which lives on a template such as $\mathcal{L}$ is difficult to classify: it is certainly not to be thought of as *small*, since there are an infinite number of distinct knot types on $\mathcal{L}$. On the other hand, the property of being a fibred knot (as is the case for knots on $\mathcal{L}$) is relatively special, so we cannot view the link of $\mathcal{L}$ as being *large*. Many other templates have similar restrictions: see Figure 6(b) for instance [26].

The question of whether there exists a template which contains *every* knot type on it was raised in [7]. The answer to this question was the result of examining *subtemplates*, or, subsets of a template which are themselves templates. Consider the embedded template $\mathcal{V}$ of Figure 6(c).

Theorem 3.3 (Ghrist, [20] 1995) *The template $\mathcal{V} \subset \mathbb{R}^3$ contains an isotopic copy of every finite link.*

Definition 3.4 A *universal template* is an embedded template $\mathcal{T} \subset S^3$ among whose closed orbits can be found knots of every type.

The template $\mathcal{V}$ is thus universal in a very strong sense. In fact, more recent results [23] lead to some surprising facts. The template $\mathcal{V}$ contains an infinite number of isotopic copies of itself, all disjoint and completely unlinked. Furthermore, $\mathcal{V}$ contains isotopic copies of *every* possible (orientable) template; hence, it rightfully deserved the name "universal."

Several examples of topologically curious flows on S^3 naturally follow. For instance, it is possible to define a flow on S^3 for which the link of periodic orbits is dense and spans all possible knot and link types. From Theorem 3.3 we conclude that the same "space of links" that fills S^3 densely can be compressed onto an ostensibly simple branched surface. This surprising result opens the door to the possibility of realizing extremely complicated orbit links in flows. For more dynamically relevant examples, see [23, Ch. 4], or the following section.

4 Applications: ODE's and bifurcation invariants

Theorem 3.2 is very useful for an abstract treatment of knots in flows, but given a specific differential equation, it is next to impossible to say whether it satisfies the conditions of the theorem (namely, whether there exists a hyperbolic structure). Thus, the theorem is difficult to apply in practice. However, several important instances do exist, and new techniques are being developed to apply template theory to ODE models for physical systems.

First, we should reiterate that the system at hand must be of third order; that is, the space of solutions to the dynamical system is three-dimensional. This occurs, for example, in the case of three time-independent first-order differential equations, as might appear when examining a set of three coupled oscillators. Or, whenever one has a two-dimensional map, there is lurking about an associated three-dimensional flow (this is the *suspension* procedure). Such examples frequently arise in the problem of *billiards*, where area-preserving annulus maps abound [12].

So given a three-dimensional ODE (or suspended two-dimensional map), we can investigate the link of closed orbits. The canonical example of such a system with rich structure is the Lorenz system, alluded to in Example 1.4. This system, which began the initial inquiries into knotted dynamics [6], is surprisingly resistant to analysis: there is still much unknown about the precise nature of the flow over thirty years after its advent. In particular, although the Lorenz template is an excellent model of the global features, there are severe obstacles to applying Theorem 3.2 — hyperbolicity is an elusive property. It is known that a certain (in some sense large) subclass of the knots which live on $\mathcal{L}$ do not live within the Lorenz attractor at standard parameter values.

Thus it would appear that trying to find a given template within a set of ODE's is also a challenge; however, in [22], the author, along with Phil Holmes, outline a general method for finding lots of templates within certain classes of ODE's exhibiting a type of behaviour known as *Shil'nikov dynamics*. These sorts of flows are characterized by a "spiraling" in the phase space. When coupled with a symmetry, it is shown that there is an abundance of universal templates within the flow.

Example 4.1 The set of equations which model the so-called *Chua circuit*,

$$\begin{array}{rcl} \dot{x} & = & 7[y - \phi(x)], \\ \dot{y} & = & x - y + z, \\ \dot{z} & = & -\beta y, \\ \phi(x) & = & \frac{2}{7}x - \frac{3}{14}\left[|x+1| - |x-1|\right], \end{array} \qquad (3)$$

give rise to a flow on $\mathbb{R}^3$ which has a universal template in it for an open set of parameters $\beta \in [\frac{13}{2}, \frac{21}{2}]$. That is, there is a subset of the flow which, upon collapsing out stable manifolds, yields a universal template. Since the periodic orbits are preserved, the periodic solutions to this equation contain every sort of knot and link.

The question of which knots live within a given ODE is by no means the complete picture. Consider a system with parameters, as is the case in most physical systems being modeled. Then, upon varying parameters, the system may undergo discontinuous changes, or *bifurcations* [24]. Except when at a bifurcation, periodic orbits persist upon change of parameters. By the uniqueness theorem for ODE's, varying the parameter in this manner gives rise to a continuous deformation of the periodic orbits. It is only at a bifurcation when a closed orbit may change its topology. Hence, if we (1) classify the topological action of a bifurcation; and (2) dress the periodic orbits with their knot and link data, we obtain *bifurcation invariants* derived from knot theory.

Example 4.2 (saddle-node bifurcation) Out of all possible bifurcations of periodic orbits, only certain types are *generic*, or "typical" within 1-parameter families. One example of such a bifurcation is the *saddle-node bifurcation* of orbits. Upon increasing the parameter μ, a pair of closed orbits draws close together. At the bifurcation value $\mu = \mu_0$, the knots collide into a single periodic orbit of the same knot type. For $\mu > \mu_0$, the periodic orbits are nonexistent — they have annihilated each other. By the Uniqueness Theorem for ODEs, the two closed orbits implicated must necessarily be isotopic. Furthermore, if *any* other orbit links the first component, it must necessarily link the second in precisely the same manner, again to respect uniqueness of solutions.

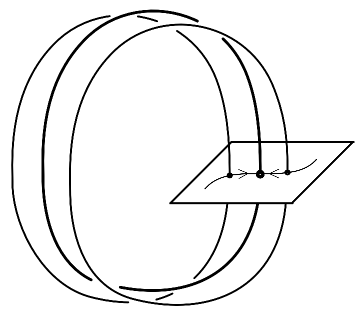


Figure 8: The topological action of a saddle-node bifurcation (left) and a period-doubling bifurcation (right).

Example 4.3 (period-doubling bifurcation) Another typical sort of bifurcation occurs when a periodic orbit of period T lies as the core of an invariant Möbius band. On one side of the bifurcation point $\mu = \mu_0$, the Möbius band is split along the core, resulting in a new periodic orbit with period $2T$: this is referred to as a *period-doubling bifurcation*. Topologically, the effect of this bifurcation is to change the knot into a 2-cable of the original. For example, an unknot may period-double into a trefoil, but not into a figure-eight.

Figure 8 expresses these two bifurcations pictorially. Given these examples, it is easy to find simple systems whose bifurcations can be characterized by knot-theoretic data, both in terms of knot types implicated and in terms of linking numbers. A fantastic application of this principal was devised by Phil Holmes and Bob Williams, who showed in [26] how to find new global restrictions on bifurcations in a family of *Hénon maps* by carefully considering knots in the suspension flows. The conclusions of their analysis include the statement that there are "infinitely many different routes to chaos" within the dynamics of this family of maps, in contrast to the search for universality among bifurcation sequences sought by many.

Such knot-theoretic methods are among the few truly global techniques available to analyze bifurcation sequences away from hyperbolic systems. Even so, this field has not seen its full potential realized, and several interesting problems await attention.

5 The quest for knots in four-dimensional dynamics

In many fields, including dynamical systems and topology, the techniques, perspectives, and approaches are confined to problems of a certain dimension: *e.g.*, the Euler characteristic classifies surfaces, but does nothing to classify three-manifolds. Likewise, orbits of flows can be knotted *only* in dimension three. Any two periodic orbits in a flow of dimension four or higher are *always* isotopic to one another: knots fall apart.[5] Knot theory, however, does not. Just as one can knot a circle (or 1-sphere) in three-dimensions, one may knot a 2-sphere in four-dimensions, a 3-sphere in five-dimensions, etc. In other words, there is a rich knot theory whenever the codimension of an embedding is two. See [39] for a survey of some results in high-dimensional knot theory.

[5] To see why, think of taking a crossing in a knot in $\mathbb{R}^3$ and undoing it by pushing one strand off into the fourth dimension, moving it freely, then returning it to $\mathbb{R}^3$ in a different position.

Hence, it would appear as though we could do some very interesting topology with ODEs of order four, if only we could find some sort of 2-sphere which is embedded in the flow. At least, this is what several physicists and mathematicians have suggested [34, 32]. Alas, this is futile — there are no sphere-like objects which are natural to dynamics, other than 1-spheres (periodic orbits)! However, a (heretofore ignored) 2-dimensional object which *is* dynamically relevant is the *torus*, $T^2 = S^1 \times S^1$. Invariant tori (subsets invariant under the flow which are homeomorphic to $S^1 \times S^1$) are relatively common objects in solutions to ODEs, especially in Hamiltonian systems and in problems involving coupled oscillators.

The question is, then, how can one knot a torus? The way that one typically thinks of first does not work. Consider a knotted curve in $\mathbb{R}^3$ along with a thin tubular neighborhood. The boundary of this neighborhood is a torus which is knotted in $\mathbb{R}^3$, *e.g.*, Figure 4 (left). However, this torus is unknotted in $\mathbb{R}^4$ for the same reason its core is: whatever a knotted torus is, it cannot "fit" inside of $\mathbb{R}^3$.

To obtain a truly knotted object, let K be a knotted curve in a solid ball B^3. Then let T_K denote the torus $K \times S^1 \subset B^3 \times S^1 \subset \mathbb{R}^4$. Using some algebraic topology (computing the fundamental group of the complement of T_K), one may show that when K is a nontrivial knot, T_K is not isotopic to the "standard" torus in $\mathbb{R}^4$. Thus, assuming that invariant tori of this form do appear in a dynamical context, we really can use information about the knot-types (or linking, etc.) to distinguish bifurcations, for example. We now give a class of examples in which lots of knotted invariant tori appear:

An *oscillator* is a dynamical system that has a unique attracting periodic orbit — examples of oscillators manifest themselves in countless systems, from neural activity to heartbeats. Placing a number of oscillators together with a small coupling between them can often produce fascinating behavior, *e.g.*, the sounds of crickets at night. (See [43] for a number of delightful examples and results in this area.)

If we couple N oscillators together with a weak enough coupling, the dynamics takes place on a phase space homeomorphic to the product of the individual oscillators: $T^N = S^1 \times \ldots \times S^1$. Hence, in the case of $N = 4$, one is concerned with flows on the four-dimensional torus T^4. What does an invariant torus represent in such a system? An invariant attracting torus represents a *partial mode-locking*, in which the four characteristic frequencies of the oscillators are slaved down to two frequencies, and the entire system behaves as a *pair* of oscillators.

Example 5.1 Generalizing the systems studied in [4], consider the return map on $T^3 \subset T^4$:

$$\dot{\theta}_i = \Omega_i + \frac{\epsilon}{2\pi}(\sin\theta_{i+1(\mathrm{mod}\ 2)} \Leftrightarrow \sin\theta_i), \tag{4}$$

where $i = 0, ..., 2$. This is a return map to the flow on T^4 associated to a rather general class of four coupled oscillators. For the values $\epsilon = 0.5$, $\Omega = (0.400023, 0.2002, 0.7002)$, the return map has an invariant attracting circle which is trivially knotted (see Figure 9, left). Upon increasing Ω_0 slightly, the invariant curve undergoes a sequence of period-doubling bifurcations which forces the curve to become knotted, culminating in what certainly appears to be a strange attractor.[6] If so, this attractor is filled with invariant curves of many different knot-types. Hence, there is a family of invariant knotted tori in the full phase space which perform complicated sequences of bifurcations — all carefully constrained by the topology of the embedding.

[6] In Figure 9, we have shown only a projection of the invariant curves onto a plane in T^3. These curves are homotopically trivial, and can be shown to be nontrivially knotted after sufficiently many period-doublings.

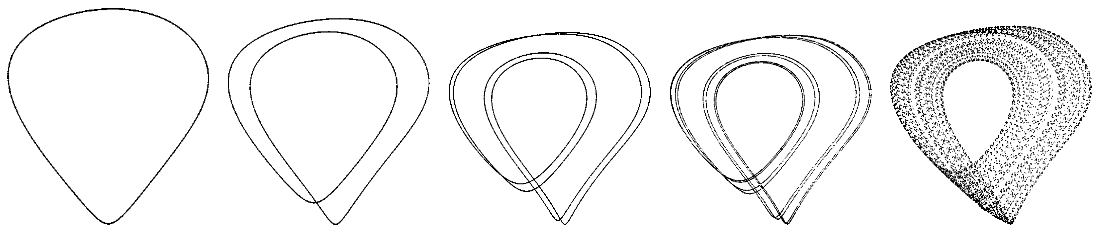


Figure 9: Invariant curves for a return map on T^3, projected to the plane. The curves undergo a period-doubling sequence, leading to knotted curves in the return map; hence, to knotted invariant tori in the flow on T^4.

Remark 5.2 The construction of knotted 2-tori in systems of four oscillators extends to higher dimensions: n coupled oscillators may have within their phase space knotted invariant tori of dimension $n \Leftrightarrow 2$.

6 Open problems (with commentary)

We close this survey with a collection of unsolved problems and new directions whose path is yet unlit. Although the principal ideas and technology for working with knots and templates in flows appeared in the early 80s, the field is, in this author's opinion, relatively untapped. It is hoped that the following questions will challenge the creativity of future researchers in this area:

6.1 Forcing theory of knots

The following question is rather broad, but gives the flavor of the forcing theory that we believe exists.

Question 1 *Given a link L and a special type of flow (*e.g.*, Hamiltonian, hyperbolic, Legendrian, Reeb), is there a nonsingular flow of this type on S^3 having precisely L as the link of closed orbits?*

Question 2 *What are the minimal conditions to force a hyperbolic flow to contain all knot types?*

A reasonable answer to the above question is that you must have an infinite unlink of untwisted unknots. Nothing less than this will do.

6.2 Universal templates

Recall, a universal template is one which contains all knots as periodic orbits.

Question 3 *Given a picture of a template, how can you tell if it is universal or not?*

This of course is a minor version of what is certainly the most pressing (and most difficult) problem in template theory:

Question 4 *When are two templates equivalent?*

One must, however be careful as to the definition of equivalence. See [23] for a discussion, along with a proof that a complete template invariant would yield a complete knot invariant (*i.e.*, this is a *hard* problem!).

An affirmative answer to the following question would agree with the observation that every template known to be universal (contains all knots) also contains all links and indeed all templates.

Question 5 *Does every universal template contain the template $\mathcal{V}$ from Figure 6(c) as a subtemplate?*

The following question was suggested by Ken Millet. The intuitive idea is that unknots should be relatively rare among knots with very large numbers of crossings. However, universal templates can be rather counterintuitive:

Question 6 *Are the unknots dense among periodic orbits on a universal template?*

6.3 Bifurcation sequences

In the Chua equations (3), varying the parameter β changes the system from having a periodic orbit link consisting of two unknotted, unlinked attractors (β small) to having a universal template.

Question 7 *What is the order in which knots and links are created in the birth of a universal template? Is there any sort of knot-theoretic analogue of the Sharkovskii ordering? See [9] for an affirmative answer to the related question for maps.*

It is much easier to compute the linking of orbits than it is their knot types [39]. Although linking is intimately related to bifurcation constraints, few applications in real problems have arisen. Also, there are several beautiful ways to define an *asymptotic linking number*, see [18] — these have been related to certain quantities in fluid dynamics (*e.g.*, "helicity" [3]).

Question 8 *In what ways can the (asymptotic, average) linking of orbits be utilized more practically in understanding the topology of bifurcations?*

Question 9 *In Hamiltonian systems, how do orbits in integrable systems bifurcate at breakdown (KAM)? What happens to the knotting and linking of near-integrable systems?*

6.4 Higher-dimensional knot theory and oscillators

Question 10 *In how many ways can one knot a torus in T^4? How can one represent these in a sensible way (as we do with planar diagrams for knotted circles)?*

Question 11 *Is there a physically motivated example of four coupled oscillators with knotted invariant tori of every knot type?*

Question 12 *Can knot-theoretic data be used to classify bifurcation sequences of invariant tori in the four coupled oscillators problem?*

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