Instructions: This exam consists of ten problems. Do all ten, showing your work and explaining your assertions. Give yourself two hours. Each problem is worth 20 points, for a total of 200.

1. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f(x) > 0$ for all $x > 0$ and such that $f(-10) < 0$.

a) Prove that the set $S = \{x \in \mathbb{R} \mid f(x) < 0\}$ has a supremum in $\mathbb{R}$.

b) Let $c = \sup(S)$. What is the value of $f(c)$? Justify your assertion.

2. a) If f is a continuous function on the open interval $a < x < b$, must f always achieve a maximum and a minimum on this interval? Give either a proof or a counterexample.

b) If f is an increasing (not necessarily continuous) function on the closed interval $[0, 1]$, prove that $f(0) \le \int_0^1 f(x)dx \le f(1)$.

3. Let $0 \le a \le 2\pi$, and let $v = (\cos a, \sin a) \in \mathbb{R}^2$.

a) Show that the function $f(t) = v \cdot (\cos t, \sin t)$ achieves a maximum and a minimum on the interval $[0, 2\pi]$.

b) Find where the maximum and the minimum occur, and interpret your answer geometrically.

4. a) Show that if $v, w \in \mathbb{R}^n$ are each orthogonal to a certain vector $z \in \mathbb{R}^n$, then every vector in the span of v, w is also orthogonal to z.

b) Let $S = \{v_1, \ldots, v_m\}$ be a linearly independent set in $\mathbb{R}^n$. Suppose that $w \in \mathbb{R}^n$ is not in the span of S. Prove that the set $\{v_1, \ldots, v_m, w\}$ is linearly independent.

5. Let $P, Q \in \mathbb{R}^2$ be distinct points in the plane, and let L be the (closed) line segment connecting P and Q.

a) Show that if $R \in \mathbb{R}^2$ lies on L then there is no ellipse in $\mathbb{R}^2$ that passes through R and has foci at P and Q.

b) Show that if $R \in \mathbb{R}^2$ does not lie on L then there is a unique ellipse in $\mathbb{R}^2$ that passes through R and has foci at P and Q.

6. Consider the curve C in $\mathbb{R}^2$ parametrized by $x = t$, $y = t^3$.

a) Find the velocity and acceleration as functions of t. For what values of t are these two vectors linearly independent?

b) For $u > 0$, let $s(u)$ be the arclength of the portion of C parametrized by $0 \le t \le u$. Express $s(u)$ as an explicit definite integral, and evaluate $s'(1)$.

7. a) Find all differentiable functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x)f'(x) = x$ for all $x \in \mathbb{R}$.

b) If f is such a function in part (a), and if $f(0) = -1$, what is $f(1)$?

8. a) Find all solutions to the differential equation $y'' + 2y' + 2y = 0$. Do the solutions form a vector space? If so, what is its dimension?

b) Do the same for the differential equation $y'' + 2y' + 2y = e^x$.

(continued)

9. Let $W \subset \mathbb{R}^4$ be the set of vectors $(x, y, z, t) \in \mathbb{R}^4$ such that $x + 2y + 2z + 4t = 0$. Let $W^\perp$ be the set of vectors $v \in \mathbb{R}^4$ such that $v \perp W$ (i.e. $v \perp w$ for all $w \in W$).

a) Determine whether W and $W^\perp$ are subspaces of $\mathbb{R}^4$; and if so, find their dimensions and their intersection.

b) Find the point of $W^\perp$ that is closest to $(2, 1, 1, 1)$.

10. Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be the linear transformation taking $(1, 0)$ to $(1, 2, 3)$ and taking $(0, 1)$ to $(-1, -2, -3)$.

a) Find the matrix of T with respect to the standard bases of $\mathbb{R}^2$ and $\mathbb{R}^3$. Also evaluate $T(2, 1)$.

b) Find the rank and nullity of T.