1. Which of the following R-modules are finitely generated? Which are free? Which are R-algebras? Among the R-algebras, which are finitely generated as R-algebras?

a) $R = \mathbb{Z}$, $M = \mathbb{Z}/5 \times \mathbb{Z}/7$
b) $R = \mathbb{Z}$, $M = (5) =$ ideal generated by 5
c) $R = \mathbb{Z}$, $M = \mathbb{Z}[\sqrt{3}]$
d) $R = \mathbb{Z}$, $M = \mathbb{Z}[\pi]$
e) $R = \mathbb{Z}$, $M = \mathbb{Z}[1/2, 1/3]$
f) $R = \mathbb{Z}$, $M = \mathbb{Q}$
g) $R = \mathbb{Z}$, $M = \frac{1}{2}\mathbb{Z} = \{\frac{n}{2} \mid n \in \mathbb{Z}\}$
h) $R = \mathbb{Z}[\sqrt{-5}]$, $M = (2, 1 + \sqrt{-5}) \subset R$
i) $R = \mathbb{R}[x]$, $M = \mathbb{R}[x, y]$
j) $R = \mathbb{R}[x]$, $M = \mathbb{R}[x, y]/(y^2 - x)$
k) $R = \mathbb{R}[x, y]$, $M = (x, y) \subset R$.

2. Let M be a module over R, and let $0 \to N' \to N \to N'' \to 0$ be an exact sequence of R-modules. Show by example that the induced sequences

$$0 \to \mathrm{Hom}(M, N') \to \mathrm{Hom}(M, N) \to \mathrm{Hom}(M, N'') \to 0 \tag{1}$$

$$0 \to \mathrm{Hom}(N'', M) \to \mathrm{Hom}(N, M) \to \mathrm{Hom}(N', M) \to 0 \tag{2}$$

$$0 \to M \otimes N' \to M \otimes N \to M \otimes N'' \to 0 \tag{3}$$

are *not* necessarily exact, unlike the case of vector spaces over a field. (Compare Math 602, fall 2004, Problem Set 11, problem 4.)

3. a) If R is a commutative ring, find an isomorphism $R \otimes_{\mathbb{Z}} \mathbb{Z}[\sqrt{2}] \xrightarrow{\sim} R[x]/(x^2 - 2)$.

b) Determine whether $\mathbb{Z}[\sqrt{3}] \otimes_{\mathbb{Z}} \mathbb{Z}[\sqrt{2}]$ and $\mathbb{Z}[\sqrt{2}] \otimes_{\mathbb{Z}} \mathbb{Z}[\sqrt{2}]$ are integral domains.

c) Simplify each of the following $\mathbb{Z}$-modules (up to isomorphism): $\mathrm{Hom}(\mathbb{Z}/10, \mathbb{Z})$ (the dual of the $\mathbb{Z}$-module $\mathbb{Z}/10$), $\mathrm{Hom}(\mathbb{Z}, \mathbb{Z}/10)$, $\mathrm{Hom}(\mathbb{Z}/10, \mathbb{Z}/6)$, $\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/10$, $\mathbb{Z}/10 \otimes_{\mathbb{Z}} \mathbb{Z}/6$, $\mathbb{Z}/10 \otimes_{\mathbb{Z}} \mathbb{Q}$, $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}$.

4. Let R be the ring of real polynomial functions on the circle $x^2 + y^2 = 25$. Let P be the point $(3, 4)$, and let I be the ideal of functions in R vanishing at P.

a) Show that I is generated by the elements $x - 3$, $y - 4$.

b) Show that if $I = (f)$ where $f \in R$, then f divides $x - 3$ and $y - 4$ in R. Deduce that f cannot vanish at any point of the circle except for P. Also, deduce that f cannot vanish to order ≥ 2 at P, as a function on the circle.

c) Deduce that I cannot be principal. (Hint: Let $F(x, y) \in \mathbb{R}[x, y]$ represent the function $f \in R$. What does the graph of $F(x, y) = 0$ look like? Where does it meet the circle? Is it tangent or transversal to the circle?)

d) Show that no two elements in I are linearly independent over the ring R.

e) Using (c) and (d), conclude that I is not a free R-module.

(continued on next page)

5. In the notation of problem 4, let Q be the point $(-3, 4)$, and let J be the ideal of functions in R vanishing at Q. Consider the R-module $I \oplus J$.

a) Show that every element $(i, j) \in I \oplus J$ can be uniquely expressed in the form

$$\big((x-3)f + (y-4)g,\ (x+3)f + (y-4)g\big),$$

for some $f, g \in R$. (Hint: First solve for f, g as linear combinations of i, j. Then use 4(a) and the identity $x^2 + y^2 = 25$ in R, to show that $\frac{x+3}{y-4} i \in R$.)

b) Show conversely that every element of $R \times R$ of the above form must lie in $I \oplus J$.

c) Deduce that $I \oplus J$ is a free R-module of rank 2, even though I is not free. (Is J free?)