1. a) Find the degree of $\alpha = \sqrt{2} + \sqrt{3}$ over $\mathbb{Q}$, and also find its minimal polynomial.
   b) Do the same for $\beta = \sqrt{3 + \sqrt[3]{2}}$.
   c) Is $\mathbb{Q}(\alpha)$ normal over $\mathbb{Q}$? Is $\mathbb{Q}(\beta)$?

2. Let $F = \mathbb{C}(x)$. For $a \in \mathbb{C}$, view $\mathbb{C}((x-a))$ as a field extension of F.

a) Show that if $a, b \in \mathbb{C}$, then there is a square root of $x - a$ in $\mathbb{C}((x-b))$ if and only if $a \neq b$.

b) For each non-negative integer n, let $F_n = F[\sqrt{x}, \sqrt{x-1}, \ldots, \sqrt{x-n}]$. Show that each F_n is a field extension of F; and that F_n can be embedded in $\mathbb{C}((x-m))$ as an F-algebra if and only if $n < m$. (Here m is a non-negative integer.) Deduce that the inclusions $F_0 \subset F_1 \subset F_2 \subset \cdots$ are strict.

c) Show that $F_\infty := F[\sqrt{x}, \sqrt{x-1}, \sqrt{x-2}, \ldots]$ is a field of infinite degree over F.

d) Is there an integer d such that every element of F_∞ satisfies a polynomial of degree at most d over F?

3. Let K be a field, and $f(x) \in K[x]$. Assume that K has characteristic 0. Let $n \geq 1$.

a) Let L be a finite field extension of K, and let $\alpha \in L$. Show that α is a root of f with multiplicity n if and only if $0 = f(\alpha) = f'(\alpha) = \cdots = f^{(n-1)}(\alpha) \neq f^{(n)}(\alpha)$.

b) Show that f has a root (in some extension of K) of multiplicity at least n if and only if $(f(x), f'(x), \ldots, f^{(n-1)}(x))$ is a proper ideal of $K[x]$.

c) What if instead K has non-zero characteristic?

4. For each of the following fields K, explicitly find the group $\operatorname{Aut} K$ of all automorphisms of K (as a field): $\mathbb{Q}$, $\mathbb{Q}[\sqrt{2}]$, $\mathbb{Q}[\sqrt[3]{2}]$, $\mathbb{Q}[\zeta_7]$, $\mathbb{Q}[\zeta_8]$, $\mathbb{Q}[\zeta_3, \sqrt[3]{2}]$. (Here $\zeta_n = e^{2\pi i/n}$, a primitive nth root of unity.)

5. Let $K = \mathbb{Q}[\sqrt{2}]$ and $L = \mathbb{Q}[\sqrt{2+\sqrt{2}}]$.

a) Find the multiplicative inverse of $\sqrt{2+\sqrt{2}}$ in L (as a polynomial in $\sqrt{2+\sqrt{2}}$).

b) Show $K \subset L$. What is $[K : \mathbb{Q}]$? $[L : K]$? $[L : \mathbb{Q}]$?

c) Let ϕ be an automorphism of L. What can you say about the restriction $\phi|_{\mathbb{Q}}$?

d) Let ϕ be an automorphism of L. What can you say about the restriction $\phi|_K$?

e) Find an element of order 4 in $\operatorname{Aut} L$. What is the group $\operatorname{Aut} L$ abstractly?

f) Replace $\sqrt{2}$ by $\sqrt{3}$, and $\sqrt{2+\sqrt{2}}$ by $\sqrt{3+\sqrt{3}}$. Try to redo parts (a) - (e). Do the results still hold?

6. Find all algebraic field extensions of $\mathbb{R}$. Justify your assertions. (You may assume that $\mathbb{C} = \mathbb{R}[i]$ is algebraically closed.)

7. Let K be a field with algebraic closure $\bar{K}$. Let $K^{\mathrm{s}} = \{a \in \bar{K} \mid a \text{ is separable over } K\}$.

a) Show that K^{s} is a subfield of $\bar{K}$ (called the *separable closure* of K).

b) Show that for every separable polynomial $f(x) \in K[x]$, the field K^{s} contains a root of f, and $f(x)$ factors over K^{s} as the product of linear factors.

c) Show that K^{s} is normal over K.