1. Let p be a prime number.

a) Use Eisenstein's Irreducibility Criterion to show that the polynomial

$$f(x) = x^{p-1} + x^{p-2} + \cdots + x + 1$$

is irreducible over $\mathbb{Q}$. [Hint: First set $y = x - 1$.]

b) Give another proof of the same assertion, by first showing that $f(x) = \Phi_p(x)$ (in the notation of Problem Set 9 #2).

2. Under the notation of Problem Set 9 #2:

a) Describe $\mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q})$ in terms of n. In particular, what is this Galois group when $n = 5$? 6? 7? 8? 12?

b) For which n is this extension abelian? cyclic? of order 2? of order 3? For which n does it have a cyclic quotient of order 3?

c) Let $K_7^+ = \mathbb{Q}(\zeta_7 + \zeta_7^{-1})$. Find $[K_7 : \mathbb{Q}]$, $[K_7 : K_7^+]$, and $[K_7^+ : \mathbb{Q}]$. Also find $\mathrm{Gal}(K_7/K_7^+)$ and $\mathrm{Gal}(K_7^+/\mathbb{Q})$.

d) Find a Galois extension of $\mathbb{Q}$ having degree 5. Find another of degree 7. [Hint: See part (c).]

3. Find the Galois group of (the splitting field of) each of the following polynomials.

a) $x^3 - 10$ over $\mathbb{Q}$.

b) $x^3 - 10$ over $\mathbb{Q}(\sqrt{2})$.

c) $x^3 - 10$ over $\mathbb{Q}(\sqrt{-3})$.

d) $x^4 - 5$ over $\mathbb{Q}$, $\mathbb{Q}(\sqrt{5})$, $\mathbb{Q}(\sqrt{-5})$, $\mathbb{Q}(i)$.

e) $x^4 - t$ over $\mathbb{R}(t)$, $\mathbb{C}(t)$.

4. Show that for every finite group G, there are field extension $\mathbb{Q} \subset K \subset L$ such that L is a finite Galois extension of K with $\mathrm{Gal}(L/K) = G$. (Remark: $[K : \mathbb{Q}]$ is allowed to be infinite.) [Hint: First show the result for $G = S_n$, using Problem Set 9 #3.]

5. a) Find a Galois extension of $\mathbb{Q}$ with Galois group $C_6 \times C_{15}$.

b) Do the same over the field $\bar{\mathbb{F}}_5(t)$ (where $\bar{\mathbb{F}}_5$ is the algebraic closure of $\mathbb{F}_5$).

c) Let $L = \mathbb{C}(x, y)$, $M = \mathbb{C}(x^2, xy, y^2) \subset L$, and $K = \mathbb{C}(x^2, y^2) \subset M$. Find $[L : M]$, $[M : K]$, $[L : K]$. Is L Galois over M? Is M Galois over K? Is L Galois over K? For those extensions that are Galois, find the Galois group.

6. Let K and L be finite extensions of a field k, and let KL be their compositum (inside some fixed algebraic closure).

a) Find a surjective k-algebra homomorphism $\pi : K \otimes_k L \twoheadrightarrow KL$.

b) Suppose that K is Galois over k. Show that π is an isomorphism if only if $K \cap L = k$. [Hint: Find $\dim_k(K \otimes_k L)$ and $\dim_k(KL)$.]

c) Does (b) still hold if K is no longer assumed Galois over k?

The following problem is optional.

7. Hilbert proved the following Irreducibility Theorem: *Let $s_1, \ldots, s_m, x_1, \ldots, x_n$ be transcendentals over $\mathbb{Q}$. If $f(s_1, \ldots, s_m, x_1, \ldots, x_n)$ is an irreducible polynomial over $\mathbb{Q}$, then*

there exist $\alpha_1, \ldots, \alpha_m \in \mathbb{Q}$ *such that* $f(\alpha_1, \ldots, \alpha_m, x_1, \ldots, x_n)$ *is an irreducible polynomial in* $\mathbb{Q}[x_1, \ldots, x_n]$*. Moreover, if a non-zero polynomial* $g \in \mathbb{Q}[s_1, \ldots, s_m]$ *is given in advance, then the* α*'s can be chosen so that* $g(\alpha_1, \ldots, \alpha_m) \neq 0$*.*

a) Verify this explicitly in the case that $m = n = 1$, $f(s, x) = x^3 - s$. Which values of α work?

b) Show that if $\mathbb{Q}(s_1, \ldots, s_m) \subset L$ is a finite field extension, then there is an irreducible polynomial $F \in \mathbb{Q}[s_1, \ldots, s_m, x]$ such that L is the fraction field of $\mathbb{Q}[s_1, \ldots, s_m, x]/(F)$.

c) Use Hilbert's theorem to show that there is an irreducible polynomial $f(x) \in \mathbb{Q}[x]$ such that the extension $\mathbb{Q} \subset \mathbb{Q}[x]/(f)$ is Galois with group S_n. [Hint: Part (b) and Problem Set 9 #3.] (Caution: You'll need to verify that $\deg(f) = n!$.)

d) Still assuming Hilbert's theorem, conclude that the field K in problem 4 above can be chosen to be finite over $\mathbb{Q}$. [Hint: Reduce to the case of part (c) above.]