A Topological Construction of Vassiliev Style Invariants of Links

by

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COLIN DAY. A Topological Construction of Vassiliev Style Invariants of Links
(Under the direction of James D. Stasheff.)

Abstract

An explicit construction of Vassiliev style link invariants is developed; a new construction of Vassiliev knot invariants is shown in the process. Inside the space of all maps $\mathcal{L}_n$ from $\coprod_{i=1}^{n} S^1$ to R^3 is the discriminant Σ, which is the set of maps that are not embeddings. Following Vassiliev, we employ a spectral sequence to find certain homology groups of the discriminant and, using a sort of infinite dimensional Alexander Duality, obtain knot and link invariants which are elements of $\tilde{H}^0(\mathcal{L}_n - \Sigma)$ and are link invariants of finite type. To calculate the value of the invariant on a knot or link we need some initial data; the needed data is entered into a table called an actuality table. Following Vassiliev, an inductive algorithm is developed that allows us to trace the progress of a given cycle through the spectral sequence without having to calculate the entire spectral sequence. We carefully investigate the the geometry of the discriminant in the course of this development.

We give examples of how the invariants are derived and how they are evaluated on links. The examples presented include examples of knot invariants, link invariants and invariants of link homotopy; we evaluate one of the invariants of link homotopy for two component links on a whole class of two component links. We also give the generalization to links of Birman and Lin's result that establishes an important relationship between invariants of finite type and a general form of the HOMFLY polynomial.

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Introduction

The main result of this paper is the generalization of Vassiliev invariants to links. In [V], V.A. Vassiliev constructs invariants of knots using machinery (mostly) from algebraic topology. Usually, knots are thought of as embedded copies of S^1 inside R^3 or S^3, while Vassiliev chose to regard knots as knotted real lines in three-space, and his invariants are of knotted lines. Birman and Lin, in [BL], have since shown that these invariants extend to invariants of knotted circles. My approach to the construction of analogous link invariants grows out of the work of Vassiliev, rather than out of the more combinatorial approach of Birman, Lin, Bar-Natan, Stanford and others; see [BL], [BL1], [B], and [S].

Before giving an outline of the organization of the dissertation, we will state the main results. Let $\mathcal{L}_n$ be the space of all maps from $\coprod_{i=1}^{n} S^1$ to R^3, and let Σ be the *discriminant* of $\mathcal{L}_n$, which is the subset of $\mathcal{L}_n$ consisting of all maps that are not embeddings. Each embedding in $\mathcal{L}_n$ can be viewed as a link of n components. We will show that there is a nested set of finite dimensional affine subspaces

$$\Gamma^1 \subset \Gamma^2 \subset \Gamma^3 \subset \cdots \mathcal{L}_n$$

whose union is dense in $\mathcal{L}_n$ and whose intersections with Σ contain only *nice* singularities. If $\mathcal{L}_n$ were an m dimensional affine space, and Σ were sufficiently nice, then the Alexander Duality theorem would tell us that

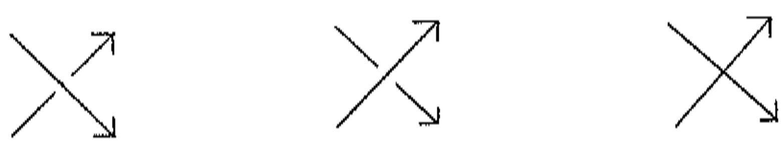
$$\tilde{H}^0(\mathcal{L}_n - \Sigma) \cong \bar{H}_{m-1}(\Sigma),$$

where $\tilde{H}^*$ is reduced cohomology and $\bar{H}_*$ is closed homology. Unfortunately, $\mathcal{L}_n$ is infinite dimensional and Σ is not that nice. Fortunately, Γ^d is finite dimensional and can be constructed so that $\Gamma^d \cap \Sigma$ is nice. Next, we inflate the space $\Gamma^d \cap \Sigma$ so that the inflated space Ω can be given a bounded filtration

$$\Omega_1 \subset \Omega_2 \subset \cdots \subset \Omega$$

determined by the complexity of singular behaviour. The homology of Ω is the same as that of $\Gamma^d \cap \Sigma$, so the above filtration gives rise to a spectral sequence $E_r^{p,q}(d)$ which converges to $\bar{H}_*(\Gamma^d \cap \Sigma)$.

Let's call an immersion L^j in Γ^d *nice* if it has no singularities and its only self intersections are j transverse double points. Let (L_+, L_-, L_x^1) be the triple consisting of two links L_+ and L_- and a nice immersion L_x^1, and assume that L_+, L_- and L_x^1 vary in the location of a single crossing in the manner pictured below:



L_+ $\qquad$ L_- $\qquad$ L_x^1.

Let $(L^j_+, L^j_-, L^{j+1}_x)$ be the triple consisting of two nice immersions L^j_+ and L^j_- with j tranverse double points and a nice immersion L^{j+1}_x with $j+1$ tranverse double points, and assume that these three immersions vary exactly as do L_+, L_- and L^1_x above.

Let $(\mathcal{V}, \mathcal{I})$ be the pair consisting of a Z-valued invariant $\mathcal{V}$ of n component links and a Z-valued invariant $\mathcal{I}$ of nice immersions of n circles. $(\mathcal{V}, \mathcal{I})$ is called a *link invariant of finite type of order I* if it satisfies the following four axioms:

V1. $\mathcal{V}(\text{unlink of } n \text{ components}) = 0$,

V2. $\mathcal{V}(L_+) - \mathcal{V}(L_-) = \mathcal{I}(L^1_x)$

$$\mathcal{I}(L^j_+) - \mathcal{I}(L^j_-) = \epsilon(L^j_+, L^j_-, L^{j+1}_x)\mathcal{I}(L^{j+1}_x), \text{ where } \epsilon(L^j_+, L^j_-, L^{j+1}_x) = \pm 1,$$

V3. $\mathcal{I}(L^m) = 0$ if $m > I$,

V4. $\mathcal{I}($  $) = 0$,

and $(\mathcal{V}, \mathcal{I})$ has some additional data in the form of a table called an *actuality table*. The actuality table gives the value of $\mathcal{I}$ on a certain set of model immersions with j double points, $1 \leq j \leq I$, and this set depends on $(\mathcal{V}, \mathcal{I})$. Since all of our invariants satisfy the four axioms, they are in some sense determined by their actuality tables. We can now state our main results.

Theorem 3.11 *Let n and d be fixed positive integers. Let*

$$\gamma \in E^{-I,I}_\infty(d) \subseteq \bar{H}_{3N-1}(\Omega_I) \subseteq \bar{H}_{3N-1}(\Omega) \cong \bar{H}_{3N-1}(\Gamma^d \cap \Sigma),$$

where $N = n(2d+1)$ and $I \leq \frac{3N+1}{5}$. Then there is a link invariant $\mathcal{V}_\gamma$ and an invariant $\mathcal{I}_\gamma$ of nice immersions such that the pair $(\mathcal{V}_\gamma, \mathcal{I}_\gamma)$ is a link invariant of finite type of order I.

We should note that $\mathcal{V}_\gamma$ is the Alexander dual of γ and is an element of $\bar{H}^0(\Gamma^d - (\Gamma^d \cap \Sigma))$. We also note that $3N$ is the dimension of Γ^d. These invariants stabilize as $d \to \infty$.

Theorem 3.22 *Let n, d, I and γ be as above. Let $\bar{d} > d$. Then for each $r = 1, 2, \ldots, \infty$, we have*

$$E^{-I,I}_r(\bar{d}) \cong E^{-I,I}_r(d).$$

Moreover, if $\bar{\gamma} \in E^{-I,I}_\infty(\bar{d})$ corresponds via this isomorphism to $\gamma \in E^{-I,I}_\infty$, then for each link L and each nice immersion L^j in $\Gamma^d \subset \Gamma^{\bar{d}}$ we have

$$\mathcal{V}_{\bar{\gamma}}(L) = \mathcal{V}_\gamma(L)$$

and

$$\mathcal{I}_{\bar{\gamma}}(L^j) = \mathcal{I}_\gamma(L^j).$$

We also give the generalization to links of Birman-Lin's main theorem from [BL]. Our generalization establishes a relationship between link invariants of finite type and a generalized form of the HOMFLY polynomial.

Theorem 4.41 *Let L be an n component link and let $H_{m,t}(L)$ be its m^{th} generalized HOMFLY polynomial. Let $W_{m,y}(L)$ be obtained by replacing t by e^y. Suppose that*

$$W_{m,y}(L) = \sum_{i=0}^{\infty} w_{m,i}(L) y^i$$

is the power series expansion that results from replacing e^y by its Taylor series about $y = 0$. Let c_i be the i^{th} coefficient of the power series expansion of

$$\left(\frac{e^{y(m+1)/2} - e^{-y(m+1)/2}}{e^{y/2} - e^{-y/2}}\right)^{n-1}$$

where the series is obtained by replacing e^y by its Taylor series about $y = 0$. Then

$$u_{m,i} = w_{m,i} - c_i$$

is a link invariant of finite type of order i.

The machinery needed to develop link invariants is considerably stickier than that which is needed to develop invariants of knotted circles, which from my point of view simply constitute the special case where the number of components is equal to one. Chapter one begins with the preliminary definitions and topological constructions needed to put together the spectral sequence from which the invariants will come. In §1.1-§1.4, we construct the sequence $\{\Gamma^d\}$ of finite dimensional subspaces of $\mathcal{L}_n$ mentioned above. §1.5 ends chapter one and is devoted to inflating $\Gamma^d \cap \Sigma$ to obtain Ω and then constructing the filtration of Ω that gives the spectral sequence.

Chapter two is an explicit construction of the machinery necessary to obtain an invariant of knotted circles and proves our main result for the case $n = 1$. In a sense, Vassiliev pushed certain boundary terms (boundary in the homological sense) out to infinity. In §2.1 and §2.2 we give a CW decomposition of $\Omega_i - \Omega_{i-1}$ that includes cells that live out at infinity (and are therefore not seen) in Vassiliev's construction. An investigation of the initial differential d_0 on the generating cells of the CW decomposition follows in §2.3. In §2.4, we give a procedure for calculating a certain coefficient, called a *relative orientation coefficient*, that determines how the cells in our decomposition fit together. There are some cells in $\Omega_i - \Omega_{i-1}$ that lie in the common boundary of certain pairs of cells in our decomposition; these cells are not present in Vassiliev's construction and the relative orientation coefficients concern these cells. These coefficients are crucial to our link invariants later. In §2.5, we explain how the data in the actuality table is arranged and how it is used to realize a knot invariant. As stated above, the invariant is realized as a cycle γ in the homology of Ω_i and the initial data needed for the actuality table turns out to be the coefficients with which cells, of our decomposition of $\Omega_j - \Omega_{j-1}$ for $j \leq i$, enter into chains which will insure the survival of γ through the spectral sequence. In §2.6, we state our main results in the case $n = 1$. Following Vassiliev, we give an algorithm by which the data of the nontrivial survival of a single given cycle γ through the spectral sequence can be recorded in the actuality table without having to work out the entire spectral sequence. In order to explain the algorithm we must explore the geometry of Ω and we will see that there are geometric reasons why the invariants satisfy the four axioms, and why the information contained in the actuality table gives an invariant. This exposition will prove most of our first main result, with the proofs of some facts postponed until chapter three. Our second main result is stated but not proved until chapter three. This exposition is designed to make the generalization in section three more natural.

Chapter three develops link invariants using the foundations laid in chapter two. In §3.1, §3.2 and §3.3 we construct the CW decompostion of $\Omega_j - \Omega_{j-1}$ that generalizes the one in chapter two. This decomposition is considerably more complicated than the analog for the case of knots. In the boundary of $3N-1$ dimensional cells of $\Omega_j - \Omega_{j-1}$ there exist cells of a certain type called *fixed point cells.* These cells do not occur in Vassliev's construction in the case of knots and, in our construction in chapter two, they cancel out so nicely that they can be ignored. For each $3N-1$ dimensional generating cell in the cellular chain group $C_{3N-1}(\Omega_j - \Omega_{j-1})$, let $d_0(e)$ be expressed as

$$d_0(e) = B_1(e) + B_2(e)$$

where d_0 is the initial differential in the spectral sequence, $B_1(e)$ is the contribution of all fixed point cells to the boundary of e and $B_2(e)$ is the rest of the differential of e. In the case $n = 1$, if $\gamma \in C_{3N-1}(\Omega_j - \Omega_{j-1})$ is a chain such that $B_2(\gamma) = 0$, then it turns out that this is sufficient to insure that γ is a cycle. This is to say that $B_2(\gamma) = 0 \Rightarrow B_1(\gamma) = 0$. *This is not the case for $n \geq 2$.* In the beginning of §3.4, we give a simple example, for $n = 2$, of a chain $\gamma \in C_{3N-1}(\Omega_j - \Omega_{j-1})$ such that $B_2(\gamma) = 0$ but $B_1(\gamma) \neq 0$. We show explicitly that this chain cannot possibly give a link invariant. The result is that when $n \geq 2$, the cancellation of these additional boundary cells, the ones not present in Vassiliev's construction, is a necessary condition for obtaining a link invariant. In §3.4, we state and then prove our main results. First, we will give the analog for the inductive algorithm from chapter two. Secondly, given a cycle in $E_{\infty}^{-j,j}$, we will show that our method for assigning numbers to n component links and nice immersions does indeed yield an invariant. The second main result, that the invariants stabilize as $d \to \infty$, is easy to show after the machinery is set up, and §3.4 concludes with this result.

Chapter four begins with an exposition of some of the combinatorial properties of the invariants. In §4.2, we generalize a lemma of Birman and Lin and then exploit this result to show that our invariants include two classes of invariants of link homotopy, which are link invariants that do not detect the knotting of one component of a link with itself. We also prove that the combinatorial properties of invariants of n component links, for $n \geq 2$, insure certain kinds of symmetry within the collection of cells that enter with nonzero coefficients into the corresponding cycle back in $E_1^{-j,j}$. In §4.3, we give several examples in depth. First, we give an example of how the actuality tables are filled out and then used. We have spared no detail in this example and have presented the process using a diagramatic approach. Next, we present several examples of actuality tables of link homotopy, and we give an example illustrating the fact that sometimes there are choices to be made when filling in the table. We then evaluate one of these invariants on a whole class of two component links. We conclude §4.3 with an example of an invariant of two component links that is not an invariant of link homotopy. This invariant distinguishes the Whitehead link from the unlink, and we show that calculation. In §4.4, we prove the generalization to Birman and Lin's result mentioned above.

Chapter five contains the proofs of lemma 1.6, theorem 3.4, and theorem 3.6. These proofs are not in the sections in which the results are stated because the statements are technical results, and their proofs do not contribute to the exposition.

As much as possible this paper sticks to the notation developed by Vassiliev in [V1] and later by Birman and Lin in [BL]. When the two sets of notation did not agree, I often went with the Birman-Lin notation as it seemed more natural. I have also proved (the generalization of) many facts that were simply stated in Vassiliev's paper. My hope is that

that this paper will aid in the understanding of Vassiliev's.

I wish to thank Dror Bar-Natan, Masahico Saito, Scott Carter, Xiao-Song Lin, Joan Birman, Ted Stanford, Lars Kjeseth and Christine Scharlach for the useful discussions I have had with them in connection with this work. I would also like to thank Jennifer Browning, Gentry Gibson and Pete Betjemann for the moral support they have provided me with. I am particularly indebted to my advisor James Stasheff for the many talks we have had, for the guidance he has given me and for the patience and care he has shown in doing so.

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1 Tools and Preliminaries

1.1 The Space of Links

DEFINITION: A *knot* is an oriented one dimensional C^1 submanifold of S^3 or R^3 that is diffeomorphic to the circle S^1. A *link* is an oriented one dimensional C^1 submanifold of S^3 or R^3 that is diffeomorphic to a disjoint union of n copies of S^1 for some n. Each of the n copies of S^1 making up the disjoint union is called a *component*. We call a link *colored* if its components are labelled in a one to one manner by the integers $1, 2, \ldots, n$.

We can view colored links as the images of embeddings $f : \coprod_{i=1}^{n} S^1 \hookrightarrow S^3$ or R^3, where the component labelled by $m \in \{1, 2, \ldots, n\}$ is the image of the m^{th} copy of S^1 in $\coprod S^1$. We will assume link means colored link.

DEFINITION: Two links are called *equivalent* if there is an orientation preserving isotopy of S^3 or R^3 taking one link to the other that preserves coloring.

Now let

$$\mathcal{L}_n = \{f : \textstyle\coprod_{i=1}^{n} S^1 \to \mathrm{R}^3 \mid f \text{ is an immersion}\}$$

and let

$$\Sigma_n = \{f \in \mathcal{L}_n \mid f \text{ is not an embedding}\}.$$

We will call Σ_n the *discriminant.*

Let n be fixed. We will omit n unless necessary. Note that connected components of $\mathcal{L}\backslash\Sigma$ are in one to one correspondence with the equivalence classes of links.

1.2 Approximations

Following Vassiliev, we want to construct our link invariants initially for finite dimensional subspaces of the space of all maps $\coprod S^1 \to S^3$. We will choose a family of finite dimensional affine subspaces of $\mathcal{L}$ whose union is a dense countably infinite dimensional subspace of $\mathcal{L}$ and will construct our invariants for maps in these spaces.

Let

$$\Gamma^d = \{p = \textstyle\coprod p_i \mid \coprod S^1 \to \mathrm{R}^3,\ p_i = (p_{i1}, p_{i2}, p_{i3})\}.$$

Each p_{ij} has the form:

$$p_{ij}(t) = a_{ij0}/2 + \sum_{k=1}^{d} (a_{i,j,2k-1} \cos kt + a_{i,j,2k} \sin kt)$$

where $t \in [0, 2\pi]$.

Γ^d is a vector subspace of dimension $3n(2d+1)$ with natural coordinates (a_{ijk}), $1 \le i \le n$, $1 \le j \le 3$, $0 \le k \le 2d$.

Note that Γ^d is not transverse to the discriminant. For example any $p \in \Gamma^d$ such that $a_{1jk} = a_{2jk} = \cdots = a_{njk}$, $\forall j, k$, has an uncountable number of self intersections. We would like Γ^d to be transverse to Σ. To do this we embed Γ^d into Γ^{2d} by sending

$$\begin{aligned}(\quad & a_{110}, a_{111}, a_{112}, \ldots, a_{1,1,2d-1}, a_{1,1,2d}, \\ & a_{120}, a_{121}, a_{122}, \ldots, a_{1,2,2d}, \\ & a_{130}, a_{131}, a_{132}, \ldots, a_{1,3,2d}, \\ & a_{210}, a_{211}, a_{212}, \ldots, \\ & a_{n,3,2d-1}, a_{n,3,2d} \;)\end{aligned}$$

in Γ^d to the point in Γ^{2d} with coordinates:

$$\begin{aligned}(\quad & a_{110}, a_{111}, a_{112}, a_{113}, a_{114}, \ldots a_{1,1,2d-1}, a_{1,1,2d}, 0, \ldots, 0, \\ & a_{120}, a_{121}, a_{122}, a_{123}, a_{124}, \ldots a_{1,2,2d-1}, a_{1,2,2d}, 0, \ldots, 0, \\ & a_{130}, a_{131}, a_{132}, a_{133}, a_{134}, \ldots a_{1,3,2d-1}, a_{1,3,2d}, 0, \ldots, 0, \\ & a_{210}, a_{211}, a_{212}, \ldots, a_{n,3,2d-1}, a_{n,3,2d}, 0, \ldots, 0 \;).\end{aligned}$$

Lemma 1.1 *$\Gamma^d \cap \Sigma$ is a stratified semialgebraic set.*

Proof: This follows from the Tarski-Seidenberg theorem. □

Corollary 1.2 *$\Gamma^d \cap \Sigma$ can be triangulated.*

It is a consequence of the (weak) transversality theorem [AVG] that in the set of embeddings $\Gamma^d \hookrightarrow \Gamma^{2d}$, the ones that are transverse to $\Gamma^{2d} \cap \Sigma$ are everywhere dense. By perturbing the image of Γ^d we can assume that it is transverse to $\Gamma^{2d} \cap \Sigma$. From now on we will assume that Γ^d refers to an embedded copy of $\Gamma^d \subset \Gamma^{2d}$ chosen transverse to $\Gamma^{2d} \cap \Sigma$.

The functions $p \in \bigcup_{d \in Z_+} \Gamma^d$ are dense in $\mathcal{L}$ because the Fourier series of any function $f : \coprod S^1 \to \mathrm{R}^3$ converges to f, so that any link is equivalent to one in Γ^d for d large enough.

1.3 Configurations

For a given $p \in \Gamma^d \cup \Sigma$, we would like to have a way of encoding how singular p is. Now we will describe certain types of configurations of points on $\coprod S^1$ that do this.

DEFINITION: An *A-configuration* on $\coprod S^1$ is a finite collection of r points partitioned into s subsets which we will call *groupings.* Each grouping is required to have cardinality ≥ 2 and has the property that the points of the grouping are all on the same copy of $S^1 \in \coprod S^1$. We define $|A| = r$ and $\sharp A = s$.

DEFINITION: Suppose $n \geq 2$. A *C-configuration* on $\coprod S^1$ is a collection of $|C|$ points partitioned into $\sharp C$ groupings. Each grouping must have cardinality ≥ 2, and in every grouping there must be two points that lie on different copies of S^1 in $\coprod S^1$.

DEFINITION: A *B-configuration* on $\coprod S^1$ is a set of b points on $\coprod S^1$.

DEFINITION: An *(A, B, C)-configuration* is a triple consisting of an A-configuration, a B-configuration and a C-configuration. The A and C configurations do not intersect, but the B-configuration may overlap with either or both of the A or C configurations.

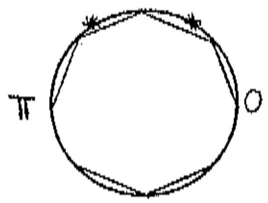
Example with Figure 1.3 *These diagrams will have more significance later. We will fully explain them for the case $n = 1$ in §2.2 and for the general case in §3.3. In this, and all other pictures of circles to come, we will assume that the circle is oriented counterclockwise. Let n=1 and assume that S^1 is parameterized by $[0, 2\pi]$ in the usual way. Let*

$$\{0, \pi/4, \pi/2, 3\pi/4, \pi\} \text{ and } \{5\pi/4, 3\pi/2, 7\pi/4\}$$

be the two groupings of an A-configuration on S^1. Let

$$\{0, \pi/3, 2\pi/3\}$$

be a B-configuration on S^1.
$|A| = 8$, $\sharp A = 2$ *and* $b = 3$:



Example with Figure 1.4 *Let $n = 2$ and again assume that each of the two copies of S^1 are parameterized by $[0, 2\pi]$. We will subscript $\theta \in [0, 2\pi]$ to indicate inclusion in the first or second copy of $S^1 \subset S^1 \coprod S^1$. Let*

$$\{0_1, (\pi)_1\}, \{(\pi/6)_2, (\pi)_2\} \text{ and } \{(\pi/2)_2, (3\pi/2)_2\}$$

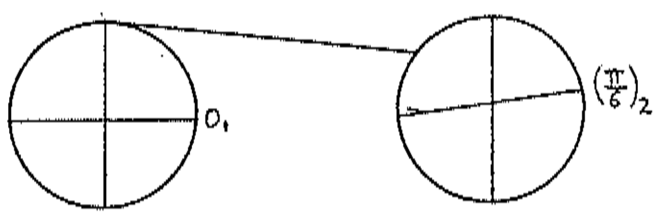
be the groupings of the A-configuration. Let

$$\{(\pi/2)_1, (3\pi/2)_1, (3\pi/4)_2\}$$

be the grouping of the C-configuration, and let

$$\{(\pi/3)_1, (\pi/2)_2, (2\pi/3)_2\}$$

be the B-configuration.
$|A| = 6$, $\sharp A = 3$, $|C| = 3$, $\sharp C = 1$ *and* $b = 3$:



From now on let us assume that each copy of S^1 in $\coprod S^1$ is parameterized by $[0, 2\pi]$

DEFINITION: Let J be an (A, B, C)-configuration. If $n = 1$, then we call $\min\{x \in S^1 \cap J\}$ the *first point* of J, where the minimum is taken relative to the standard ordering of $[0, 2\pi)$. If $n \geq 2$, then $\min\{x \in J \cap (S^1)_i\}$ is called the *first point* of J on $(S^1)_i$.

Example 1.5 *Using the Example with Figure 1.4, $(0)_1$ is the first point of the (A, B, C)-configuration on $(S^1)_1$ and $(\pi/6)_2$ is the first point of the (A, B, C)-configuration on $(S^1)_2$.*

DEFINITION: We say that two (A, B, C)-configurations J and J' are *equivalent* if there is an orientation preserving diffeomorphism of $\coprod_{i=1}^n S^1$ that sends $(S^1)_i$ to itself for each i and takes J to J' preserving first points. We write $J' \sim J$ if J and J' are equivalent.

DEFINITION: Suppose $p \in \Gamma^d$ and suppose J is an (A, B, C)-configuration. Let

$$\{g_1, g_2, \ldots, g_{\sharp A}\}$$

be the groupings of points in the A-configuration of J,

$$\{h_1, h_2, \ldots, h_{\sharp C}\}$$

be the groupings of points of the C-configuration of J and

$$\{\beta_1, \beta_2, \ldots, \beta_b\}$$

be the points of the B-configuration of J. If for every x and y in g_i we have $p(x) = p(y)$, $1 \leq i \leq \sharp A$ and for every u and v in h_j we have $p(u) = p(v)$, $1 \leq j \leq \sharp C$, and for $1 \leq k \leq b$ we have $p'(\beta_k) = 0$ then we say that p *respects* J.

This is to say that a map p respects a configuration J if for each grouping of points in the A or C configurations of J, the immersion p maps every point in that grouping to the same point, and p has vanishing derivative at each point in the B-configuration of J.

Let J be an (A, B, C)-configuration. We define

$$M(\Gamma^d, J) = \{p \in \Gamma^d \mid p \text{ respects } J\}$$

and we define the *complexity* of J to be the number

$$I = |A| + |C| + \sharp A + \sharp C + b.$$

If $p \in M(\Gamma^d, J)$, and the image of p is a curve that posseses no self intersections or singularities beyond those mandated by J, then we say that the curve *realizes* J. This terminology is due to Bar-Natan in [B] and will be very useful.

1.4 A Lemma and More Definitions

Note that p is in $M(\Gamma^d, J)$ if and only if p is a solution of $3I$ equations that are linear in the coordinates a_{ijk} of Γ^d. These equations come in three forms:

1. For some i, $1 \leq i \leq n$ and $p = \coprod p_i$, we have $p_i(s) - p_i(t) = 0$, where s and t are in the same grouping of points in the A-configuration of J.

2. For some i and j, $1 \leq i, j \leq n$, $p_i(s) - p_j(t) = 0$, where s and t are in the same grouping of points in the C-configuration of J.

3. For some i, $1 \leq i \leq n$, $p_i'(t) = 0$, where t is a point of the B-configuration of J.

Note that our choice of Γ^d insures that Γ^d does not contain the zero map in Γ^{2d}, as it is too singular for our purposes. Each Γ^d is an affine $3N$ dimensional plane in Γ^{2d} that misses zero. It follows that p is in $M(\Gamma^d, J)$ if and only if p is a solution to the matrix equation $\Phi_J(\vec{x}) = 0$, where $N = n(2d+1)$, $\Phi_J \in \text{Hom}_{\mathbb{R}}(\mathbb{R}^{3N+1}, \mathbb{R}^{3I})$ is determined by J, while Γ^d is determined J and Γ^d is identified with $\mathbb{R}^{3N}$.

The following lemma establishes the relationship between the complexity of a given (A,B,C)-configuration J and the dimension of $M(\Gamma^d, J')$ for some $J' \sim J$.

Lemma 1.6 *For any (A,B,C)-configuration J we have that:*

A) For almost every $J' \sim J$, the set $M(\Gamma^d, J)$ has codimension $3I$ inside Γ^d and almost all $M(\Gamma^d, J')$ are empty when $I > N$.

B) If $I \leq N$ then

$$\{J' \mid J' \sim J \text{ and } M(\Gamma^d, J') \text{ has codimension } 3I - t \text{ in } \Gamma^d, t \geq 1\}$$

is a subset in $\{J' \mid J' \sim J\}$ of codimension $\geq t(3N - 3I + t + 1)$. If $I \leq (3N+1)/5$ then, for each $t \geq 1$, the set

$$\{J' \mid \ J' \sim J \text{ and } M(\Gamma^d, J') \text{ has codimension } 3I - t\}$$

is empty.

C) If $I > N$ then, in $\{J' \mid J' \sim J\}$, the set of J' such that $\dim_R M(\Gamma^d, J) = s \geq 0$ is of codimension $(s+1)(3I - 3N + s)$. If $I \geq 3N - 1$ then $\{J' \mid J' \sim J\}$ is empty.

Proof: The proof uses some basic linear algebra, as well as the Product of Coranks theorem [AVG], and can be found in Chapter five. □

Part A) of the lemma says that for a generic $J' \sim J$, the space $M(\Gamma^d, J)$ has the codimension one would expect, i.e. $3I$. Part B) says that, for $I \leq (3N+1)/5$, $M(\Gamma^d, J)$ always has codimension $3I$, and part C) says that if $I \geq 3N - 1$ then $M(\Gamma^d, J)$ is zero dimensional.

For our purposes some configurations are special, so we will introduce some more terminology, following Birman-Lin and Vassiliev.

DEFINITION: An (A,B,C)-configuration J is called *inadmissible* if it contains a grouping of cardinality two in the A-configuration of J, and these two points bound a segment on some $(S^1)_i \subset \coprod S^1$ that contains no other points of J. Otherwise J is called *admissible.*

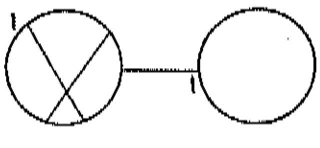
DEFINITION: An (A,B,C)-configuration J is called *noncomplicated* if J has any of the following forms:

- The groupings of the A and C configurations all have cardinality two and the B-configuration is empty. A configuration of this type, of complexity I, is called an $[I]$-configuration.

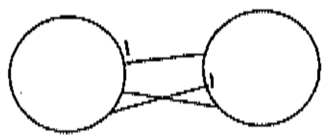
- One grouping of the A-configuration has cardinality three, all of the other groupings of the A-configuration and all of the groupings of the C -configuration have cardinality two, and the B-configuration is empty. This type of configuration is called an $\langle I\rangle_A$-configuration.
- One grouping of the C-configuration has cardinality three, all of the other groupings of the C-configuration and all of the groupings of the A-configuration have cardinality two, and the B-configuration is empty. This kind of configuration is called an $\langle I\rangle_C$-configuration.
- All of the groupings of the A and C configurations have cardinality two. The B-configuration consists of a single point that is distinct from all of the other points in the configuration. This type of configuration is called an $[I-1]^*$-configuration.

A configuration is called *complicated* otherwise.

Example with Figure 1.7 *Let $n = 2$. Pictured are two sets of two circles, each representing a copy of S^1 in $S^1 \amalg S^1$. The circle to the left is the first circle. We will describe a diagram that encodes the given (A, B, C)-configuration. The configurations are pictured by denoting points of the A and C configurations by little dots on the circles and by denoting points on the B-configuration by little stars. Groupings of points are connected by chains of arcs of* chords. *These chords have added significance later. The first point on each circle is denoted by a 1 next to the point.*
$|A| = 4$, $\sharp A = 2$, $|C| = 2$, $\sharp C = 1$ *and* $b = 1$*:*



$|A| = 0$, $\sharp A = 0$, $|C| = 6$, $\sharp C = 3$ *and* $b = 0$*:*



Each of the examples above are of noncomplicated configurations. The diagrams actually describe the whole equivalence class of configurations containing the given one.

1.5 Generating Collections

Fix the integer d. Later on we will be trying to find certain homology groups of the space $\Gamma^d \cap \Sigma$. Following Vassiliev, we will first construct a topological space whose homology agrees with that of $\Gamma^d \cap \Sigma$. It will be easier to calculate the homology of the new space. First we need Vassiliev's notion of a generating collection.

DEFINITION: Let $W \subset \coprod S^1$ be a finite set. A *generating collection* T of W is a family of pairwise distinct unordered pairs $(t_1, s_1), (t_2, s_2), \ldots, (t_l, s_l)$ where $t_i, s_i \in W$ and $t_i \neq s_i$ for $1 \leq i \leq l$ and where any map $p : \coprod S^1 \to \mathbb{R}^3$ has the following two properties:

1. p maps each point in W to the same point,

2. for every t and s in W, there is a finite nonrepeating sequence $t = t_0, t_1, t_2, \ldots, t_r = s$ of elements of W such that $(t_0, t_1), (t_1, t_2), \ldots, (t_{r-1}, t_r)$ are pairs of T.

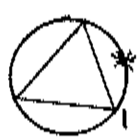
DEFINITION: The triple (T, V, R) is a *generating collection* for an (A, B, C)-configuration J if T is the union of generating collections for each of the groupings of the A-configuration of J, R is the union of generating collections for each of the groupings of the C-configuration of J and V is the set of pairs

$$\{(\beta_1, \beta_1), (\beta_2, \beta_2), \ldots, (\beta_b, \beta_b)\}$$

where $\{\beta_1, \beta_2, \ldots, \beta_b\}$ are the points of the B-configuration of J.

There are, in general, several different generating collections for a given (A, B, C)-configuration, and there is a unique maximal generating collection for each (A, B, C)-configuration.

Example with Figure 1.8 *The type of diagram that we used to carry the information of an (A, B, C)-configuration can also carry the additional data of a generating collection of the (A, B, C)-configuration. The arcs, that connect pairs of points, can themselves be viewed as unordered pairs. A star, placed at the location of a singular point β of the B-configuration, can also denote the pair (β, β) corresponding to that point in a generating collection.*
A maximal generating collection:



Three submaximal generating collections:

Now let's create the space whose homology agrees with that of $\Gamma^d \cap \Sigma$. To do this we will inflate the maps of $\Gamma^d \cap \Sigma$ in such a way that the subspaces $M(\Gamma^d, J)$ are inflated in inverse proportion to the complexity of J. The result, we shall see, is to create a space which has a nice CW decomposition in which the generating cells will correspond to equivalence classes of (A, B, C)-configurations. The reason to inflate is to insure that the cells corresponding to $[j]$-configurations, for $1 \leq j \leq I$, all have the same dimension; this leads us to a filtration of the CW decomposition by complexity, and finally to a spectral sequence. We need the following lemma.

Let $\iota : \coprod S^1 \times \coprod S^1 \to \coprod S^1 \times \coprod S^1$ be the involution taking (x, y) to (y, x). Let $\Phi = (\coprod S^1 \times \coprod S^1)/\iota$.

Lemma 1.9 *For any integer k, there is an integer M and an embedding $\lambda : \Phi \to R^M$ with the property that no set of k distinct points of Φ are mapped by λ to a $(k-2)$ dimensional affine plane in R^M.*

Proof: This is an application of the lemma found in the proof of theorem 1.5 in [V2]. □

For M dependent on $k = 3N(3N-1)$ where $N = n(2d+1)$, fix such an embedding λ. Fix an (A, B, C)-configuration J and let (T, V, R) be a generating collection for J. The collection (T, V, R) consists of the following sets of unordered pairs of points:

$$\{(t_1, s_1), (t_2, s_2), \ldots, (t_l, s_l)\}$$
$$\{(r_1, q_1), (r_2, q_2), \ldots, (r_m, q_m)\}$$
$$\{(\beta_1, \beta_1), (\beta_2, \beta_2), \ldots, (\beta_b, \beta_b)\}$$

from the A, C and B configurations of J. Suppose that $a_1, a_2, \ldots, a_{\sharp A}$ and $c_1, c_2, \ldots, c_{\sharp C}$ are the cardinalities of the groupings of the A and C configurations, respectively. From the inequalities $l \leq \sum a_i(a_i - 1)$ and $m \leq \sum c_j(c_j - 1)$ we obtain:

$$\begin{aligned} l + m + b &\leq \sum a_i(a_i - 1) + \sum c_i(c_i - 1) + b \\ &\leq [\max_i(a_i)]\,[\sum \frac{a_i - 1}{2}] + [\max_j(c_j)]\,[\sum \frac{c_j - 1}{2}]/2] + b. \end{aligned}$$

Now, if $I = |A| + |C| - \sharp A - \sharp C + b$ is the complexity of J, then $I + 1 \geq \max_i(a_i)$ and $I + 1 \geq \max_j(c_j)$. From the inequalities above we get:

$$\begin{aligned} l + m + b &\leq (I+1)\sum_{i,j}[\frac{a_i - 1}{2} + \frac{c_j - 1}{2}] + [\frac{I+1}{2}]b \\ &= [\frac{I+1}{2}]I \leq \frac{3N(3N-1)}{2} \end{aligned}$$

as long as $I \leq 3N$. The result is that the points

$$\lambda(t_1, s_1), \ldots, \lambda(t_l, s_l), \lambda(r_1, q_1), \ldots, \lambda(r_m, q_m), \lambda(\beta_1, \beta_1), \ldots, \lambda(\beta_b, \beta_b)$$

are in general position, so that their convex hull is an $l + m + b - 1$ dimensional simplex.

Let p be in $M(\Gamma^d, J)$. To the pair $(p, (T, V, R))$ we assign the subset of $\Gamma^d \times R^M$ consisting of p in the first coordinate and the interior of the above obtained $l + m + b - 1$ dimensional simplex in the second coordinate. We will call this simplex the *standard simplex* associated to (T, V, R).

Lemma 1.10 *If $(p_1, (T_1, V_1, R_1)) \neq (p_2, (T_2, V_2, R_2))$ then the corresponding standard simplices have no common interior points.*

Proof: If $p_1 \neq p_2$, then (T_1, V_1, R_1) and (T_2, V_2, R_2) lie in different copies of R^M. If $(T_1, V_1, R_1) \neq (T_2, V_2, R_2)$ are generating families of the same configuration then their standard simplices are either both faces (not necessarily of codimension one!) of the standard simplex of the unique maximal generating collection, or one is a face of the other. In either case they share no interior points. Finally, suppose (T_1, V_1, R_1) and (T_2, V_2, R_2) are generating collections for configurations J and J', respectively, and $p \in M(\Gamma^d, J) \cap M(\Gamma^d, J')$. The total number of points spanning the two standard simplices is less than $2(\frac{3N(3N-1)}{2}) = k$. It follows that these k points are in general position so that the interiors of the two standard simplices cannot overlap. □

If J is an (A, B, C)-configuration, let $calS_J$ refer to the standard simplex of the maximal generating collection of J. Let S_J be the union of the interiors of the standard simplices of all of the generating families of J. Note that $S_J \subset \mathrm{S}_J$ and also that S_J does not contain the interior of every face of S_J. Define

$$\Omega = \bigcup M(\Gamma^d, J) \times \mathrm{S}_J$$

where the union is taken over all configurations J of complexity less than or equal to $3N$. Let $\bar{H}_*$ refer to *closed homology*, which is the homology of the one point compactification of the space modulo the compactifying point.

Theorem 1.11 *If* $F : \Omega \to \Sigma \cap \Gamma^d$ *is given by* $(p, x) \mapsto p$ *then* F *induces an isomorphism* $F_* : \bar{H}_*(\Omega) \to \bar{H}_*(\Sigma \cap \Gamma^d)$.

Proof: This is proved for the case $n = 1$ as lemma 1 in section 3.3 of Chapter three in [V1] and the same proof works in this more general setting. A different wording of the same type of proof can also be found as the proof of proposition 3.4 in [V3]. The idea is that we are inflating Σ in such a way that the inflated space Ω can retract back to Σ. □

Because $\Gamma^d = \mathrm{R}^{3N}$, the one point compactification of Γ^d is homeomorphic to S^{3N}. Using Alexander duality, we obtain the result that

$$\bar{H}_{3N-q-1}(\Sigma \cap \Gamma^d) \cong \tilde{H}^q(\Gamma^d - \Sigma),$$

where $\tilde{H}^*$ refers to reduced cohomology.

Any link L can be approximated by a link L_d in Γ^d for d large enough. A cocycle of $\tilde{H}^0(\Gamma^d \backslash \Sigma)$ evaluated on the component of L_d in $\Gamma^d \cap \Sigma$ gives an invariant of that component. Since this invariant stabilizes as $d \to \infty$, it gives an invariant of L.

Alexander duality says that to find $\tilde{H}^0(\Gamma^d - \Sigma)$, we can find $\bar{H}_{3N-1}(\Gamma^d \cap \Sigma)$ and so by the theorem above finding $\bar{H}_{3N-1}(\Omega)$ becomes our goal.

To obtain a filtration by complexity, let

$$\Omega_I = \{(p, x) \in \Omega \mid p \in M(\Gamma^d, J) \text{ and the complexity of } J \text{ is } \leq I\}.$$

Then $\Omega_1 \subset \Omega_2 \subset \cdots \Omega_{3N} = \Omega$ is an increasing bounded filtration. This filtration determines an homology spectral sequence $E^r_{s,t}(d)$ converging to $\bar{H}_*(\Omega)$.

Following Vassiliev, transform the homology spectral sequence $E^r_{s,t}(d)$ to a spectral sequence $E_r^{p,q}(d)$ by letting $p = -s$ and $q = 3N - 1 - t$. The spectral sequence $E_r^{p,q}(d)$ operates like a comohomology spectral sequence with its differentials. This is to say that d_0 will be a set of arrows pointing upwards, d_1 a set of arrows pointing left to right and so forth. We call $E_r^{p,q}(d)$ the *principal spectral sequence*, and it is a second quadrant spectral sequence. We make this change so that the groups that we are interested in, which are the groups $E_r^{-I,I}(d)$, lie on the upper negative diagonal. Thus to find $\bar{H}_{3N-1}(\Gamma^d \cap \Sigma)$ we need to find the groups $E_\infty^{-I,I}(d)$.

2 A New Construction for the Case of Knots

2.1 J-Blocks and the First Reductions

Vassiliev constructed his knot invariants for *noncompact* knots, which are knotted embedded real lines in $\mathbb{R}^3$ with fixed asymptotic directions. There is a one to one correspondence between embeddings of S^1 in $\mathbb{R}^3$, S^1 in S^3, and $\mathbb{R}$ in $\mathbb{R}^3$ that have fixed asymptotic directions; thus Vassiliev's construction yields a knot invariant. Vassiliev's construction depends on this representation in several ways. In particular, the configuration space of α points on a line is homeomorphic to $(0,1)^\alpha$, and so the result of theorem 2.5, that a certain bundle is trivial, comes for free. Additionally, and perhaps more importantly, Vassiliev's space $\sigma_I - \sigma_{I-1}$, corresponding to our space $\Omega_I - \Omega_{I-1}$, has a much simpler CW decomposition than ours does. Certain cells, which we will call *fixed point cells*, are *pushed out to infinity* in some sense. While this simpler decomposition is easier to work with, it has the disadvantage of not being generalizable to links and does not carry some of the useful information that the more complex one gains from how fixed point cells glue the space together. In order to generalize Vassiliev's work to links, we will have to create machinery to treat the case of knotted circles rather than knotted lines. For the remainder of this chapter we will restrict our attention to the case of knots, so the number n of components is fixed at 1. When $n = 1$, any configuration J is only an (A, B)-configuration, since there is no C-configuration in J.

Recall that the equivalence class of an (A, B)-configuration J determines a diagram (see examples 1.7 and 1.8) and that the first point of J is marked by a 1 on the diagram. In general, here are other equivalence classes of configurations that determine a diagram that is pictorially the same as the one that the class of J determines, except that its first point lies at some other point.

Example with Figure 2.1 *Pictured are two chord diagrams that are identical, except for the location of first points. We will explain the significance of chord diagrams in §2.2.*

DEFINITION: Let J be an (A, B)-configuration. We say that J^r is *related* to J (the superscript r is for related) if the equivalence class of J^r determines a diagram that is identical to the diagram associated with $\{J' \mid J' \sim J\}$ except that it varies in the location of its first point. We denote this by $J \leftrightarrow J^r$ We call the set $\{J^r \mid J^r \leftrightarrow J\}$ the *family* of J. Let

$$B_J = \bigcup M(\Gamma^d, J^r) \times S_{J^r}$$

where the union is taken over all J^r related to J. The union B_J is called a J-*block*.

Lemma 2.2 *If J_1 and J_2 are nonrelated configurations of complexity I, then $\mathcal{B}_{J_1} \cap \mathcal{B}_{J_2} = \emptyset$*

Proof: Lemma 1.4 implies that

$$M(\Gamma^d, J_1^r) \times \mathcal{S}_{J_1^r} \cap M(\Gamma^d, J_2^r) \times \mathcal{S}_{J_2^r} = \emptyset$$

for every J_1^r related to J_1 and J_2^r related to J_2. □

To begin computing the spectral sequence, we must first compute $\bar{H}_{3N-1}(\Omega_I - \Omega_{I-1})$. Let $Z_I = \cup \mathcal{B}_J$, where the union is taken over all complicated configurations of complexity I.

Lemma 2.3 *Z_I is closed in $\Omega_I - \Omega_{I-1}$.*

Proof: Suppose that $\bar{Z}_I - Z_I \neq \emptyset$ and $q \in \bar{Z}_I - Z_I$. Since q is not in Z_I, we know that $q \in \mathcal{B}_J$ for some noncomplicated J and $q = (p, x)$ for some $p \in M(\Gamma^d, J)$ and $x \in \mathrm{S}_J$.

The point x lies in the interior of a standard simplex $S_\Psi \subsetneq \mathrm{S}_J$ for some generating collection Ψ of J. By lemma 1.10, the interior of S_Ψ doesn't intersect the interiors of any other standard simplices. Since $q \in \bar{Z}_I$, it follows that S_Ψ must be a face of some standard simplex $\mathcal{S}$ corresponding to a complicated configuration $\mathcal{J}$ of complexity I.

Let $\{(s_i, t_i)\}$ be the collection of pairs in Ψ and, for each i, let $x_i = \lambda(s_i, t_i)$ where λ is as in lemma 1.9. The simplex S_Ψ is spanned by $\{x_i\}$ and this collection of points must be a subcollection of the set $\{y_i\}$ of points spanning $\mathcal{S}$.

It follows that either $\{x_i\} = \{y_i\}$ or $\{x_i\} \subset \{y_i\}$. In the first case $J = \mathcal{J}$ by lemma 2.2 and so $\mathcal{J}$ is noncomplicated, which is a contradiction. In the second case $\mathcal{J}$ must have complexity strictly greater than I, again a contradiction. The result is that Z_I is closed. □

Theorem 2.4 *For each I, $\bar{H}_k(Z_I) = 0$ when $k \geq 3N - 2$.*

Proof: This is proved as theorem 3.1.2 in [V]. The more general result is theorem 3.4, which we will prove in Chapter five. □

Corollary 2.5 *The map $\Phi : \bar{H}_k(\Omega_I - \Omega_{I-1}) \to \bar{H}_k(\Omega_I - (\Omega_{I-1} \cup Z_I))$, induced by the projection map $C_k(\Omega_I - \Omega_{I-1}) \to C_k(\Omega_I - (\Omega_{I-1} \cup Z_I))$, is an isomorphism for $k \geq 3N - 1$ and injective for $k \geq 3N - 2$.*

Proof: We have an isomorphism

$$C_k(\Omega_I - (\Omega_{I-1} \cup Z_I)) \to C_k(\Omega_I - \Omega_{I-1})/C_k(Z_I)$$

which gives the short exact sequence of complexes:

$$0 \to C_k(Z_I) \to C_k(\Omega_I - \Omega_{I-1}) \to C_k(\Omega_I - (\Omega_{I-1} \cup Z_I)) \to 0.$$

The result follows from the induced long exact sequence of the corresponding homology groups and theorem 2.4. □

For the remainder of this Chapter, let d be a fixed large positive integer. Suppose $I \leq \frac{3N+1}{5}$ and J has complexity I. By lemma 1.4.B, $M(\Gamma^d, J)$ has codimension exactly equal to $3I$ inside Γ^d. Because $S_J \cong S_{J^r}$ and $M(\Gamma^d, J) \cong M(\Gamma^d, J^r)$ for every J^r related to J, $\mathcal{B}_J$ can be viewed as the total space of a locally trivial fibration over $\{J^r \mid J^r \leftrightarrow J\}$ with fiber $M(\Gamma^d, J) \times S_J$. Let $\rho(J)$ be the number of geometrically distinct points in J. We will see in §2.2 that the configuration space $\{J^r \mid J^r \leftrightarrow J\}$ is homeomorphic to $S^1 \times (0,1)^{\rho(J)-1}$. In order to calculate homology groups of $\Omega_I - \Omega_{I-1}$, we need to know how twisted this fibration is. It turns out that it is not twisted at all.

Theorem 2.6 *$\mathcal{B}_J$ is a trivial fibration; as a result, $\mathcal{B}_J$ is homeomorphic to*

$$\{J^r \mid J^r \leftrightarrow J\} \times M(\Gamma^d, J) \times S_J.$$

Proof: The proof uses some basic linear algebra and shows explicitly that each twisting map is the identity map. The more general result is given as theorem 3.6 and is proved in Chapter five. □

2.2 Chord diagrams and the CW Decomposition of $\Omega_I - \Omega_{I-1}$

Let J be an (A, B)-configuration of complexity I and assume that d is fixed large enough that $I \leq \frac{3N+1}{5}$, where $N = n(2d+1)$. We will design and explain a type of diagram that we have already seen in the examples. These diagrams encode the topological structure of $\mathcal{B}_J$. The diagram corresponds to the triple consisting of the family of configurations related to J, the interior of a face S_Ψ of S_J arising from a generating family $\Psi = (T, R)$ of J and the space $M(\Gamma^d, J)$.

Start with a circle pictured on the plane and with counterclockwise orientation. On the circle are $\rho(J)$ points with the first point of J marked by a 1. When two points on the circle are contained in the same grouping of the A-configuration of J and the pair consisting of those points appears in Ψ, we join the two points by a single chord. When a point on the circle belongs to the B-configuration we distinguish it with a $*$.

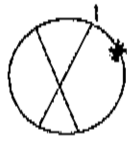
We will see that each of these diagrams corresponds to a cell in a certain CW decomposition of $\Omega_I - \Omega_{I-1}$. The cell corresponding to one of these diagrams has dimension:

$$\rho(J) + (3N - 3I) + (\text{the number of chords}) + b - 1$$

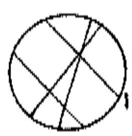
where $\rho(J)$ is the dimension of the configuration space $\{J' \mid J' \sim J\}$, while $3N - 3I$ is the dimension of $M(\Gamma^d, J)$ and (the number of chords)$+b-1$ is the dimension of the simplex corresponding to a generating family Ψ of J.

Example with Figure 2.7 *The four types of diagrams which can occur for noncomplicated configurations are shown in the following four figures.*

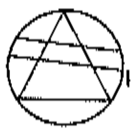
The first is for a $[2]^*$-configuration; the associated cell has dimension $3N-2$:



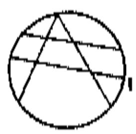
The following diagram is for a $[4]$-configuration; the associated cell has dimension $3N-1$:



The next is for a $\langle 4\rangle$-configuration; the cell has dimension $3N-1$:



The last is for a $\langle 4\rangle^*$-configuration; the cell has dimension $3N-2$:



We will now describe precisely the CW decomposition of the one point compactification of $\Omega_I - \Omega_{I-1}$ that we will use. We take the one point compactification, following Vassiliev, because the closures of some of the cells that make up $\Omega_I - \Omega_{I-1}$ are not contained in $\Omega_I - \Omega_{I-1}$. It is sufficient to decompose the one point compactification of each B_J. This decomposition allows a generalization of Vassiliev's work to the case of knotted circles; it is this decomposition which has a natural extension to the case of links.

Recall that our (A,B)-configurations are finite sets of points on circles parameterized by $[0,2\pi]$. Each configuration J has a point that occurs first as we move in the counterclockwise direction from 0. We call this point the *first point* of J, and we include the case where this first point actually is 0. For each J^r related to J, let $q_1(J^r)$ be the parameter value of the first point of J^r. Continuing around the circle counter clockwise, let $q_2(J^r)$ be (the parameter value of) the next point after $q_1(J^r)$. Let $q_3(J^r),\ldots,q_{\rho(J)}(J^r)$ be the other points of J^r in order taken the same way. We obtain coordinates

$$Q(J^r) = (q_1(J^r), q_2(J^r), \ldots, q_{\rho(J)}(J^r))$$

for each J^r in the configuration space $\{J^r \mid J^r \leftrightarrow J\}$.

Let e^k denote an open cell of real dimension k. Decompose $\{J' \mid J' \sim J\}$ into two cells c_0 and c_1 as follows: let

$$\begin{aligned} c_0(J) &= \{(q_1, q_2, \ldots, q_{\rho(J)}) \mid q_1 \in (0,2\pi)\} \cong e^{\rho(J)} \\ c_1(J) &= \{(0, q_2, q_3, \ldots, q_{\rho(J)})\} \cong e^{\rho(J)-1}. \end{aligned}$$

This decomposition leads to CW decomposition on the one point compactification of $\mathcal{B}_J$, which we will call the *canonical decomposition.* Let $\mu = \rho(J)$. This number μ is the number of equivalence classes of configurations that are subsets of the family $\{J^r | J^r \leftrightarrow J\}$ of J. Let $J_1, J_2, \ldots J_\mu$ be representatives of these equivalence classes. Aside from the compactifying point, the decomposition of $\mathcal{B}_J$ has the following cells:

$$\begin{aligned} e(\Psi, J_j) &= \{(Q,p,x) | Q \in c_0(J), p \in M(\Gamma^d, J_j), x \in S_\Psi\} \\ f(\Psi, J_j) &= \{(Q,p,x) | Q \in c_1(J), p \in M(\Gamma^d, J_j), x \in S_\Psi\} \end{aligned}$$

where S_Ψ is the interior of the simplex associated to Ψ and where $1 \leq j \leq \mu$. We call each $e(\Psi, J_j)$ a *best cell* of $\mathcal{B}_J$ and each $f(\Psi, J_j)$ a *fixed point cell* of $\mathcal{B}_{J_i}$. The number of best cells and the number of fixed point cells are both equal to μ times the number of distinct generating families of J and are both equal to the number of chord diagrams that have $J_1, J_2, \ldots, J_{\mu-1}$ or J_μ as an underlying configuration. The best cells are connected to the fixed point cells in an obvious way. We will discuss how the cells are oriented and how they are glued together in the next section. We will need to use the terms *best cell* and *fixed point cell* very frequently, so from now on we will call any best cell a BC and any fixed point cell a FPC.

Since $\Omega_I - \Omega_{I-1}$ is the union of J-blocks over all J of complexity I, we have described a cellular structure on the one point compactification of $\Omega_I - \Omega_{I-1}$.

We will now describe an orientation of these cells. Let J be an (A, B)-configuration. First, order the points of $\rho(J)$ points of J as they were given in the above coordinate expression $Q(J)$. We will call this the *standard ordering of the points* of J.

Let $g_1, g_2, \ldots, g_{\sharp A}$ be the groupings of points in the A-configuration and let a_i be the cardinality of g_i, $1 \leq i \leq \sharp A$. Let $\beta_1, \beta_2, \ldots, \beta_b$ be the b points of the B-configuration taken in order.

Order the groupings lexicographically, setting $g_i \leq g_j$ if and only if $a_i \leq a_j$ and if $a_i = a_j$ setting $g_i < g_j$ if and only if $\min(g_i) < \min(g_j)$, where the min is taken relative to the standard ordering of points.

Following 3.3 in [V], an orientation of $e(\Psi, J)$ or $f(\Psi, J)$ consists of orientations of S_Ψ, of $M(\Gamma^d, J)$ and of c_0 or c_1, respectively.

To orient S_Ψ we order its vertices. Suppose (t_1, s_1) and (t_2, s_2) are pairs from Ψ and $t_1, s_1, t_2, s_2 \in g_i$ for some i. Set $(t_1, s_1) \leq (t_2, s_2)$ if and only if $\min(t_1, s_1) \leq \min(t_2, s_2)$, and if $\min(t_1, s_1) = \min(t_2, s_2)$, then set $(t_1, s_1) \leq (t_2, s_2)$ if and only if $\max(t_1, s_1) \leq \max(t_2, s_2)$. If (t_1, s_1) and (t_2, s_2) are pairs in Ψ where $t_1, s_1 \in g_i$ and $t_2, s_2 \in g_j$, then $(t_1, s_1) \leq (t_2, s_2)$ if and only if $g_i \leq g_j$. The pairs $(\beta_1, \beta_1), \ldots (\beta_b, \beta_b)$ appear after the pairs from the A-configuration and in the order that $\beta_1, \ldots, \beta_b$ appear on S^1. We will call this ordering the *standard ordering of the pairs* of J.

We orient $S^1 \times (0,1)^{\rho(J)}$ in the standard way. This gives an orientation of c_0 and c_1.

To orient $M(\Gamma^d, J)$, first fix an orientation of $\Gamma^d \cong \mathbb{R}^{3N}$. Recall that J determines $3I$ functions linear in the coordinates (a_{ijk}) of Γ^d. Let $p \in \Gamma^d$ and let p_1, p_2, p_3 be the coordinate functions of $p : S^1 \to \mathbb{R}^3$. The $3I$ functions come in two forms:

1. $p_r(t) - p_r(s)$ for $1 \le r \le 3$ and $s, t \in J$, and
2. $\frac{\partial p_r}{\partial x}|_{x=t}$ for $1 \le r \le 3$ and $t \in J$.

Now let

$$\begin{array}{rcl} t_{11}, t_{12}, \ldots, t_{1a_1} & \in & g_1, \\ t_{21}, t_{22}, \ldots, t_{2a_2} & \in & g_2, \\ \vdots & & \vdots \\ t_{\sharp A1}, t_{\sharp A2}, \ldots, t_{\sharp A a_{\sharp A}} & \in & g_{\sharp A}, \\ \beta_1, \ldots, \beta_b & \in & \text{the } B\text{-configuration of } J \end{array}$$

be the points of J, written such that the groupings $g_1, \ldots, g_{\sharp A}$ are in the order given above, and both the points of each grouping and the points of the B-configuration of J inherit their order from the standard ordering of the points of J.

We will define a $3I$-form w_J that orients the orthogonal complement of $M(\Gamma^d, J) \subset \Gamma^d \cong \mathbb{R}^{3N}$. Each grouping $g_i = \{t_{i1}, t_{i2}, \ldots, t_{ia_i}\}$ contributes a $3(a_i - 1)$-form to the form w_J. This $3(a_i - 1)$-form is the wedge product, in ascending order, of the 3-forms

$$d(p_1(t_{i,k+1}) - p_1(t_{i,k})) \wedge d(p_2(t_{i,k+1}) - p_2(t_{i,k})) \wedge d(p_3(t_{i,k+1}) - p_3(t_{i,k}))$$

where $1 \le k \le a_i - 1$. Let $d(t_{i,k+1} - t_{i,k})$ be a shorthand for this 3-form. Then

$$d(t_{i2} - t_{i1}) \wedge d(t_{i3} - t_{i2}) \cdots d(t_{a_i} - t_{a_i - 1})$$

is a shorthand for the $3(a_i - 1)$-form. Similiarly, if β_j is in the B-configuration of J, let $d(\partial\beta_j)$ be a shorthand for the 3-form

$$d\left(\frac{\partial p_1}{\partial t}|_{t=\beta_j}\right) \wedge d\left(\frac{\partial p_2}{\partial t}|_{t=\beta_j}\right) \wedge d\left(\frac{\partial p_3}{\partial t}|_{t=\beta_j}\right).$$

The result is that w_J can be expressed as follows:

$$w_J = d(t_{12} - t_{11}) \wedge d(t_{13} - t_{12}) \wedge \cdots \wedge d(t_{\sharp A, a_{\sharp A}} - t_{\sharp A, a_{\sharp A} - 1}) \wedge d(\partial\beta_1) \wedge \cdots \wedge d(\partial\beta_b).$$

Now we will define an orientation on $M(\Gamma^d, J)$. Fix an orientation on all of Γ^d, which we recall is $3N$ dimensional. The $3I$-form w_J defines a basis B_1 of the orthogonal complement of the $3N - 3I$ dimensional space Γ^d. Now pick a basis B_2 of $3N - 3I$ vectors for $M(\Gamma^d, J)$ in such a way that the basis $B = B_1 \cup B_2$ is a positively oriented basis for Γ^d. We give $M(\Gamma^d, J)$ the orientation it inherits from B_2.

2.3 The Differential d_0

To describe the action of d_0 on the cellular chain group $C_*(\Omega_I - \Omega_{I-1})$, it is sufficient to describe its action on the generating cells of that group. First, let's describe the boundary of $e(\Psi, J)$ in B_J for some (A, B)-configuration J. Three types of cells can occur in the boundary of $e(\Psi, J)$.

1. $f(\Psi, J)$ is in the boundary of $e(\Psi, J)$ as q_1 tends to zero. Let J^r be a configuration related to J and let Ψ^r be the generating family of J^r corresponding to Ψ. Let $f(\Psi^r, J^r)$ be the FPC corresponding to J^r. If the first point of J^r corresponds to the last point of J, then $f(\Psi^r, J^r)$ is in the boundary of $e(\Psi, J)$ as $q_{\sharp A}$ tends to zero. The incidence coefficient of $f(\Psi, J)$ in $d_0(e(\Psi, J))$ is $(-1)^{a+1}$, where $a = \dim(S_J)$. The incidence coefficient of $f(\Psi^r, J^r)$ in $d_0 e(\Psi, J)$ is $(-1)^a\eta$, where η is the coefficient that makes the orientations of $e(\Psi, J)$ and $f(\Psi^r, J^r)$ compatible. We will see shortly that $\eta = 1$ if J and J^r are $[I]$-configurations. The integer $\eta = \pm 1$ is called the *relative orientation coefficient*, or ROC for short, of the BC's corresponding to J and J^r.

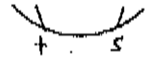
2. A generator corresponding to the BC e^* of an $[I-1]^*$ or $\langle I \rangle_A$ configuration $\bar{J}$ occurs in the boundary of $e(\Psi, J)$ when the configuration $\bar{J}$ underlying e^* is obtained from J by an edge contraction. Diagramatically this is realized by contracting the edge on S^1 between two points. Let's describe the incidence coefficient of e^* in $d_0(e_{\Psi,J})$ when e^* arises from an allowed edge contraction. Let

   - t and s be two adjacent points of J contracting to a point r in $\bar{J}$,
   - $a = \dim(\mathcal{S}_\Psi)$,
   - $b = \max(n(t), n(s))$, where $n(t)$ is the number, in the standard ordering of points, of the point t,
   - p =the number, in the standard ordering of pairs, of the pair (t, s), provided that (t, s) is a pair in Ψ,
   - q =the number, in the standard ordering of pairs, of the pair (r, r), provided that (t, s) is a pair in Ψ,
   - $\zeta = \pm 1$ be the orientation compatibility of $M(\Gamma^d, J)$ and $M(\Gamma^d, \bar{J})$,
   - $\epsilon = \pm 1$ be the parity of renumbering the pairs as we pass from the standard ordering of the pairs of J to the standard ordering of the pairs of $\bar{J}$; this is the orientation compatibility between S_J and $S_{\bar{J}}$.

   In the pictured contractions below, we assume that s and t are the points that bound the contracting edge in the left hand figures and in the right hand figures that r is the point that replaces s and t when the edge contracts.

    contracts to  with incidence coefficient $\zeta(-1)^{p-q+a+b}$,

    contracts to  with incidence coefficient $\zeta(-1)^{p-q+a+b}$,

    contracts to  with incidence coefficient $\epsilon\zeta(-1)^{a+b}$,

    contracts to  with incidence coefficient $\epsilon\zeta(-1)^{a+b}$.

   In the third case above, the pictured contraction is allowed if s and t are not in the same grouping or are in the same grouping but are not connected by a chain of less than three chords. In the last case, the contraction is allowed if s and t are not in the same grouping. We note that in all of the figurs except the second, there may be more than one chord attaching either to s or to t. Any other type of edge contraction is not allowed, since other types of edge contraction have either no meaning in the context of contracting segments of curves that lie between self intersection points, or the boundary is not in $\Omega_I - \Omega_{I-1}$. The contraction that occurs when t is the last point of the configuration and s is the first point is also not allowed, because the last point is moving towards a fixed first point at 0. The cell that is obtained has codimension two and lies in the boundary of the FPC associated to the given diagram.

3. Lastly, a generator $f(\Psi', J)$ occurs in the boundary of $e(\Psi, J)$ where Ψ' is a generating family obtained from Ψ by deleting a chord. The restriction is that Ψ' still be a generating family of J. This boundary is a boundary in S_Ψ, and the incidence coefficient is $(-1)^{p-1}$, where p is the number of the pair corresponding to the deleted arc.

The boundary of a fixed point cell is defined in the same way but we are allowed to contract the last edge between the last and first points, since in this case the codimension of the bounding cell is one.

Let X_I be the free abelian group generated by the BC's of all $[I]$ and $\langle I\rangle_A$ configurations and let Y_I be the free abelian group generated by the BC's of the $[I-1]^*$ and $\langle I\rangle_A$ configurations and the FPC's of $[I]$ and $\langle I\rangle$-configurations.

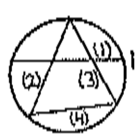
By definition Y_I is equal to the cellular chain group $C_{3N-2}(\Omega_I - (\Omega_{I-1} \cup Z_I))$. Now let

$$\pi : C_{3N-2}(\Omega_I - \Omega_{I-1}) \to C_{3N-2}(\Omega_I - (\Omega_{I-1} \cup Z_I))$$

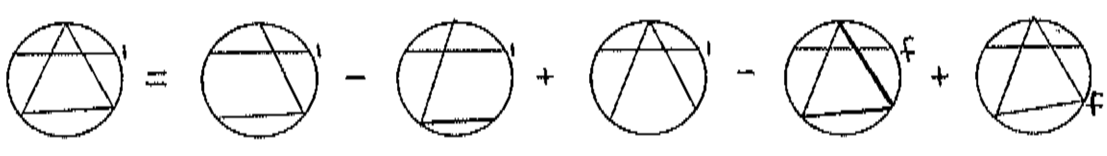
be projection. Define $h_I : X_I \to Y_I$ as the composition $\pi \circ d_0$. By corollary 2.2,

$$\ker(h_I) \cong \bar{H}_{3N-1}(\Omega_I - \Omega_{I-1}).$$

Example with Figure 2.8 *See the pictured chord diagram. The pairs, in order, are* $(1,3), (2,4), (2,5)$ *and* $(4,5)$. *We've numbered the pairs by integers* $(1), (2) \ldots$ *on the corresponding chords.*



If we use the diagrams to represent generating best cells, then h_I can be calculated entirely diagramatically. When a diagram is to stand for a FPC, we have put an f (to stand for fixed) next to the first point, which has been fixed at zero.



2.4 Relative Orientation Coefficients

Because we are working with immersed circles in R^3 rather than immersed lines, the $3N-1$ dimensional cells from noncomplicated configurations, our best cells, contain additional boundary terms, i.e. the fixed point cells. Suppose a FPC f lies in the boundaries of two

BC's e_1 and e_2 corresponding to related configurations J_1 and J_2. By construction f inherits an orientation from the BC that it bounds, as the first point of the BC moves to zero. We can compare this orientation with the one it receives as a bounding cell of the other BC. We call the compatibility ± 1 of these two orientations a *relative orientation coefficient* or ROC for short. If two BC's do not share a common FPC in their boundary, then we extend the notion of a ROC to be the product of a chain of ROC's taken between a chain of BC's between the two given ones.

Let's call two BC's *adjacent* if their underlying configurations are related and the BC's share a common FPC in their boundary. We want to develop a shorthand, using the chord diagrams, which will allow us to calculate ROC's quickly.

We will identify BC's with the chord diagrams that describe them. If a chord diagram is to stand for a FPC, we will say so explicitly. Fix a configuration J and let e be its BC. First, label the points of a chord diagram by $1, 2, 3, \ldots, \rho(J)$ starting with the first point and proceeding counterclockwise. Now, label the chords of the diagram by $(1), (2), \ldots$ in the order assigned to the corresponding pairs in Ψ. Finally, write down the shorthand for the $3I$-form w_J that induces the orientation on $M(\Gamma^d, J)$ (see section 2.2).

Let e_1 and e_2 be adjacent BC's of related configurations J_1 and J_2, so that the first point of J_1 corresponds to the second point of J_2. First, we record how the points are renumbered as we switch from the ordering of the points of J_1 to the ordering of the points of J_2. Now let

- $\mathrm{PP}(e_1, e_2) = (-1)^{\rho(J)-1}$, which is the parity between the orderings of points on the two BC's,
- $\mathrm{CP}(e_1, e_2)=$ the parity between the orderings of chords on (the diagrams of) the two BC's,
- $\mathrm{FP}(e_1, e_2)=$ the parity of the orientation switch between $M(\Gamma^d, J_1)$ and $M(\Gamma^d, J_2)$. FP is the product of the parities of reordering the points in each of the groupings.

The ROC between e_1 and e_2 is the product $\mathrm{ROC}(e_1, e_2)=\mathrm{PP}(e_1, e_2)\mathrm{CP}(e_1, e_2)\mathrm{FP}(e_1, e_2)$.

Example with Figure 2.9 *Pictured are two adjacent BC's, unlabelled above and labelled below. In this case*

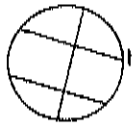
$$PP = (-1)^5 = -1$$

$$CP = sgn\begin{pmatrix} (1) & (2) & (3) \\ (3) & (1) & (2) \end{pmatrix} = 1$$

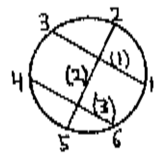
$$FP = sgn\begin{pmatrix} 1 & 3 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 2 & 5 \\ 2 & 5 \end{pmatrix}\begin{pmatrix} 6 & 4 \\ 4 & 6 \end{pmatrix}$$

therefore ROC$= 1$.

Unlabelled:



Labelled:



Lemma 2.10 *If e_1 and e_2 are adjacent best cells of related $[I]$-configurations J_1 and J_2, then the $ROC(e_1, e_2) = 1$.*

Proof: Assume that the first point of e_1 corresponds to the second point of e_2, then PP$= -1$ since $\rho(J)$ is even. CP is equal to the factor of FP that arises from changing the order in which the 3-forms making up w_{J_1} are wedged. The other factor of FP is a -1, because the pair $(1, t)$ in e_2 is switched to the pair $(\rho(J), t-1)$ in e_1 and must be reordered to $(t-1, \rho(J))$. This switch is a kind of *change in polarity* in the given 3-form. The result is that (CP)(FP)$= -1$, hence ROC$(e_1, e_2) = 1$. □

Remark: In the last example e_1 would correspond to the unmarked diagram on the right and e_2 to the one on the left. The pair $(1, 5)$ in e_2 becomes $(4, 6)$ in e_1. Since $1 \mapsto 6$ and $5 \mapsto 4$, the pair $(1, 5)$ becomes $(6, 4)$, which is written in the wrong order, so order of the points in the pair switches from $(6, 4)$ to $(4, 6)$ and results in the -1 that occurs as the second factor of FP.

2.5 Actuality Tables and Their Use

For each cycle $\gamma \in E_1^{-I,J}$, we hope to fill out an actuality table for γ. If γ survives nontrivially all the way to $E_\infty^{-I,J}$, then we can complete an actuality table for γ and γ will give us a knot invariant. If γ fails to make it nontrivially to into $E_r^{-I,J}$ for some r, $1 < r \leq \infty$, then we are not able to complete an actuality table for γ, we do not obtain an invariant and we must scrap the project for γ. In some sense the completed actuality table for an element in $E_\infty^{-I,J}$ gives a history of how that element progressed through the spectral sequence. We will explain how this works in §2.6. We will, following Vassiliev, give an inductive algorithm in §2.6 that will allow us to discover whether or not a cycle makes it nontrivially from $E_r^{-I,J}$ to $E_{r+1}^{-I,J}$, for any r, $1 < r \leq \infty$. If it does, then we will see how the needed data is entered in the associated actuality table, and if it does not we will see why it fails. It is worth noting that we need not work out the entire spectral sequence to use the algorithm; rather, we can study the progress of γ alone *without* having to calculate anything else.

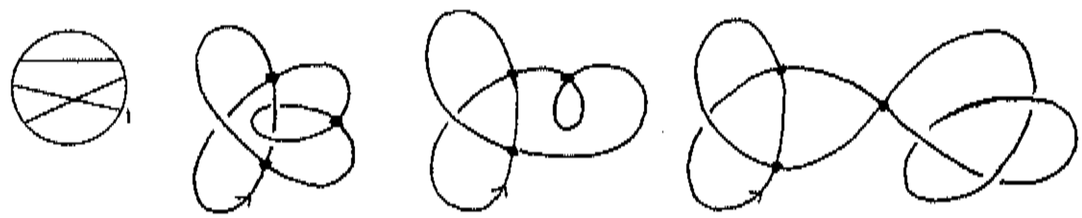
In this section we will assume that we have filled out an actuality table T_γ for some $\gamma \in E_\infty^{-I,J}$. Let us now describe what is in the table and how we use the table.

T_γ has I rows,numbered $1, 2, \ldots, I$ from bottom to top. In the j^{th} row there is one box for each equivalence class of $[j]$-configurations. In each box in the top row, there are two pieces of information:

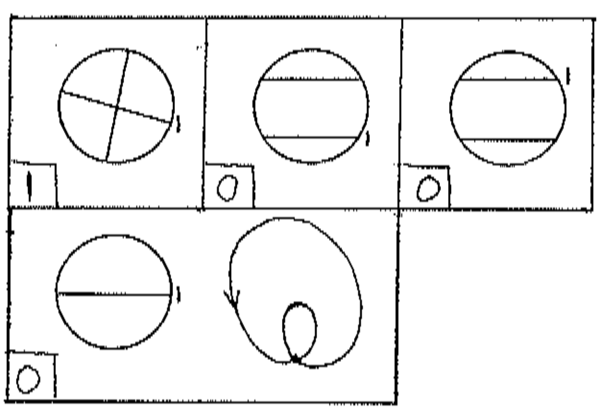
1. the diagram corresponding to (the BC of) the given configuration type,
2. an integer, called the *actuality index*; the actuality index is calculated when we calculate $E_{I-j}^{-I,I}$.

In each box in the j^{th} row, $1 \leq j \leq I-1$, there are three pieces of information. Like the boxes in the top row, each of these boxes contains the chord diagram of a noncomplicated configuration type of complexity j and an actuality index. Additionally, each box contains a *representative curve*, which is a curve that realizes some configuration in the configuration class. There are only two restrictions. First, if the configuration is inadmissable, then the loop of the curve associated to the part of S^1 that lies between the adjacent pair must not be knotted to any other part of the curve. Such a curve is called a *good model* for the configuration. Secondly, two related configurations must always be represented by the same curve. We will see shortly why no representative curves are needed for the boxes in the top row of the table.

Example with Figure 2.11 *Here is the chord diagram of an inadmissable configuration J and three immersed curves K_1, K_2 and K_3 respecting J. The curve K_1 is not a good model for J, while the other two both are.*



Example with Figure 2.12 *Pictured is the actuality table of the simplest nontrivial invariant, which corresponds to a cycle in $E_1^{-2,2} \cong E_\infty^{-2,2}$.*

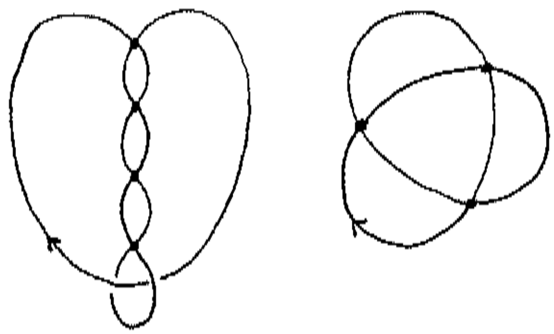


It is easiest to describe the calculation of the invariant axiomatically. This explanation will show how the table is used.

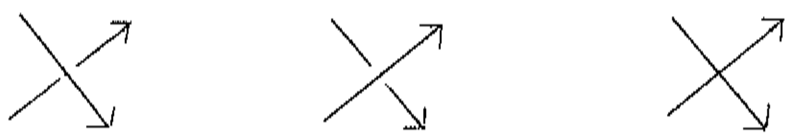
Let Σ_1 be the nonsingular part of $\Sigma \cap \Gamma^d$, i.e. only those points of Σ that respect configurations of complexity one and have no other singularities or self intersections. We define Σ_2 to be the points of $\Sigma \cap \Gamma^d$ that respect configurations of complexity two and don't have additional self intersections or singular points. We define $\Sigma_3, \Sigma_4, \ldots$ similiarly. For a more careful treatment of the following notion, see Stanford, [S], or Chapter two of Birman-Lin, [BL].

DEFINITION: Let $K^m \in \Sigma_m$ be an immersion with m transverse double points. We will call such immersions *nice immersions*. A *graph* of K^m is a drawing of a planar representation of K^m in which all crossings are tranverse, only two strands cross at any crossing and the m double points appear as crossings without crossing information. If two curve segments cross in the planar drawing of K^m but do not touch in K^m itself, then one of the segments is drawn with a small break to indicate that it crosses under the other one. When $m = 0$, K^0 is a knot and the graph of K^0 is a knot diagram in the usual sense.

Example with Figure 2.13 *Two knotted graphs K^4 and K^3:*



Let (K_+, K_-, K_x^1) be the triple consisting of two links K_+ and K_- and a nice immersion K_x^1. Assume that K_+, K_- and K_x^1 vary in the location of a single crossing in the manner pictured below:



K_+ $\qquad$ K_- $\qquad$ K_x^1.

The invariant $\mathcal{V}_\gamma$ satisfies the following recursive relation:

$$\mathcal{V}_\gamma(K_+) - \mathcal{V}_\gamma(K_-) = \mathcal{I}_\gamma(K_x^1)$$

with the initial condition

$$\mathcal{V}_\gamma(U) = 0$$

where U is an unknotted circle and where $\mathcal{I}_\gamma(K_x^1)$ is a number, called the *index* of K_x^1, associated to the immersion K_x^1. This index can be calculated from the table, as we will explain. The condition that $\mathcal{V}_\gamma(U) = 0$ is by choice.

What have we done? If K is any knot, we connect K to U by a homotopy in general position relative to Σ. This homotopy is a curve in $\mathcal{L}_1$ and passes through Σ only at some points $K_1^1, K_2^1, \ldots, K_s^1$ of Σ_1. Each of these passages through Σ can be viewed as crossing changes. Using the recursive formula we find that

$$\mathcal{V}_\gamma(K) = \sum_{i=1}^{s} \epsilon_i \mathcal{I}_\gamma(K_i^1)$$

where $\epsilon_i = \pm 1$ depending on how the recursion formula is used. To find $\mathcal{I}_\gamma(K_i^1)$ we need to calculate again using a new set of recursive formulas. For $1 \leq j \leq I - 1$ and the triple $(K_+^j, K_-^j, K_x^{j+1})$ analogous to (K_+, K_-, K_x^1) above, the general recursion formula is:

$$\mathcal{I}_\gamma(K_+^j) - \mathcal{I}_\gamma(K_-^j) = (-1)^{n(t)+n(s)+j+1} \mathcal{I}_J(K_x^{j+1})$$

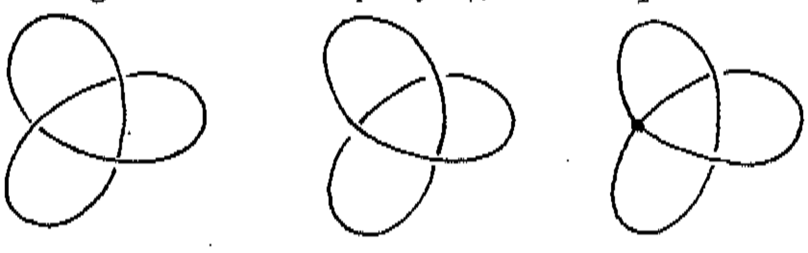
with an initial condition

$$\mathcal{I}_\gamma(C) = \mathcal{I}$$

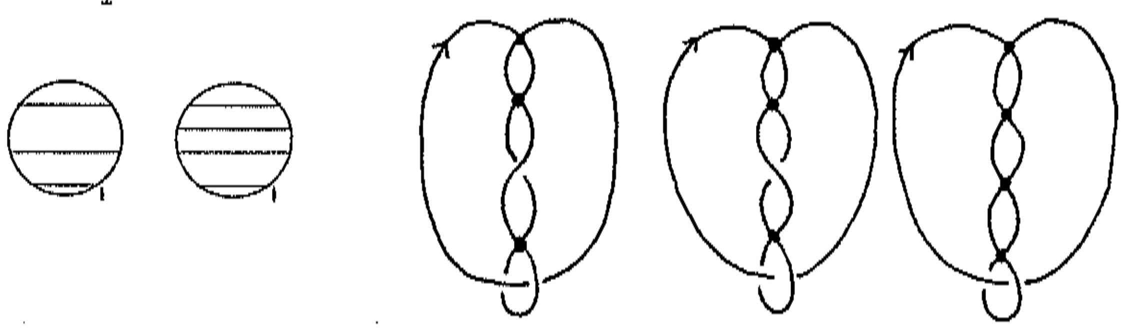
for each curve C in the j^{th} row of the actuality table and associated actuality index $\mathcal{I}$. The curves $K{+}^j$ and K_-^j are nice immersions that both realize some $[j]$-configuration J, and the curve K_x^{j+1} realizes a $[j+1]$-configuration $\bar{J}$ whose chord diagram can be obtained from that of J by adding a pair of points t and s and a chord connecting them to the chord diagram of J. Let $n(t)$ and $n(s)$ be the numbers assigned to t and s in the standard ordering of the points of $\bar{J}$.

So, to find $\mathcal{I}_\gamma(K^j)$ for an arbitrary curve realizing a $[j]$-configuration J, we connect K^j to C^j by a homotopy in Σ_j that is in general position relative to Σ_{j+1}, where C^j is the curve representing J in the table. $\mathcal{I}_\gamma(K^j)$ is then the sum of $\mathcal{I}_\gamma(C^j)$ with signs obtained by the directions of the passages through Σ_{j+1}. The number $\mathcal{I}_\gamma(C^j)$ is the actuality index in the box corresponding to J. One of the things we must show is that this assignment of numbers to curves is independent of the choice of homotopy from either K to U or from K^j to C^j.

Example with Figure 2.14 *An example of K_+, K_- and K_x^1:*



An example of J, $\bar{J}$, K_+^3, K_-^3 and K_x^4. The curves K_+^3 and K_-^3 both realize J. The curve K_x^4 realizes $\bar{J}$.



At the I^{th} stage of calculation, the index $\mathcal{I}_\gamma(K^I)$ of every curve K^I realizing a given $[I]$-configuration J is assigned the actuality index of J from the table. This is why no representative curves are needed for the top row. This actuality index is the coefficient with which the BC of J enters the cycle γ. By the general recursion formula above, any curve with more than I self intersections is assigned the index 0. This last property, that for $m > I$

$$\mathcal{I}_\gamma(K^m) = 0,$$

is the main property which defines an invariant of finite type. A formal definition of link invariants of finite type will be presented in §4.4.

Recall that a $\langle j \rangle_A$-configuration is an (A, B)-configuration that has an empty B-configuration, and whose A-configuration has $j-2$ groupings of cardinality 2 and a single grouping of cardinality 3. An *extended actuality table* is an actuality table augmented by the addition of enough boxes in the j^{th} row, $1 \leq j \leq I$, to include diagrams, representative curves and actuality indices for all of the $\langle j \rangle_A$-configurations. The only restriction is that the representative curve, for a given $\langle j \rangle_A$-configuration, is to be chosen so that the frame of vectors given by the tangent directions at the triple point will be positively oriented.

2.6 The Main Results in the Case $n = 1$

We can now prove our first main result. In the course of the proof we will learn how to enter the actuality indices in an arbitrary row of the extended actuality table; we will see that there are geometric reasons why the invariant is of finite type and that it is indeed an invariant.

First, let's recall the notion of a knot invariant of finite type.

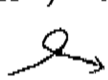
DEFINITION: Let $(\mathcal{V}, \mathcal{I})$ be the pair consisting of a Z-valued invariant $\mathcal{V}$ of n component links and a Z-valued invariant $\mathcal{I}$ of nice immersions of n circles. Let U, L_+, L_-, L_x^1, L_+^j, L_-^j, L_x^{j+1}, s, t, $n(t)$ and $n(s)$ all be as in §2.5. The pair $(\mathcal{V}, \mathcal{I})$ is called a *link invariant of finite type of order* I if it satisfies the following four axioms:

V1. $\mathcal{V}(U) = 0$,

V2. $\mathcal{V}(K_+) - \mathcal{V}(K_-) = \mathcal{I}(K_x^1)$

$$\mathcal{I}(K_+^j) - \mathcal{I}(K_-^j) = (-1)^{n(s)+n(t)+j+1}\mathcal{I}(K_x^{j+1}),$$

V3. $\mathcal{I}(K^m) = 0$ if $m > I$,

V4. $\mathcal{I}($  $) = 0$,

and $(\mathcal{V}, \mathcal{I})$ has the additional data of an actuality table.

Theorem 2.15 *Let d be a fixed positive integer. Let*

$$\gamma \in E_\infty^{-I,J}(d) \subseteq \bar{H}_{3N-1}(\Omega_I) \subseteq \bar{H}_{3N-1}(\Omega) \cong \bar{H}_{3N-1}(\Gamma^d \cap \Sigma)$$

where $N = 2d+1$ and $I \leq \frac{3N-1}{5}$. Then there is a knot invariant $\mathcal{V}_\gamma$ and an invariant $\mathcal{I}_\gamma$ of nice immersions such that the pair $(\mathcal{V}_\gamma, \mathcal{I}_\gamma)$ is a knot invariant of finite type of order I.

Proof: First, we will study the geometry of Ω and give an inductive procedure describing how the actuality indices are entered into the actuality table. Within the description of that procedure we will see why our invariants satisfy axioms V1, V2, V3 and V4. Following this discussion we will state a theorem which shows that the invariants are indeed invariants. The proof of the generalization of this theorem to links is postponed until Chapter three.

Suppose that $\gamma \in E_1^{-I,I}$ and we want to enter the actuality indices in the top row of a prospective extended actuality table for γ.

DEFINITION: Let $\gamma \in E_1^{-I,I}$. Let γ be expressed as

$$\gamma = \sum_i \alpha_i e_i$$

when written as a chain of BC's from the standard decomposition of $\Omega_I - \Omega_{I-1}$, where each e_i is a $3N-1$ dimensional BC and each α_i is an integer. For each i, let J_i be the $[I]$ or $\langle I \rangle_A$ configuration corresponding to e_i. Then α_i is the *actuality index* of J_i entered into the box, corresponding to J_j, in the prospective extended actuality table of γ.

Now let's describe an inductive algorithm for finding and entering actuality indices in rows $I-1, I-2, \ldots, 1$ in the extended actuality table. If this algorithm can be carried out at each step, then we will have shown that γ survives the spectral sequence nontrivially, and we will have completed the extended actuality table for γ.

For the remainder of this section, fix $\gamma \in E_r^{-I,I}$. In calculating the spectral sequence to this point, we have entered the actuality indices into the table on rows $I, I-1, I-2, \ldots, I-r+1$. Let's see how we'll calculate the indices for row $I-r$. We want to find the boundary of γ in $\Omega_{I-r} - (\Omega_{I-r-1} \cup Z_{I-r})$, and express this boundary as a linear combination of $3N-2$ dimensional BC's and FPC's. An inductive algorithm for obtaining this linear combination is explained here, and can also be found in section 4.5 of [V]. An example of this algorithm, as well as the application of our discussion in this section, is given in §4.3.

Suppose that $\bar{J}$ is an $[I-r+1]$ or $\langle I-r+1 \rangle_A$ configuration. Let $\bar{e}$ be the BC of $\bar{J}$. Let J be a $[I-r]$ or $\langle I-r \rangle_A$ configuration (resp) obtained by deleting a pair of points from J. Let e be the BC of J. Inside e the set

$$\{(p,x) \mid p \in M(\Gamma^d, J) \ \cap M(\Gamma^d, \bar{J}), x \in S_J \}$$

forms a ($3N-2$ dimensional) hypersurface, and this hypersurface is the boundary of $\bar{e}$ inside e. The union H_e of these hypersurfaces, over all possible $\bar{J}$, subdivides e into $3N-1$ dimensional *elementary components*. The $3N-2$ dimensional cells of this subdivision correspond to certain components of Σ_{I-r+1}. These $3N-2$ dimensional cells inherit their orientation from e. Each e in $\Omega_{I-r} - (\Omega_{I-r-1} \cup Z_{I-r})$ is similiarly subdivided. Note that this subdivision induces a *new* cellular decomposition of $\Omega_{I-r} - (\Omega_{I-r-1} \cup Z_{I-r})$; this is the subject of the next lemma.

Define the *geometric boundary* $\partial_r \gamma$ of γ by the above and by linearity, with multiplicity given by the index of the given component of Σ_{I-r+1}. This index can be found using of the actuality table and the crossing change formulas. $\partial_r \gamma$ is a cycle by an argument given in 4.5.1 in [V]. The following definitions can be found in 4.5.2 in [V].

DEFINITION: A chain of elementary components is called *compatible* with $\partial_r \gamma$ if in its intersection with any $3N-1$ dimensional BC e, its boundary is equal to the intersection of $\partial_r\gamma$ and e.

To get a better understanding of what a compatible chain is, let

- e be a BC in $\Omega_{I-r} - (\Omega_{I-r-1} \cup Z_{I-r})$,
- $\{c_i\}$ be the set of elementary components of e.

The boundary of each c_i consists of a sum, with appropriate coefficients, of whichever $3N-2$ dimensional hypersurfaces of H_e surround c_i, and also a sum of elementary components of $3N-2$ dimensional BC's that lie in the intersection of c_i with the boundary of e inside $\Omega_{I-r}-(\Omega_{I-r-1} \cup Z_{I-r})$. Suppose that ξ is a compatible chain, and that a_i is the coefficient of c_i in ξ, for each i. Then $\partial_r\gamma \cap e$ is equal to the boundary of $\sum a_i c_i$ in $\Omega_{I-r}-(\Omega_{I-r-1} \cup Z_{I-r})$.

DEFINITION: Let J be an $[I-r]$ or $\langle I-r\rangle_A$ configuration and let K^{I-r} be the curve that represents J in the actuality table. Let e be the associated BC. We will abuse notation here by talking about an elementary component of e *containing a curve*. We really mean that e contains all points (p,x) where p is the curve and $x \in S_J$. Given that, the elementary component of e that contains K^{I-r} is called the *main component* of e. The FPC's are similiarly subdivided. Each FPC is associated to two BC's, both of which are represented by the same curve in the actuality table. The elementary component of any FPC that contains the curve which represents the associated BC's in the table is called the main component of the FPC.

Lemma 2.16 *The set of chains compatible with $\partial_r\gamma$ is isomorphic to X_{I-r}.*

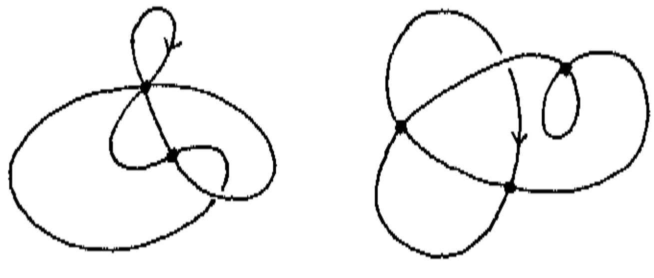
Proof: This is proved as the (only) lemma in section 4.5 in [V]. The coefficient with which the main component enters into any compatible chain determines, by means of the crossing change formulas, the coefficients with which all the other elementary components enter the chain.

Remark: *This idea is really the central idea behind the Vassiliev knot invariants and this is where the crossing change formulas enter.*

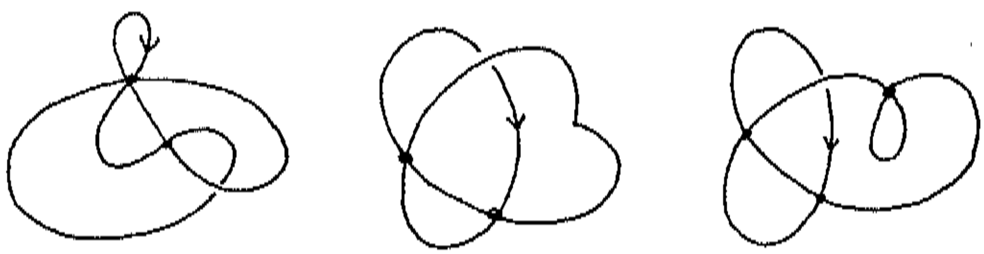
Initially, choose a chain ξ that is compatible with $\partial_r\gamma$ and has the property that the main component of each BC enters ξ with coefficient zero.

There are three types of $3N-2$ dimensional cells from the canonical decomposition of the space $\Omega_{I-r}-(\Omega_{I-r-1} \cup Z_{I-r})$: the BC's from $\langle I-r\rangle_A^*$-configurations, those from $[I-r]$ -configurations and the FPC's from $[I-r]$ and $\langle I-r\rangle_A$ configurations. Each of these is subdivided into elementary components by γ. Define the main components of these cells as follows: the main component of a BC from an $\langle I-r\rangle_A^*$-configuration is defined to be the elementary component containing the curve that represents the associated $\langle I-r\rangle_A$-configuration in the actuality table. Similiarly, define the main component of a $3N-2$ dimensional FPC to be the component that contains the curve representing the associated family of $[I-r]$-configurations on the table. For the BC of an $[I-r-1]^*$-configuration, let the main component be the component of the chosen curve realizing the configuration family. □

Example with Figure 2.17 *In the first figure are examples of curves chosen to represent a $\langle 3\rangle_A$-configuration and an inadmissable* [3]*-configuration.*



In this figure are curves chosen to represent the associated $\langle 3\rangle_A^*$-configuration, $[2]^*$*-configuration and FPC of the above* [3]*-configuration.*



The boundary of ξ is $\partial_r\gamma$ plus a linear combination of elementary components from the $3N-2$ dimensional cells above. Call this linear combination $\tilde{d}_r\gamma$. The coefficients with which the main components of the BC's and FPC's appear in $\tilde{d}_r\gamma$ will determine the coefficients with which the other elementary components enter.

Since the main components of each $3N-1$ dimensional BC enter ξ with coefficient zero, the corresponding main components of all $3N-2$ dimensional FPC's enter $\tilde{d}_r\gamma$ with coefficient zero. The main components of BC's from $[I-r-1]^*$-configurations also enter $\tilde{d}_r\gamma$ with coefficient zero. We'll need a lemma of Birman and Lin, lemma 3.4 in [BL1], to prove this.

Lemma 2.18 *If J is an inadmissible $[I]$-configuration and $\gamma_0 \in E_1^{-I,I}$, then the BC of J enters γ_0 with coefficient zero.*

Proof: Following Birman-Lin, suppose that the BC of an inadmissable configuration entered γ_0 with nonzero coefficient. In general, this cell has the BC from an $[I-2]^*$-configuration in its boundary. If this boundary cell is to be cancelled, then some other BC, with nonzero coefficient in γ_0, must have the same cell in its boundary. This cannot happen, since the configurations of these two cells would be identical and, therefore, the cells would not be distinct. The result is that such a BC must enter γ_0 with coefficient zero. □

Remark: We said *in general.* If J is an inadmissible configuration that has its only isolated chord between first and last points, then there is no BC from an $[I-1]^*$-configuration in its boundary. However, if the BC from J enters the cycle γ_0 with nonzero coefficient, then BC's from each related configuration class must enter with the same coefficient; otherwise the FPC's will not cancel appropriately. This is one of the central ideas behind the proof of theorem 3.16, which generalizes theorem 2.21.

Lemma 2.19 *Any BC e^* from an $[I-r-1]^*$-configuration enters $\tilde{d}_r\gamma$ with coefficient zero.*

Proof: Inductively, assume that the result holds for $t = 1, 2, 3, \ldots, r-1$. The main component of e^* lies in the boundary of some elementary component C of the BC e of a unique inadmissable configuration $\bar{J}$. The main component C_p of e has coefficient zero in ξ by hypothesis. It follows that C also has zero coefficient in ξ, since in travelling from C to C_p the *jumps* across the walls of the subdivision occur in components of Σ_{I-r+1} that have index zero by induction. Note that each of these jumps is given by the crossing change formula. This is the case where the curve $K_{\mp}^{j+1}$ has index zero (by induction) and either K_+^j or K_-^j has index zero. The result will be that the remaining curve will have index zero. □

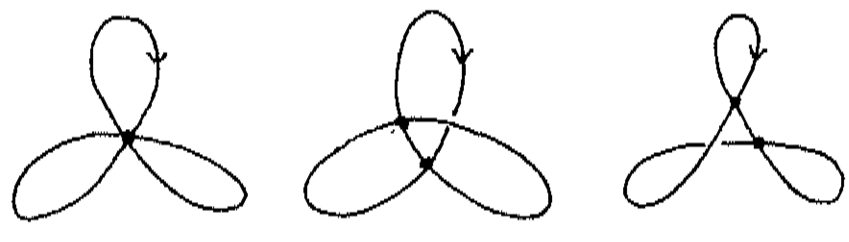
Suppose f is a $3N-2$ dimensional FPC lying in the boundary of some $3N-1$ dimensional BC e. The main component of f has coefficient zero in $\tilde{d}_r\gamma$ because the main component of e enters ξ with coefficient zero.

We need to calculate the coefficient with which (the main component of) a BC of an $\langle I-r\rangle_A^*$-configuration enters $\tilde{d}_r\gamma$. What follows can also be found in section 4.5.3 in [V], as well in section 1.6 in [BL1].

Suppose that e is the BC of an $\langle I-r\rangle_A^*$- configuration J. Let $\bar{J}$ be the $\langle I-r\rangle_A$ configuration obtained diagramatically by adding the missing chord to the diagram for J. Let K be the curve assigned to $\bar{J}$ in the table. There are six ways to "spread" or *resolve* the triple point of the curve into two double points, thus obtaining six curves respecting $[I-r]$-configurations. Some two of these curves, $\bar{K}_1$ and $\bar{K}_2$, respect $[I-r]$-configurations J_1 and J_2 that degenerate under a particular edge contraction to J. Let e_1 and e_2 be the BC's of J_1 and J_2, respectively, and let K_1 and K_2 be the curves representing J_1 and J_2 on the actuality table, respectively.

Example with Figure 2.20 *Here are examples of J, $\bar{J}$, J_1 and J_2:*

Examples of K, K_1 and K_2:



Next, using the crossing change formulas, calculate the indices of $\bar{K}_1$ and $\bar{K}_2$ with the assumption that K_1 and K_2 have index zero. The elementary components of e_1 and e_2 that contain $\bar{K}_1$ and $\bar{K}_2$ enter ξ with these indices. If these indices are a_1 and a_2, respectively, and b is the incidence coefficient of e in $h_{I-r}(e_1)$ and in $h_{I-r}(e_2)$ (which are the same), then the coefficient of e in $\tilde{d}_r\gamma$ is $b(a_1 + a_2)$.

Now, if possible, find a chain α, of $3N-1$ dimensional BC's from $[I-r]$ and $\langle I-r\rangle_A$ configurations, with the property that $h_{I-r}(\alpha) + \tilde{d}_r\gamma = 0$. The isomorphism from X_{I-r} to compatible chains sends α to a chain ξ_α compatible with $\partial_r\gamma$. The main component of each BC appears in ξ_α with the same coefficient with which the given BC appeared in α. This new choice of a compatible chain, if such a choice exists, insures that $\tilde{d}_r\gamma$ is equal to the boundary of *something* and thereby insures that γ_r survives as a nonzero cycle to the next stage. Let J be an $[I-r]$ or $\langle I-r\rangle_A$ configuration, and let e be its BC. The coefficient, with which e enters α is entered into the actuality table as the actuality index for J. If no such chain α can be found, then γ does not survive, since the homology class $d_r\gamma$ of $\tilde{d}_r\gamma$ is not zero. We must then scrap the project. Kontsevich, in [K1], has claimed to have shown that in the case $n=1$ the spectral sequence collapses at $E_1^{p,q}$, which implies that such an α can always be found.

DEFINITION: Let J be an $[I-r]$ or $\langle I-r\rangle_A$ configuration and let e be its BC. Let α be a chain chosen such that

$$h_{I-r}(\alpha) + \tilde{d}_r\gamma = 0.$$

The *actuality index* of J is defined to be the coefficient with which e enters the chain α.

DEFINITION: Let γ, α, and ξ_α be as above. Let K^{I-r} be a nice immersion, with $I-r$ double points, that realizes some $[I-r]$-configuration J and lies in the BC e corresponding to J. Let c be the elementary component of J that contains K^{I-r}. Let a be the coefficient with which c enters ξ_α; then we define

$$\mathcal{I}_\gamma(K^{I-r}) = a.$$

It is immediate that for any other curve $\bar{K}^{I-r}$ in c, we have

$$\mathcal{I}_\gamma(\bar{K}^{I-r}) = \mathcal{I}_\gamma(K^{I-r}).$$

We can also see that the invariant satisfies V4 by lemma 2.19. It satisfies V2 by lemma 2.16. If γ survives nontrivially to the end and we complete an actuality table for γ, then no cells from $\Omega_m - \Omega_{m-1}$, for $m > I$, contribute to the progress of γ through the spectral sequence. Hence, it is natural to define $\mathcal{I}_\gamma(K^m) = 0$ for $m > I$, thereby satisfying V3. The axiom V1 is satisfied by definition. The next theorem will conclude the proof of theorem 2.15 and will establish that our assignment of numbers to knots and nice immersions is an invariant.

Theorem 2.21 *Let* $\gamma \in E_\infty^{-I,I} \subset \bar{H}_{3N-1}(\Omega)$.

1. *If $j \leq I$ and J_1 and J_2 are related $[j]$-configurations, then the actuality indices of J_1 and J_2 are equal.*

2. *If K_1^j and K_2^j realize J_1 and J_2, respectively, and both have the same image in R^3, then*

$$\mathcal{I}_\gamma(K_1^j) = \mathcal{I}_\gamma(K_2^j).$$

3. *If K^j realizes a $[j]$-configuration J, then the number $\mathcal{I}_\gamma(K^j)$ is independent of the choice of path taken from K^j to C^j, where C^j represents J in the actuality table T_γ.*

Proof: We will prove the more general result for links as theorem 3.17. □

This concludes the proof of theorem 2.15. Note that our calculations did not depend on d, which was the maximum degree of the Fourier polynomials making up the coordinate functions of the maps of Γ^d. It has been implicit up to now that for each d, there is an associated spectral sequence $E_r^{p,q}(d)$. Our discussion up to now is valid for the groups $E_r^{-I,I}(d)$ as long as $I \leq \frac{3N+1}{5}$, where $N = 2d+1$.

Theorem 2.22 *Let d be a fixed positive integer, $I \leq \frac{3N+1}{5}$ where $N = 2d+1$. Let $\gamma \in E_\infty^{-I,I}$ and let $\bar{d} > d$. Then for each $r = 1, 2, \ldots, \infty$, we have*

$$E_r^{-I,I}(\bar{d}) \cong E_r^{-I,I}(d).$$

Moreover, if $\bar{\gamma} \in E_\infty^{-I,I}(d)$ corresponds via this isomorphism to $\gamma \in E_\infty^{-I,I}$, then for each knot K and each nice immersion K^j in $\Gamma^d \subset \Gamma^{\bar{d}}$ we have

$$\mathcal{V}_{\bar{\gamma}}(K) = \mathcal{V}_\gamma(K)$$

and

$$\mathcal{I}_{\bar{\gamma}}(K^j) = \mathcal{I}_\gamma(K^j).$$

A Vassiliev knot invariant of order I is defined to be a cycle in

$$\lim_{d \to \infty} \bar{H}_{3N-1}(\Omega_I(d))$$

where d is taken sufficiently large to begin with, and evaluation on a knot is as given above.

3 Generalizing the Construction to Links

3.1 Our Goal

Our goal is to generalize the construction of the knot invariants of Chapter two to obtain an invariant of links. As before, our desire is to complete an actuality table T_γ for $\gamma \in E_1^{-I,I}$.

In 3.2, we will give the generalizations of the definitions of relatedness and J-block. We will construct a more general type of chord diagram, and then give generalizations of the results of section 2.1. In section 3.3, we give the much more complicated CW-decomposition of the space $\Omega_I - \Omega_{I-1}$ and discuss the action of the initial differential d_0 on this space. The relative orientation coefficients (ROC) play a greater role with links than they do in the case of knots, particularly in terms of calculating the cancellation of FPC's in the boundary of chains of BC's. We will devote section 3.4 to this discussion. Finally, in section 3.5, we will explore the higher order differentials in an effort to understand how to fill out the actuality table for a given $\gamma \in E_1^{-I,I}$.

3.2 Chord Diagrams, Relatedness, J-blocks and Generalizations of the Preliminary Results

For the remainder of this chapter, fix an integer $n \geq 2$, a large integer d and let I be an integer such that $I \leq \frac{3N+1}{5}$, where $N = n(2d+1)$. We will assume that all curves lie in $\Gamma^d \subset \mathcal{L}_n$, and that all configurations are configurations on $\coprod_{i=1}^{n} S^1$. Recall that $\mathcal{L}_n$ consists of maps from an *ordered* disjoint union of n circles to R^3.

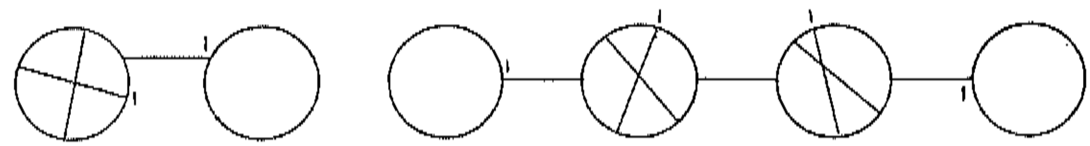
Let J be an (A, B, C)-configuration, and let Ψ be a generating collection of J. As before, a *chord diagram* can be employed to encode the information carried by the triple consisting of:

- The equivalence class $\{J' \mid J' \sim J\}$.
- A generating collection Ψ of J.
- The set of maps $M(\Gamma^d, J)$ that respect J.

Let $\rho_i(J)$ be the number of geometrically distinct points on $J \cap (S^1)_i$. The chord diagram will consist of n circles. The circles are pictured in order left to right and are assumed to have counterclockwise orientation. On the i^{th} copy of S^1, there are $\rho_i(J)$ points. If $\rho_i(J) \neq 0$, then the configuration J has a *first point* on $(S^1)_i$, and this point is marked on the chord diagram by a 1. The points of each grouping of the A and C configurations of J are interconnected by a collection of chords. These chords are given in the following way: if s and t are a pair of points in either the A or C configuration of J and the pair (s, t) is one of the pairs of Ψ, then there is a chord running from s to t in the chord diagram. The points of the B-configuration of J are each marked by a $*$. Recall that the C-configuration of J

consists of all groupings of points that have some two points on different copies of S^1. It follows that any chord that connects two different circles will correspond to a pair of points from some grouping in the C-configuration of J.

Example with Figure 3.1 *Pictured are two chord diagrams, one for $n = 2$ and one for $n = 4$.*



DEFINITION: Let J be an (A, B, C)-configuration. If J^r determines an equivalence class of configurations whose diagrams are identical to the diagrams associated with $\{J' \mid J' \sim J\}$ except that they vary by the location of some of their first points, then we say that J and J^r are *related*, and we denote this by $J^r \leftrightarrow J$. We call the set of configurations related to J the *family* of J.

Let

$$\mathcal{B}_J = \bigcup M(\Gamma^d, J) \times \mathrm{S}_J)$$

where the union is taken over the family of J. This union, as before, is called a *J-block.* We will now state the generalizations of the results from section 2.1.

Lemma 3.2 *If J_1 and J_2 are nonrelated configurations of complexity I, then $\mathcal{B}_{J_1} \cap \mathcal{B}_{J_2} = \emptyset$.*

Proof: The proof is exactly the same as that of lemma 2.2. □

Again, let Z_I be the union of all the J-blocks from complicated configurations.

Lemma 3.3 *Z_I is closed in $\Omega_I - \Omega_{I-1}$.*

Proof: The proof is exactly the same as that of lemma 2.3. □

Theorem 3.4 *For each I, $\bar{H}_k(Z_I) = 0$ when $k \geq 3N - 2$.*

Proof: The proof of this theorem is a long but straightforward generalization of theorem 3.1.2 in [V] and is found in Chapter five. □

The exact sequence of chain complexes:

$$0 \to C_k(Z_I) \to C_k(\Omega_I - \Omega_{I-1}) \to C_k(\Omega_I - (\Omega_{I-1} \cup Z_I)) \cong C_k(\Omega_I - \Omega_{I-1}, Z_I) \to 0$$

induces the long exact sequence of homology groups:

$$\cdots \to \bar{H}_k(Z_I) \to \bar{H}_k(\Omega_I - \Omega_{I-1}) \xrightarrow{\Phi} \bar{H}_k(\Omega_I - (\Omega_{I-1} \cup Z_I)) \to \bar{H}_{k-1}(Z_I) \to \cdots$$

Using theorem 3.4 we obtain:

Corollary 3.5 *The map $\Phi : \bar{H}_k(\Omega_I - \Omega_{I-1}) \to \bar{H}_k(\Omega_i - (\Omega_{I-1} \cup Z_I))$ is an isomorphism for $k \geq 3N - 1$ and injective for $k \geq 3N - 2$.*

Let $\rho(J) = \sum_{i=1}^{n} \rho_i(J)$ be the number of geometrically distinct points in J. The configuration space $\{J^r \mid J^r \leftrightarrow J\}$ is homeomorphic to $(S^1)^j \times (0,1)^{\rho(J)-j}$, where $j \le n$ is the number of circles on which J has points.

We assumed that $I \le \frac{3N+1}{5}$, so each $M(\Gamma^d, J^r)$ is of dimension $3N-3I$ and we can view $\mathcal{B}_J$ as a locally trivial fibration with base $\{J^r \mid J^r \leftrightarrow J\}$ and fiber $M(\Gamma^d, J) \times \mathrm{S}_J \subset \Gamma^d \times \mathrm{R}^M$.

Theorem 3.6 *$\mathcal{B}_J$ is a trivial fibration and is therefore homeomorphic to the product*

$$\{J^r \mid J^r \leftrightarrow J\} \times M(\Gamma^d, J) \times S_J.$$

Proof: The proof, found in Chapter five, uses some linear algebra and shows explicitly that the twisting maps are the identity. □

3.3 The CW-Decomposition of $\Omega_I - \Omega_{I-1}$, Best Cells and Fixed Point Cells

Recall that to give a CW-decomposition of $\Omega_I - \Omega_{I-1}$, we must first take its one point compactification. We do this, following Vassiliev, because the closures of some of the cells that make up $\Omega_I - \Omega_{I-1}$ are not contained in $\Omega_I - \Omega_{I-1}$. It is sufficient to decompose the one point compactification of each $\mathcal{B}_J$. Fix an (A, B, C)-configuration J of complexity I and assume that $I \le \frac{3N+1}{5}$. As before, each chord diagram corresponds to several cells, of several different dimensions. The unique largest dimensional one has dimension

$$\rho(J) + (3N - 3I) + (\text{the number of chords in the diagram}) + b - 1.$$

Recall that b = the number of $*$'s in the diagram. We can think of each J-block $\mathcal{B}_J$ as a cell complex obtained by glueing together all of the cells from all of the chord diagrams that come from J or from any J^r related to J.

Let us give a CW-decomposition of the configuration space $\{J^r \mid J^r \leftrightarrow J\}$. Without loss of generality, let us assume that J contains points on each copy of S^1 in $\coprod_{i=1}^{n} S^1$. Remember that each S^1 is oriented and parameterised by $[0, 2\pi)$ in the usual way. For each $J^r \leftrightarrow J$ and $1 \le i \le n$, let $q_{i1}(J^r)$ be the coordinate on $(S^1)_i$ of the first point in $J \cap (S^1)_i$, i.e. the coordinate of the first point in J^r encountered moving counterclockwise from zero on $(S^1)_i$. This point may actually be at zero. Let $q_{i2}(J^r)$ be the coordinate in the interval $(q_{i1}(J^r), 2\pi) \cong (0,1)$ of the first point encountered moving counterclockwise from q_{i1} and define $q_{i3}(J^r), q_{i4}(J^r), \ldots, q_{i\rho_i(J)}$ similiarly. Then

$$Q(J^r) = (q_{11}(J^r), q_{12}(J^r), \ldots, q_{1\rho_1(J)}(J^r), q_{21}(J^r), \ldots, q_{n\rho_n(J^r)}(J^r))$$

are coordinates for $\{J^r \mid J^r \leftrightarrow J\}$, where $q_{i1}(J^r)$ is the coordinate of the first point of J^r on the i^{th} copy of S^1 in $(S^1)^n \times (0,1)^{\rho(J)-n}$. Now we that we have coordinates for $\{J^r \mid J^r \leftrightarrow J\}$, we can give a CW-decomposition of the equivalence class $\{J' \mid J' \sim J\}$:

$$\begin{aligned}
c(J) &= \{(q_{11}, \ldots, q_{n\rho_n(J)}) \mid q_{i1} \in (0, 2\pi), 1 \le i \le n\} \\
c_1(J) &= \{(q_{11}, \ldots, q_{n\rho_n(J)}) \mid q_{11} = 0, q_{i1} \in (0, 2\pi), 2 \le i \le n\} \\
c_2(J) &= \{(q_{11}, \ldots, q_{n\rho_n(J)}) \mid q_{21} = 0, q_{i1} \in (0, 2\pi), i \in \{1, 3, \ldots, n\}\} \\
&\ \vdots \quad \vdots \quad \vdots
\end{aligned}$$

$$
\begin{aligned}
c_n(J) &= \{(q_{11},\dots,q_{n\rho_n(J)}) \mid q_{n1}=0, q_{i1}\in(0,2\pi), 1\le i\le n-1\}\\
c_{12}(J) &= \{(q_{11},\dots,q_{n\rho_n(J)}) \mid q_{11}=q_{21}=0, q_{i1}\in(0,2\pi), 3\le i\le n\}\\
\vdots\ &\ \vdots\quad \vdots\\
c_{k_1k_2}(J) &= \{(q_{11},\dots,q_{n\rho_n(J)}) \mid q_{k_11}=q_{k_21}=0, q_{i1}\in(0,2\pi),\\
&\qquad i\in\{1,2,\dots,\hat{k}_1,\dots,\hat{k}_2,\dots,n\}.
\end{aligned}
$$

Generally, for $s<n$, and $k_1,k_2,\dots,k_s\in\{1,2,\dots,n\}$ such that $k_1<k_2<\cdots<k_s$, we have

$$
\begin{aligned}
c_{k_1k_2\dots k_s}(J) &= \{(q_{11},\dots,q_{n\rho_n(J)}) \mid q_{k_11}=q_{k_21}=\cdots=q_{k_s1}=0, q_{i1}\in(0,2\pi),\\
&\qquad i\in\{1,2,\dots\hat{k}_1,\dots,\hat{k}_2,\dots,\hat{k}_s,\dots,n\}\}\\
\vdots\ &\ \vdots\quad \vdots\\
c_{123\dots n}(J) &= \{(q_{11},\dots,q_{n\rho_n(J)}) \mid qi1=0, 1\le i\le n\}
\end{aligned}
$$

Note that $c(J)\cong e^{\rho(J)}$ and $c_{k_1k_2\dots k_s}(J)\cong e^{\rho(J)-s}$.

We will define the *canonical decomposition* of $\Omega_I-\Omega_{I-1}$ analogously to the case of knots. Let μ be the number of equivalence classes of configurations that are subsets of the family of J and let $J_1,J_2,\dots J_\mu$ be representatives of these equivalence classes. $\mathcal{B}_J$ has the following cells:

$$
\begin{aligned}
e(J_i,\Psi) &= \{(Q,p,x) \mid Q\in c(J_i), p\in M(\gamma^d,J)x\in S_\Psi\}\\
f(J_i,\Psi,k_1,k_2,\dots,k_s) &= \{(Q,p,x) \mid Q\in c_{k_1k_2\dots k_s}(J_i), p\in M(\Gamma^d,J), x\in S_\Psi\}
\end{aligned}
$$

where $1\le j\le\mu$, $1\le s\le n$ and $1\le k_1<k_2<\cdots<k_s\le n$ and where Ψ ranges over all generating collections of J_j.

Each $e(J_j,\Psi)$ is called a *best cell* of $\mathcal{B}_J$ and each $f(J_j,\Psi,k_1,k_2,\dots,k_s)$ is called an *s-fixed point cell* of $\mathcal{B}_J$. As before, we will refer to best cells as BC's. When we say FPC, we will be referring to a 1-fixed point cell. We will call $e(J_j,\Psi)$ the *parent cell* of each FPC $f(J_j,\Psi(k_1k_2\dots k_s)$ and call each of these FPC's a *child* of $e(J_j,\Psi)$. Each child receives its orientation from the parent. The number of BC's in $\mathcal{B}_J$ is equal to μ times the number of distinct generating families of J and is equal to the number of distinct chord diagrams that encode one of $J_1,J_2,\dots,J_\mu$. Note that in the case of knots, each BC has only one child, whereas in the case of links, each BC has multiple children, corresponding to fixing different combinations of first points at the zero point of the respective copy of S^1.

We will initially orient all of these cells as before. Later we will find it convenient to change some of these orientations. First, we must orient $\{J^r \mid J^r\leftrightarrow J\}$, $M(\Gamma^d,J)$ and S_Ψ. To do this we need to order:

- the set of points in J; this ordering is called the *standard ordering of the points of* J,
- the groupings of J,
- the set of pairs in an arbitrary generating collection Ψ of J; this ordering is called the *standard ordering of the pairs of* Ψ,
- the $3I$ equations that determine the subspace $M(\Gamma^d,J)$ of Γ^d.

The points of J are given the same order that the coordinates $Q(J)$ of these points have. This ordering is established by first numbering the points on each circle starting with the first point and proceeding counterclockwise. The circles are numbered from 1 to n, and this imposes a dictionary order on the points. This ordering, together with the standard orientation of $(S^1)^n \times (0,1)^{\rho(J)-n}$, gives an orientation of $\{J^r \mid J^r \leftrightarrow J\}$.

The groupings of J, the pairs of S_Ψ and the $3I$ equations are ordered in exactly the same way as they were for knots (see section 2.2). These orderings lead to orientations of S_Ψ and $M(\Gamma^d, J)$ in the same way that they did when we worked with knots. These orientations give an orientation on $e(J,\Psi)$ and one on each of its FPC children.

Let's examine the action of the differential d_0 on the BC $e(J,\Psi)$. The action of d_0 on any FPC $f(J,\Psi,k_1,\ldots,k_s)$ is defined similiarly and we will point out the differences, as we come to them.

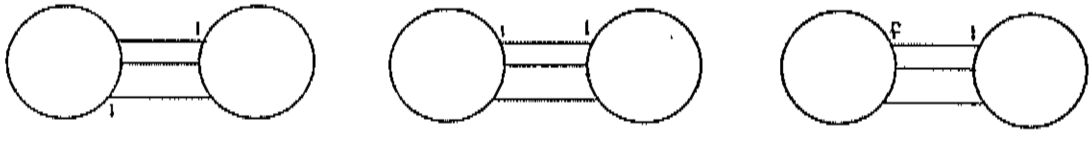
There are three ways in which a cell can occur as a boundary term in $d_0(e(J,\Psi))$:

1. If $J \cap (S^1)_i$ is nonempty, then J has a first point q_{i1} on $(S^1)_i$. Let $f(J,\Psi(i))$ be the bounding cell of $e(J,\Psi)$ obtained by moving q_{i1} to zero. A different codimension one FPC is obtained when, instead of moving the first point to zero, the *last point* $q_{i\rho_i(J)}$ is moved to zero from the other side and this cell is an FPC $f(\bar{J},\bar{\Psi}(i))$, where $\bar{J}$ determines an equivalence class of configurations whose first point on $(S^1)_i$ corresponds to the last point of J on $(S^1)_i$. The BC's of such J and $\bar{J}$ are called *adjacent*; see example 3.8. The incidence coefficient of $f(J,\Psi(i))$ in $d_0(e(J,\Psi))$ is $(-1)^{a+c_i}$, where $a = \dim(S_\Psi)$ and c_i is the number of the first point of J on $(S^1)_i$ in the standard ordering of points. The incidence coefficient of $f(\bar{J},\bar{\Psi}(i))$ is $(-1)^{a+c_i+1}\eta$, where η is the ROC between $e(J,\Psi)$ and $e(\bar{J},\bar{\Psi})$.

2. Edge contraction is exactly as in the case of knots. We do not contract the edge on $(S^1)_i$ between $q_{i\rho_i(J)}$ and q_{i1} because this contraction results in an FPC that has dimension two smaller than that of $e(J,\Psi)$.

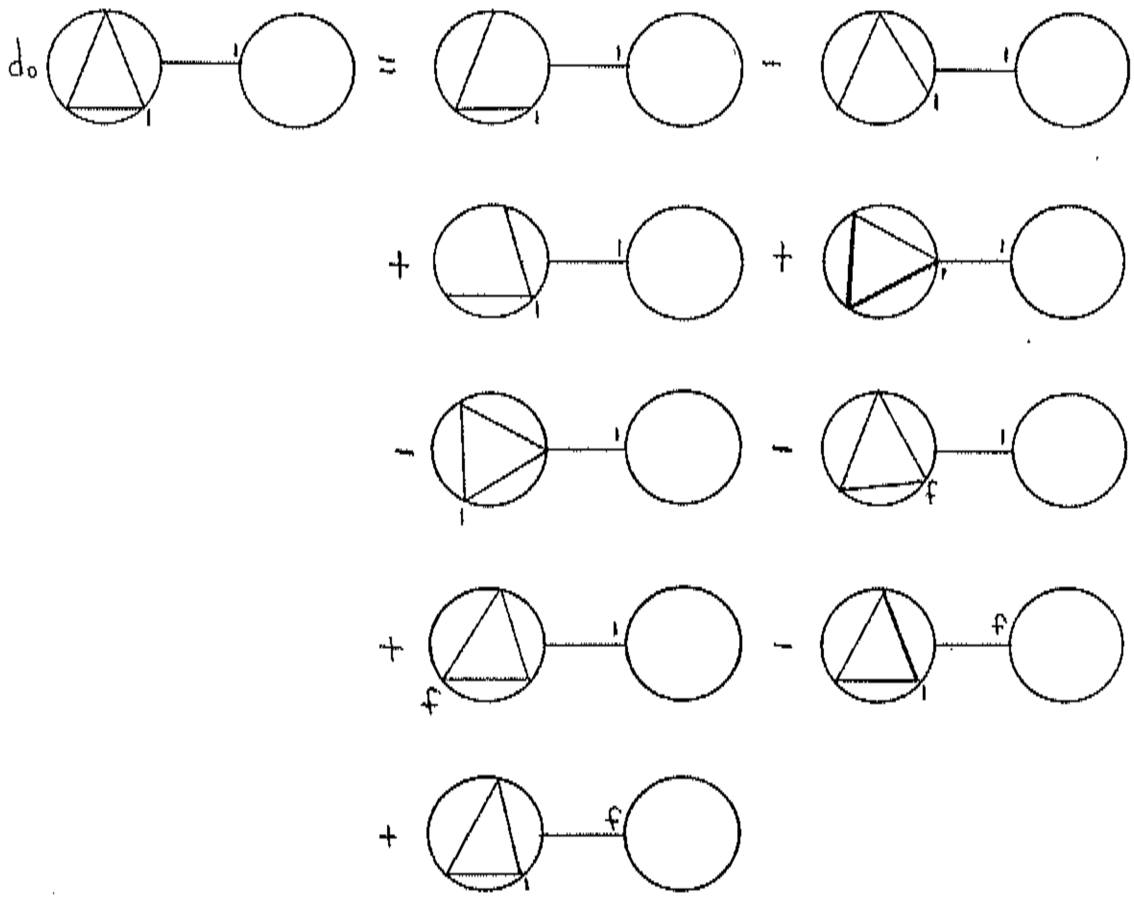
3. Chord deletion is exactly the same as in the case of knots.

Example with Figure 3.7 *The diagram on the left corresponds to the BC $e(J,\Psi)$ of J and the one in the center to the BC $e(\bar{J}\bar{\Psi})$ of $\bar{J}$. The FPC $f(\bar{J}\bar{\Psi}(i))$ inherits its orientation from $e(\bar{J}\bar{\Psi})$ and is part of the boundary of both BC's and is pictured on the right.*

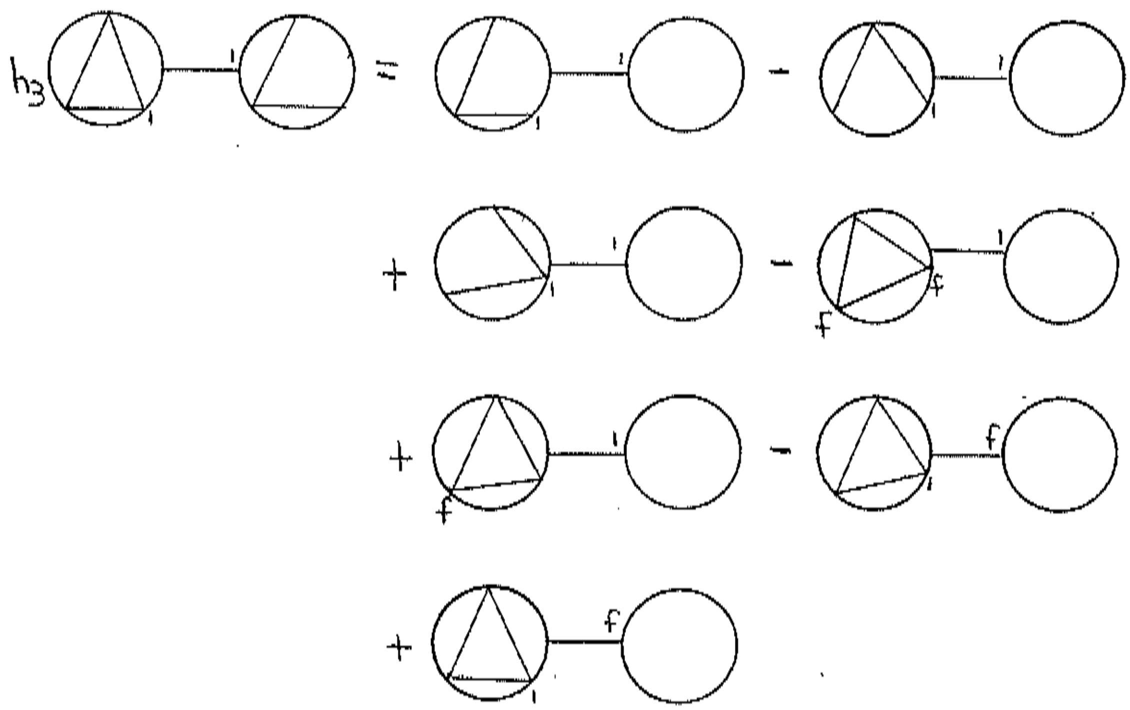


Let X_I be the chain complex generated by the chord daigrams corresponding to equivalence classes of $\langle I\rangle_A$, $\langle I\rangle_C$ and $[I]$ configurations. Let Y_I be the chain complex generated by the chord diagrams corresponding to equivalence classes of $\langle I\rangle^*$ and $[I-1]^*$ configurations and the FPC's of $\langle I\rangle_A$, $\langle I\rangle_C$ and $[I]$ configurations. We define h_I as before, i.e. h_I the map on chord diagrams given by the boundary map restricted to BC's of noncomplicated configurations, and taking values only in the cellular chain group of cells from noncomplicated configurations.

Example with Figure 3.8 *This is d_0 of the BC of a $\langle 3 \rangle_A$- configuration. When a diagram stands for a FPC, we put an f next to whichever first point is fixed at zero.*



This is h_3 of the same BC:

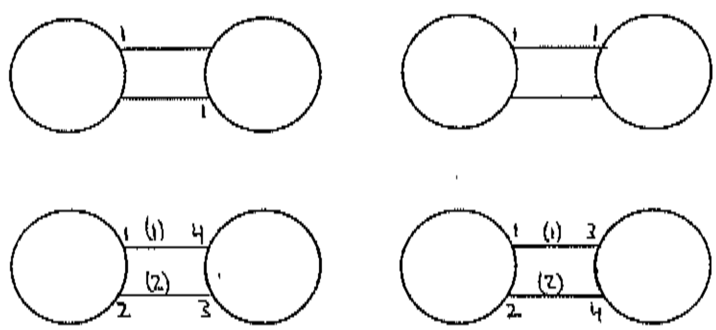


3.4 The Main Results

Let $e(J,\Psi)$ and $e(\bar{J},\bar{\Psi})$ be adjacent BC's, and suppose that J and $\bar{J}$ vary in that the first point of $\bar{J}$ on $(S^1)_i$ corresponds to the last point of J. The FPC $f(J,\Psi(i))$ is a common bounding cell to both BC's and inherits its orientation from $e_{J\Psi}$. The ROC between $e(J,\Psi)$ and $e(\bar{J},\bar{\Psi})$ is the compatibility between the given orientation of $f(J,\Psi(i))$ and the orientation it recieves as a bounding cell of $e(\bar{J},\bar{\Psi})$.

The ROC is the product of the three numbers PP, CP and FP. These numbers are defined in the same way as before. To calculate the ROC, we view the transition from $e(\bar{J},\bar{\Psi})$ to $e(J,\Psi)$ as a map which renumbers the points of $\bar{J}$ with the numbers assigned to the corresponding points in J. This renumbering induces reorderings of the standard ordering of points, the standard ordering of pairs and the ordering of the 3-forms that wedge together to become w_J and determine the orientation of $M(\Gamma^d, J)$. The parities of these three reorderings are PP, CP and FP, respectively. When $n \geq 2$, the ROC between adjacent BC's from $[I]$-configurations is not always one.

Example with Figure 3.9 *Pictured are diagrams of adjacent BC's, first unlabelled and then labelled with the numbers of all points and chords.*



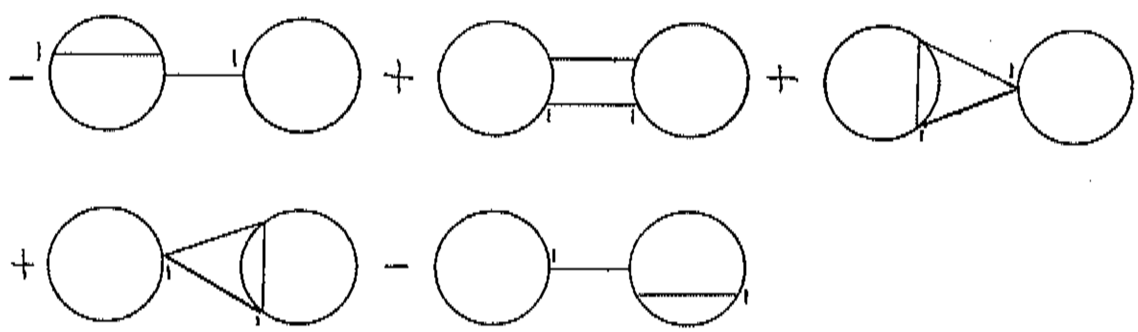
The renumbering map is given by:
$1 \mapsto 1$
$2 \mapsto 2$
$3 \mapsto 4$
$4 \mapsto 3$
Therefore:

$$\begin{aligned} PP &= sgn\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 3 \end{pmatrix} = -1 \\ CP &= sgn\begin{pmatrix} (1) & (2) \\ (1) & (2) \end{pmatrix} = 1 \\ FP &= sgn\begin{pmatrix} 1 & 3 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 2 & 4 \\ 2 & 4 \end{pmatrix} = 1 \end{aligned}$$

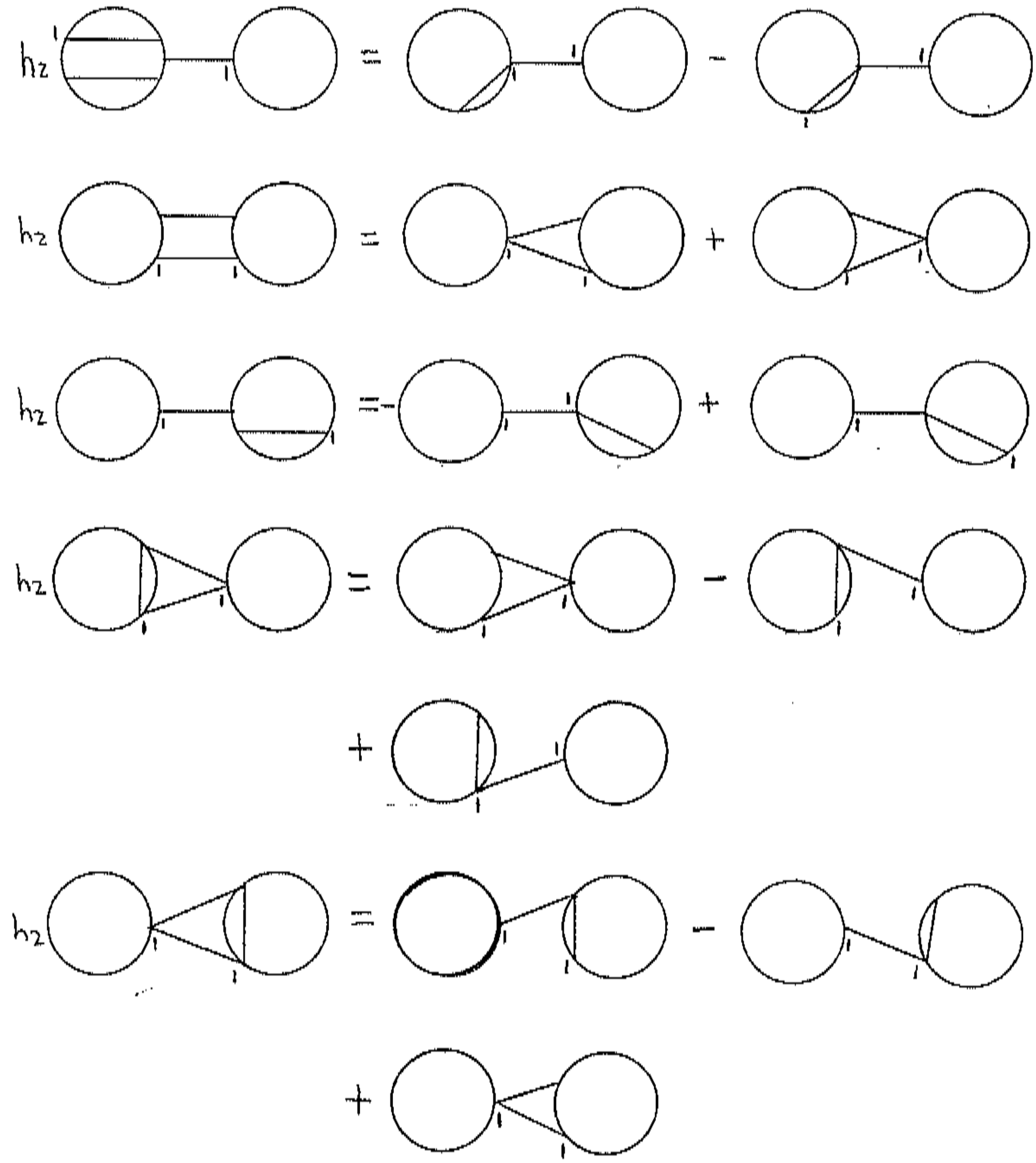
Consequently the ROC is -1.

If we compare our approach in Chapter two to Vassiliev's approach, we see that there is a one-to-one correspondence between the $3N-1$ dimensional BC's from noncomplicated configurations in our approach and the $3N-1$ dimensional cells from noncomplicated configurations in Vassiliev's canonical decomposition of $\Omega_I - \Omega_{I-1}$ (which he calls $\sigma_I - \sigma_{I-1}$). In some sense, Vassiliev pushes the FPC's *out to infinity* by considering knots (knotted graphs) to be embedded (immersed) lines rather than circles. There is a one-to-one correspondence between ambient isotopy classes of embeddedings of R in R^3 with fixed asymptotic directions, embeddings of S^1 in R^3 and embeddings of S^1 in S^3. We can view Vassiliev's construction for knots as our construction for knots in the case where the image of the zero point under each embedding or immersion of S^1 is the point at infinity in S^3. Thus all of these FPC's can be viewed, in the case of knots, as boundaries at infinity, which, when we pull them back, cancel so nicely that we could ignore them. *This is not the case with links.* If we ignore the FPC's in the boundary, certain chains of $3N-1$ dimensional BC's appear to be cycles, and the actuality indices arising from these pseudo-cycles contradict the recursive equations that the invariants satisfy. Another way of saying this is the following: for each generating cell e in the cellular chain group $C_{3N-1}(\Omega_I - (\Omega_{I-1} \cup Z_I))$, let $B_1(e)$ be the sum of all FPC terms in $d_0(e)$ and let $B_2(e) = d_0(e) - B_1(e)$. When $n = 1$ we have $B_2(\gamma) = 0 \Rightarrow B_1(\gamma) = 0$, but for $n \geq 2$ this is not the case. Hence the condition that $B_1(\gamma) = 0$ is necessary for γ to be a cycle in $E_1^{-I,I}(d)$. The following example illustrates this fact; it is an example in which the chain γ satisfies $B_2(\gamma) = 0$ but clearly cannot give an invariant. We note that our link invariants satisfy essentially the same skein relations as our knot invariants in Chapter two did; we will prove this later.

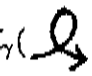
Example with Figure 3.10 *Let $n = 2$ and consider the chain of chord diagrams:*



We calculate the boundary of the corresponding cells diagramatically, ignoring the FPC boundary terms. Notice cells from two inadmissible configurations enter this chain with nonzero coefficients and that cells from related configurations enter with different coefficients:



If we count the chain in the example as a cycle γ, we are claiming the existence of a cycle that would satisfy

$\mathcal{I}_\gamma($  $) - \mathcal{I}_\gamma($  $) = (-1)^{even\ number}\mathcal{I}_\gamma($  $) \neq 0$

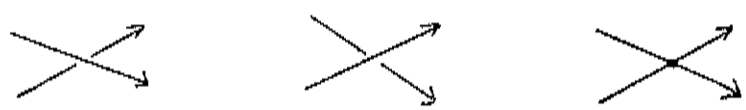
which is clearly not acceptable for a link invariant.

Let γ be an honest-to-goodness cycle. We define the actuality table T_γ, associated to γ, exactly as in §2.5. There are two levels of calculation with these invariants. First, given γ, we must fill out T_γ. Secondly, if γ is not trivial, we can evaluate γ on a link. It is very messy to evaluate the invariant on a link, so we'd like to clean up the mess. We want a *reduced actuality table* that has only one box for each family of $[j]$-configurations, for $1 \leq j \leq I$. To obtain a reduced actuality table for $\gamma \in E_1^{-I,I}$, we must still fill out the extended actuality table, but then we can construct the reduced table easily. We must first prove that the crossing change formulas and initial conditions used to define both $\mathcal{V}_\gamma$ (for links) and $\mathcal{I}_\gamma$ (for knotted graphs of n components) only depend on the configuration families and representative curves. The invariant is calculated on a link by taking a sum of certain weights over a series of crossing changes that changes the link to the unlink. We evaluate on an immersed graph the same way, but the changes turn the graph into a chosen representative graph from the actuality table.

DEFINITION: Let J be an (A, B, C)-configuration. We call the J-block $\mathcal{B}_J$ *oriented* if we have switched the orientations of some collection of BC's in $\mathcal{B}_J$ so that the ROC's between any two BC's in $\mathcal{B}_J$ is $+1$. Any BC whose orientation remains unchanged is called *representative* of the oriented J-block, and any BC whose orientation we change is called *contingent*. We will assume that each FPC inherits its orientation from its parent BC, so the FPC offspring of contingent BC's have their orientations changed, while orientations of the offspring of representative BC's remain unchanged. Note that h_I of a contingent cell, after its orientation change, is the negative of h_I on that cell before changing orientation.

From now on assume that each J-block is oriented. Let's now define invariants of finite type.
DEFINITION: Let

- U be the unlink of n components,
- L^j be any immersion in which the only singularities are j transverse double points,
- the triples (L_+, L_-, L_x^1) and $(L_+^j, L_-^j, L_x^{j+1})$ denote links and immersions that vary locally as pictured below:



- J be the $[j]$-configuration that is realized by both L_+^j and L_-^j,
- $\bar{J}$ be the $[j+1]$-configuration that is realized by L_x^{j+1},
- s and t be the points added to the chord diagram of J to obtain that of $\bar{J}$,

- $n(s)$ and $n(t)$ be the numbers of the points s and t in $\bar{J}$,
- e and $\bar{e}$ be the BC's of J and $\bar{J}$, respectively,
- ϵ be a function on BC's that is 1 on any representative cell and is -1 on any contingent cell.

A pair $(\mathcal{V}, \mathcal{I})$ of integer valued invariants of n-component links and n-component immersions, respectively, is an *n-component link invariant of finite type* if the pair satisfies the following five axioms:

V1. $\mathcal{V}(U) = 0$,

V2. $\mathcal{V}(L_+) - \mathcal{V}(L_-) = \mathcal{I}(L_x^1)$
$\mathcal{I}(L_+^j) - \mathcal{I}(L_-^j) = \epsilon(e)\epsilon(\bar{e})(-1)^{n(s)+n(t)+j+1}\mathcal{I}(L_x^{j+1})$,

V3. $\mathcal{I}(L^I)$ depends only on the actuality index of the family of the BC of the configuration that L^I realizes or, equivalently, $\mathcal{I}(L^{I+m}) = 0$ where $m \geq 1$,

V4. $\mathcal{I}(G) = 0$ for any immersion G that is a good model for an inadmissable configuration,

and $(\mathcal{V}, \mathcal{I})$ has the additional data of an actuality table.

Our main result will show that our invariants are invariants of finite type by construction and that the value of the invariant is independent of the choice of how the given link or immersion is taken to the unlink or representative immersion and depends only on the families of configurations used in the evaluation.

Theorem 3.11 *Let n and d be fixed positive integers. Let*

$$\gamma \in E_\infty^{-I,I}(d) \subseteq \bar{H}_{3N-1}(\Omega_I) \subseteq \bar{H}_{3N-1}(\Omega) \cong \bar{H}_{3N-1}(\Gamma^d \cap \Sigma)$$

where $N = n(2d+1)$ and $I \leq \frac{3N+1}{5}$. Then there is a link invariant $\mathcal{V}_\gamma$ and an invariant $\mathcal{I}_\gamma$ of nice immersions such that the pair $(\mathcal{V}_\gamma, \mathcal{I}_\gamma)$ is a link invariant of finite type of order I.

Proof: Our proof will follow the same outline as the proof of the analogous theorem for n=1 in Chapter two. We will, however, prove the analog to theorem 2.21 very carefully.

Now, in the top row of our extended actuality table, the coefficients with which each BC enters γ are entered as actuality indices. We have oriented all of the BC's of each J-block so that the indices entered for BC's from related configurations are equal. Let's see how to enter the actuality indices in the rows of the table below the top row. Let's assume that γ has survived the spectral sequence through r steps and is still nontrivial, so γ has a descendant that lives as a nonzero element of $E_r^{-I,I}$.

Abusing notation, we'll refer to the descendant of γ as γ also. Let e be any $3N-1$ dimensional BC in $\Omega_{I-r} - (\Omega_{I-r-1} \cup Z_{I-r})$. Recall the following:

- $\partial_r\gamma$ subdivides each BC e in $\Omega_{I-r} - (\Omega_{I-r-1} \cup Z_{I-r})$ into *elementary components.*
- *Compatible* chains of elementary components are those whose boundary inside any e is exactly the intersection of $\partial_r\gamma$ and e.

- The *main component* of e is the elementary component of the cell containing the curve which represents e in the actuality table T_γ.

Lemma 3.12 *Let $\bar{X}_{I-r}$ be the set of chains compatible with $\partial_r\gamma$, then $\bar{X}_{I-r}$ is isomorphic to X_{I-r}.*

Proof: We must modify our proof of lemma 2.15 to take into account the fact that we have changed the orientations of some of the cells in each J-block. The idea is still that the coefficient with which the principal component of any BC enters ξ determines the coefficients with which every other elementary component enters ξ.

Let e be a $3N-1$ dimensional BC in $\Omega_{I-r}-(\Omega_{I-r-1}\cup Z_{I-r})$. Each elementary component in e is in one-to-one correspondence with the isotopy class of some knotted graph that respects the configuration J associated with e. Recall that e actually corresponds to the entire class $\{J' \mid J' \sim J\}$. Let

- ϵ be a function which takes the value 1 on contingent cells and -1 on dependent cells,
- ξ be a chain compatible with $\partial_r\gamma$,
- e_1 and e_2 be two elementary components of e that share a common boundary in $e \cap \partial_r\gamma$,
- $\bar{e}$ be the BC in $\Omega_{I-r+1}-(\Omega_{I-r}\cup Z_{I-r+1})$, part of whose geometric boundary seperates e_1 and e_2,
- $\bar{J}$ be (a representative of) the $[I-r+1]$, $\langle I-r+1\rangle_A$, or $\langle I-r+1\rangle_C$ configuration (class) corresponding to $\bar{e}$ and
- a be the index assigned to this elementary component of $\bar{e}$, obtainable by using V2 and the $(I-r+1)^{st}$ row of the table.

If we jump from e_1 to e_2 across $\bar{e}$, then at the point of crossing we pass through a curve which identifies two points t and s that occur in $\bar{J}$, but not in J. Let $n(t)$ and $n(s)$ be the numbers of these points in the standard ordering of the points of $\bar{J}$. The coefficients with which e_1 and e_2 enter ξ differ by

$$\epsilon(e)\epsilon(\bar{e})(-1)^{n(t)+n(s)+(I-r+1)}a,$$

so if K_+ and K_- are two curves in e and vary by a jump through $\bar{e}$ at K_x, then the modified crossing change formula is:

$$\epsilon(e)(\mathcal{I}_\gamma(K_+)-\mathcal{I}_\gamma(K_-)) = \epsilon(\bar{e})(-1)^{n(t)+n(s)+(I-r+1)}\mathcal{I}_\gamma(K_x).$$

Let Υ be the map that takes each generator e of X_{I-r} to the chain in $\bar{X}_{I-r}$ in which the main component of e has coefficient 1 and all other main components enter with coefficient zero. Extending this map by linearity gives the required isomorphism. □

We have a strategy to see if γ survives nontrivially to the next stage and, if it does, fill out the $(I-r)^{th}$ row of the extended actuality table. An example of this strategy is given as the first long example in §4.2. The strategy is as follows:

1. Choose a chain ξ compatible with $\partial_r\gamma$, with the property that each main component enters ξ with coefficient zero.

2. The $3N-2$ dimensional best cells generating Y_{I-r} are subdivided into elementary components in the same way as the cells in X_{I-r} are. This subdivision gives a new cell decomposition of $\Omega_{I-r} - (\Omega_{I-r-1} \cup Z_{I-r})$.

To calculate the boundary $\tilde{d}_r\gamma$ of ξ inside $\Omega_{I-r} - (\Omega_{I-r-1} \cup z_{I-r})$, we must calculate the coefficients with which the main components of the three types of $3N-2$ dimensional cells of the canonical decomposition enter this boundary. The multiplicities of other elementary components of these cells in the boundary is determined by the multiplicities with which corresponding elementary components of $3N-1$ BC's enter ξ, i.e. everything depends on main components.

The main component of any $3N-2$ dimensional FPC is defined to be the component containing the same curve that represents any of its parent cells in the actuality table. Similiarly, the main component of the BC of an $\langle I-r\rangle^*$-configuration is defined to be the one containing the curve representing the associated $\langle I-r\rangle$-configuration in the table. It is easy to see that the BC of any $[I-r-1]^*$-configuration lies in the boundary of a unique inadmissable $[I-r]$-configuration. It simplifies matters (but makes no difference in the end) to choose the main component of the BC of an $[I-r-1]^*$-configuration to be the component containing the curve which corresponds to the boundary of the curve representing the corresponding $[I-r]$-configuration in the table.

Lemma 3.13 *If J is an inadmissable $[I]$-configuration and $\gamma_0 \in E_1^{-I,I}$, then the BC of J enters γ_0 with coefficient zero.*

Proof: The proof is exactly the same as the proof of lemma 2.17. □

Lemma 3.14 *Let $1 \leq r \leq I-1$, let $\gamma \in E_r^{-I,I}$ and let ξ be a chain compatible with $\partial_r\gamma$, then any BC e^* from an $[I-r-1]^*$-configuration enters $\tilde{d}_r\gamma$ with coefficient zero.*

Proof: The proof is exactly the same as that of lemma 2.18 in the general case, and if we have chosen the main component of e^* as suggested above, then since the main component of every $3N-1$ dimensional BC enters ξ with coefficient zero, the main component of e^* enters $\tilde{d}_r\gamma$ with coefficient zero. □

Lemma 3.15 *Let γ and ξ be as in the previous lemma. If f is the a $3N-2$ dimensional FPC, then the main component of f enters $\tilde{d}_r\gamma$ with coefficient zero.*

Proof: The main components of every $3N-1$ dimensional BC enter ξ with coefficient zero, therefore the main components of every $3N-2$ dimensional FPC enter $\tilde{d}_r\gamma$ with coefficient zero as well. □

3. Let e be the BC of an $\langle I-r\rangle^*$-configuration and $\hat{e}$ be the BC of the associated $\langle I-r\rangle$-configuration. Let e_1 and e_2 be the BC's of the two $[I-r]$-configurations whose diagrams contract to that of e under a single edge contraction. Let C be the main component of e, and let C_1 and C_2 be the elementary components of e_1 and e_2, respectively, that have C as part of their boundary. The main component of $\hat{e}$ enters ξ with coefficient zero, so the

coefficient with which C enters $\tilde{d}_r\gamma$ comes entirely from the multiplicities with which C_1 and C_2 enter ξ times the incidence coefficient of e in $h_{I-r}(e_1)$ and $h_{I-r}(e_2)$. If we determine this coefficient for each such e, then we can view $\tilde{d}_r\gamma$ as the resulting linear combination of main components of BC's of $\langle I-r\rangle^*$-configurations.

4. For γ to survive to the next stage we must show that $d_r\gamma$, which is the class of $\tilde{d}_r\gamma$, is zero in homology. So, we must show that $\tilde{d}_r\gamma$ is the boundary of *something*. The boundary of ξ is $\partial_r\gamma + \tilde{d}_r\gamma$ and $\partial_r\gamma$ is a cycle. This statement is footnoted in 4.5.1 in [V]. If we didn't have to worry about Z_{I-r}, then $\partial_r\gamma$ would certainly be a cycle, but the set Z_{I-r} is not closed in Ω_{I-r-1}. The good news is that the intersection of the closure of Z_{I-r} is of codimension two in Ω_{I-r-1}, so $\partial_r\gamma$ has no codimension one boundary in Ω_{I-r-1}. Since $\partial_r\gamma$ is a cycle, we must show that $\tilde{d}_r\gamma$ is the boundary of some chain in $\hat{X}_{I-r}$. To do this, we find a chain α_{I-r} in X_{I-r} such that

$$h_{I-r}(\alpha_{I-r}) + \tilde{d}_r\gamma = 0.$$

If such a chain can be found, then $\bar{\alpha}_{I-r} = \Upsilon(\alpha_{I-r})$ has the negative of $\tilde{d}_r\gamma$ as its boundary, so the existence of such an α_{I-r} insures the nontrivial survival of γ to the next stage.

5. Assuming that α_{I-r} can be found, enter the coefficients with which the generating BC's of X_{I-r} enter α_{I-r} in the appropriate boxes of the $I-r^{th}$ row in the extended actuality table. If γ survives all the way to the last stage, then the table is completely filled out. Suppose that no such α_{I-r} can be found. Then $\tilde{d}_r\gamma$ is *not* equal to the boundary of anything and thus its class $d_r\gamma$ is not zero in homology. In this case γ fails to survive, we cannot fill in an actulity table for γ and we cannot get an invariant from γ.

DEFINITION: Let $\gamma \in E_\infty^{-I,I}$ and let $T\gamma$ be the actuality table for γ.

1. Let L^I be a nice immersion with I self-intersections that respects an $[I]$-configuration J. Let a be the actuality index of J in the table. Then we define

$$\mathcal{I}_\gamma(L^I) = a.$$

2. Let L^m be a nice immersion with $m \geq I$ self-intersections. Then we define

$$\mathcal{I}_\gamma(L^m) = 0.$$

3. Let L^{I-r} be a nice immersion with $I-r$ self-intersections that realizes an $[I-r]$-configuration J. Let e be the BC corresponding to J and let c be the elementary component of e that contains L^{I-r}. Then we define

$$\mathcal{I}_\gamma(L^{I-r}) = a$$

where a is the coefficient with which c enters the chain $\bar{\alpha}_{I-r}$.

4. Let L be any n-component link and let U be any link equivalent to the n-component unlink. We define

$$\mathcal{V}_\gamma(U) = 0$$

and define $\mathcal{V}_\gamma(L)$ using the recursion formula

$$\mathcal{V}_\gamma(L_+) - \mathcal{V}_\gamma(L_-) = \mathcal{I}_\gamma(L^1_x).$$

By definition, $\mathcal{I}_\gamma$ satisfies V3. By the proof of lemma 3.12, $\mathcal{I}_\gamma$ satisfies V2. By lemma 3.13 and 3.14, $\mathcal{I}_\gamma$ satisfies V4. Finally, $\mathcal{V}_\gamma$ satisfies V1 and V2 by definition. We still must prove the following theorem:

Theorem 3.16 *Let $\gamma \in E_\infty^{-I,I} \subset \bar{H}_{3N-1}(\Omega)$.*

1. *If $j \leq I$, and J_1 and J_2 are related $[j]$-configurations, then the actuality indices of J_1 and J_2 in T_γ are equal.*
2. *Suppose that L_1^j and L_2^j are curves realizing J_1 and J_2 and that L_1^j and L_2^j have the same image in R^3. Then*
$$\mathcal{I}_\gamma(L_1^j) = \mathcal{I}_\gamma(L_2^j).$$
3. *If L is any n-component link and U is the n-component unlink, then $\mathcal{V}_\gamma(L)$ is independent of the sequence of crossing changes made to take L to U. Similiarly, if L^j respects a $[j]$-configuration J and K^j is the curve representing J in T_γ, then $\mathcal{I}_\gamma(L^j)$ does not depend on the sequence of crossing changes made to take L^j to K^j.*

Proof: To prove the first statement, suppose that $j = I$. Let e_1 and e_2 be adjacent BC's from related $[I]$-configurations J_1 and J_2, respectively. Let γ_0 refer to the ancestor of γ in $E_1^{-I,I}$. If e_1 enters γ_0 with coefficient c, then e_2 must enter γ_0 with coefficient c as well, otherwise the $3N-2$ FPC that glues these cells together would appear in $h_I(\gamma_0)$ with a nonzero coefficient. Note that, by construction, the issue of whether e_1 or e_2 are contingent doesn't change this argument.

Assume by induction that the result holds for $I, I-1, \ldots, j+1$. Suppose that c_1 and c_2 are the coefficients with which the BC's of J_1 and J_2 enter α_j, where α_j is as above. We know that no $3N-2$ dimensional FPC's can have nonzero coefficient in $h_j(\alpha_j)$, since none have nonzero coefficient in $\bar{d}_{I-j}\gamma$. By the argument above for $j = I$, part one follows.

The second statement holds for $j = I$ by the fact that for any curve L^I realizing an $[I]$-configuration J, the number $\mathcal{I}_\gamma(L^I)$ only depends on the configuration class of J. By the first statement of the theorem, all of these equivalence classes receive the same actuality index and that only depends on the family type of the given configuration. Assume inductively that the second and third statements hold for $I, I-1, \ldots, j+1$. We will prove the second and third statements of the theorem by proving three lemmas.

To prove the next three lemmas, let's make the following assignments: let

- J_1 and J_2 be related $[j]$-configurations,
- $\bar{J}_1$ and $\bar{J}_2$ be related $[j+1]$-configurations,
- e_1 and e_2 be the BC's of J_1 and J_2, respectively,
- $\bar{e}_1$ and $\bar{e}_2$ be the BC's of $\bar{J}_1$ and $\bar{J}_2$, respectively,
- K_+^j and K_-^j be curves lying in elementary components e_{1+} and e_{1-} of e_1, and varying by a single passage across $\partial_{I-j}e_1$ at a curve K_x^{j+1} in e_1,
- L_+^j and L_-^j be curves in e_2 with the same images as K_+^j and K_-^j, respectively,

- e_{2+} and e_{2-} be elementary components of e_2 containing L^j_+ and L^j_-, respectively,
- L^{j+1}_x be a curve in $\bar{e}_2$ with the same image as K^{j+1}_x,
- R^j_1 and R^j_2 be the curves, with the same image, representing J_1 and J_2 in the actuality table,
- t and s be the points added to the chord diagram of J_1, in order to obtain the chord diagram of $\bar{J}_1$,
- for any point t in the chord diagram of any configuration, let $n(t)$ be the number of that point in the standard ordering of the points of that configuration class. Let's assume that $n(s) < n(t)$.

Lemma 3.17 *Suppose that the jump between e_{1+} and e_{1-} can be made through either K^{j+1}_x in $\bar{e}_1$ or through L^{j+1}_x in $\bar{e}_2$. If τ and σ are the points added to the chord diagram of J_1 in order to obtain $\bar{J}_2$, then*

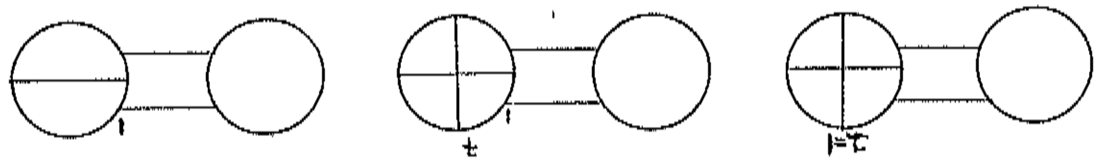
$$\epsilon(e_1)\epsilon(\bar{e}_1)(-1)^{n(t)+n(s)+j+1}\mathcal{I}_\gamma(K^{j+1}_x) = \epsilon(e_1)\epsilon(\bar{e}_2)(-1)^{n(\tau)+n(\sigma)+j+1}\mathcal{I}_\gamma(L^{j+1}_x)$$

Proof: We know by induction that $\mathcal{I}_\gamma(K^{j+1}_x) = \mathcal{I}_\gamma(L^{j+1}_x)$, so we must show that

$$\epsilon(\bar{e}_1)(-1)^{n(t)+n(s)} = \epsilon(\bar{e}_2)(-1)^{n(\tau)+n(\sigma)}.$$

We can reduce to the case where $\bar{e}_1$ and $\bar{e}_2$ are adjacent. Assume that the chord diagrams of $\bar{J}_1$ and $\bar{J}_2$ vary in the location of their first points on $(S^1)_i$ and that the first point of $\bar{J}_1$ on $(S^1)_i$ corresponds to the second point of $\bar{J}_2$ on $(S^1)_i$. The only way that chords can be added to the chord diagram of J_1 to obtain either $\bar{J}_1$ or $\bar{J}_2$ is when t is the last point (on the chord diagram) of $\bar{J}_1$ on $(S^1)_i$ and τ is the first point of $\bar{J}_2$ on $(S^1)_i$. We have also assumed that σ on $\bar{J}_2$ corresponds to t on $\bar{J}_1$ and τ corresponds to s.

Example with Figure 3.18 *An example of the chord diagrams of J_1, $\bar{J}_1$, and $\bar{J}_2$:*



Recall that $\mathrm{ROC}(\bar{e}_1, \bar{e}_2)$ is the product of the numbers $\mathrm{PP}(\bar{e}_1, \bar{e}_2)$, $\mathrm{CP}(\bar{e}_1, \bar{e}_2)$ and $\mathrm{FP}(\bar{e}_1, \bar{e}_2)$. Let's calculate these three numbers:

$$\mathrm{PP}(\bar{e}_1, \bar{e}_2) = (-1)\rho_j(\bar{J}_1) - 1 = (-1)^{n(t)-n(\tau)} = (-1)^{n(t)+n(\tau)}.$$

If s and σ are not on $(S^1)_i$, then $n(s) = n(\sigma)$, and *none* of the 3-forms, that wedge together to become $w_{\bar{J}_1}$, need to have their polarities changed in the transition to $w_{\bar{J}_2}$. It follows that CP=FP because FP is simply the product of CP times (-1) raised to the number of polarity switches. Therefore

$$\begin{aligned} &\mathrm{ROC}(\bar{e}_1, \bar{e}_2)(-1)^{n(t)+n(\tau)} = 1 \\ \Rightarrow\ &\mathrm{ROC}(\bar{e}_1, \bar{e}_2) = \epsilon(\bar{e}_1)\epsilon(\bar{e}_2) \\ \Rightarrow\ &\epsilon(\bar{e}_1)\epsilon(\bar{e}_2)(-1)^{n(t)+n(\tau)+n(s)+n(\sigma)} = 1 \end{aligned}$$

which gives the desired result. □.

Lemma 3.19 *Suppose that the jump between e_{2+} and e_{2-} can be made across the curve L_x^{j+1} of $\bar{e}_2$. If τ and σ are the points added to the chord diagram of J_2 in order to obtain the chord diagram of $\bar{J}_2$, then*

$$\epsilon(e_1)\epsilon(\bar{e}_1)(-1)^{n(t)+n(s)+j+1}\mathcal{I}_\gamma(K_x^{j+1}) = \epsilon(e_2)\epsilon(\bar{e}_2)(-1)^{n(\tau)+n(\sigma)+j+1}\mathcal{I}_\gamma(L_x^{j+1}).$$

Proof: As above, $\mathcal{I}_\gamma(K_x^{j+1}) = \mathcal{I}_\gamma(L_x^{j+1})$ by induction and we can reduce to the case where e_1 and e_2 are adjacent. Assume that the first point of J_1 on $(S^1)_i$ corresponds to the second point of J_2 on $(S^1)_i$. There are four possibilities.

First, suppose that neither t nor s lie on $(S^1)_i$. Then $n(t) + n(s) = n(\tau) + n(\sigma)$. Let's calculate $\mathrm{ROC}(e_1, e_2)$ and $\mathrm{ROC}(\bar{e}_1, \bar{e}_2)$:

$$\mathrm{PP}(e_1, e_2) = (-1)^{\rho_i(J_1)-1} = (-1)^{\rho_i(\bar{J}_1)-1} = \mathrm{PP}(\bar{e}_1, \bar{e}_2).$$

We know that

$$\mathrm{CP}(e_1, e_2)\mathrm{FP}(e_1, e_2) = (-1)^\kappa$$

where κ is the number of 3-forms that change polarity in the transition from e_1 to e_2. This number doesn't depend on the added chords (s, t) and (τ, σ) at all, so

$$\begin{aligned}
&\mathrm{CP}(e_1, e_2)\mathrm{FP}(e_1, e_2)\mathrm{CP}(\bar{e}_1, \bar{e}_2)\mathrm{FP}(\bar{e}_1, \bar{e}_2) = 1\\
\Rightarrow\ &\mathrm{ROC}(e_1, e_2) = \mathrm{ROC}(\bar{e}_1, \bar{e}_2)\\
\Rightarrow\ &\epsilon(e_1)\epsilon(e_2)\epsilon(\bar{e}_1)\epsilon(\bar{e}_2)(-1)^{n(t)+n(s)+n(\tau)+n(\sigma)} = 1
\end{aligned}$$

which gives the result.

Secondly, suppose that $s \in (S^1)_i$ and $t \notin (S^1)_i$. Then we have $n(t) = n(\tau)$, $n(s) = n(\sigma) + 1$ and

$$\mathrm{PP}(e_1, e_2) = (-1)\rho_i(J_1) - 1 = (-1)\rho_i(\bar{J}_1) = -\mathrm{PP}(\bar{e}_1, \bar{e}_2).$$

Now, if s is not the last point of $\bar{J}_1$ on $(S^1)_i$, then there is no change in polarity of the 3-form $d(t-s)$, when it tranforms to $d(\tau - \sigma)$. Therefore

$$\begin{aligned}
&\mathrm{CP}(e_1, e_2)\mathrm{FP}(e_1, e_2) = \mathrm{CP}(\bar{e}_1, \bar{e}_2)\mathrm{FP}(\bar{e}_1, \bar{e}_2)\\
\Rightarrow\ &\mathrm{ROC}(e_1, e_2)\mathrm{ROC}(\bar{e}_1, \bar{e}_2)(-1)^{n(t)+n(s)+n(\tau)+n(\sigma)} = 1
\end{aligned}$$

which is what we want. If s were the last point of $\bar{J}_1$ on $(S^1)_i$, then J_1 and J_2 are from the same equivalence class of configurations. That possibility was covered in the last lemma.

The third possibility, where $t \in (S^1)_i$ and $s \notin (S^1)_i$, is handled in the same way as the second case.

Finally, suppose that s and t are both on $(S^1)_i$. If t were a last point on $(S^1)_i$, then J_1 and J_2 would be of the same configuration class. Assume that t is not a last point of $\bar{J}_1$ on $(S^1)_i$, then $(-1)^{n(t)+n(s)} = (-1)^{n(\tau)+n(\sigma)}$, $n(\sigma) < n(\tau)$ and there is no polarity change in the transition from $d(t-s)$ to $d(\tau - \sigma)$. It follows that

$$\mathrm{CP}(e_1, e_2)\mathrm{FP}(e_1, e_2) = \mathrm{CP}(\bar{e}_1, \bar{e}_2)\mathrm{FP}(\bar{e}_1, \bar{e}_2)$$

which gives the result as above. □

Lemma 3.20 *Let K^j be any curve in e_1 and let L be any link.*

1. *Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be two paths in e_1 from K^j to R_1^j, and in general position relative to Σ_{J-1}. Then $\mathcal{I}_\gamma(K^j)$ can be calculated along either path and is independent of which path is chosen.*

2. *Let $\mathcal{Q}_1$ and $\mathcal{Q}_2$ be two paths from L to the unlink that are in general position relative to Σ_1. Then $\mathcal{V}_\gamma(L)$ can be calculated along either path and is independent of which path is chosen.*

Proof: The statement is equivalent to showing that the contribution of any closed loop to the index is zero.

Suppose that $e = e_1$, $\bar{e}_1$, $\bar{e}_2$, $J = J_1$, $\bar{J}_1$ and $\bar{J}_2$ are all given as above, except that $\bar{J}_1$ and $\bar{J}_2$ are now taken to be nonrelated. Let K_{++}, K_{+-}, K_{--} and K_{-+} be curves in e such that:

- K_{++} and K_{+-} vary by a crossing change through a curve K_{+x} in $\bar{e}_1$,
- K_{+-} and K_{--} vary by a crossing change through a curve K_{x-} in $\bar{e}_2$,
- K_{--} and K_{-+} vary by a crossing change through a curve K_{-x} in $\bar{e}_1$,
- K_{-+} and K_{++} vary by a crossing change through a curve K_{x+} in $\bar{e}_2$,
- K_{+x} and K_{-x} vary by a crossing change through the curve K_{xx} in the BC $\mathcal{E}$ of a $[j+2]$-configuration $\mathcal{J}$, and K_{x-} and K_{x+} vary by a crossing change through K_{xx} as well.

Let τ and σ be the points added to the chord diagram of J to obtain $\bar{J}_2$. The points τ and σ can also be added to the diagram of $\bar{J}_1$ to get that of $\mathcal{J}$. Recall that t and s are the points added to the chord diagram of J to get $\bar{J}_1$ and they can also be added to the chord diagram of $\bar{J}_2$ to obtain that of $\mathcal{J}$.

Let $n_1(t)$ refer to the number of the point t in the standard ordering of the points of $\bar{J}_1$, and similiarly for $n_1(s)$. Let n_2 be the assignment of numbers to points in $\bar{J}_2$ and let $\mathcal{N}$ be the assignment of numbers to points in $\mathcal{J}$. Let's assume that $n_1(s) < n_1(t)$, $n_2(\tau) < n_2(\sigma)$, $\mathcal{N}(s) < \mathcal{N}(t)$ and $\mathcal{N}(\sigma) < \mathcal{N}(\tau)$.

We can reduce to the case where the closed loop goes from K_{++} through K_{+x} to K_{+-}, through K_{x-} to K_{--}, through K_{-x} to K_{-+} and finally through K_{x+} to K_{++}. To complete the proof we need to examine the crossing change formulas:

$$\begin{aligned}
\mathcal{I}_\gamma(K_{++}) &= \mathcal{I}_\gamma(K_{+-}) + \epsilon(e)\epsilon(\bar{e}_1)(-1)^{n_1(t)+n_1(s)+j+1}\mathcal{I}_\gamma(K_{+x}) \\
&= \mathcal{I}_\gamma(K_{--}) + \epsilon(e)\epsilon(\bar{e}_1)(-1)^{n_1(t)+n_1(s)+j+1}\mathcal{I}_\gamma(K_{+x}) \\
&\quad + \epsilon(e)\epsilon(\bar{e}_2)(-1)^{n_2(\tau)n_2(\sigma)+j+1}\mathcal{I}_\gamma(K_{x-}) \\
&= \mathcal{I}_\gamma(K_{-+}) + \epsilon(e)\epsilon(\bar{e}_1)(-1)^{n_1(t)+n_2(s)+j+1}\mathcal{I}_\gamma(K_{+x}) \\
&\quad + \epsilon(e)\epsilon(\bar{e}_2)(-1)^{n_2(\tau)+n_2(\sigma)+j+1}\mathcal{I}_\gamma(K_{x-}) \\
&\quad + \epsilon(e)\epsilon(\bar{e}_1)(-1)^{n_1(t)+n_1(s)+j+1}\mathcal{I}_\gamma(K_{-x}) \\
&= \mathcal{I}_\gamma(K_{++}) + \epsilon(e)\epsilon(\bar{e}_1)(-1)^{n_2(\tau)+n_2(\sigma)+j+1}\mathrm{Ind}(K_{+x})
\end{aligned}$$

$$+ \epsilon(e)\epsilon(\bar{e}_2)(-1)^{n_2(\tau)+n_2(\sigma)+j+1}\mathcal{I}_\gamma(K_{-x})$$
$$- \epsilon(e)\epsilon(\bar{e}_1)(-1)^{n_1(t)+n_2(s)+j+1}\mathcal{I}_\gamma(K_{x-})$$
$$- \epsilon(e)\epsilon(\bar{e}_2)(-1)^{n_2(\tau)+n_2(\sigma)+j+1}\mathcal{I}_\gamma(K_{x+}).$$

We must show that the sum of these last four terms is zero and that this sum equals

$$\begin{aligned} &\epsilon(e)\epsilon(\bar{e}_1)(-1)^{n_1(t)+n_1(s)+j+1}(\mathcal{I}_\gamma(K_{+x}) - \mathcal{I}_\gamma(K_{-x})) \\ &+ \epsilon(e)\epsilon(\bar{e}_2)(-1)^{n_2(\tau)+n_2(\sigma)+j+1}(\mathcal{I}_\gamma(K_{x-}) - \mathcal{I}_\gamma(K_{x+})) \\ = \ &\epsilon(e)\epsilon(\bar{e}_1)(-1)^{n_1(t)+n_1(s)+j+1}(\epsilon(\bar{e}_1)\epsilon(\overline{\overline{e}})(-1)^{\mathcal{N}(\tau)+\mathcal{N}(\sigma)+j+2}\mathcal{I}_\gamma(K_{xx})) \\ &+ \epsilon(e)\epsilon(\bar{e}_2)(-1)^{n_2(\tau)+n_2(\sigma)+j+1}(\epsilon(\bar{e}_2)\epsilon(\overline{\overline{e}})(-1)^{\mathcal{N}(\tau)+\mathcal{N}(\sigma)+j+2}\mathcal{I}_\gamma(K_{xx})); \end{aligned}$$

thus it is sufficient to show that

$$(-1)^{n_1(t)+n_1(s)+\mathcal{N}(\tau)+\mathcal{N}(\sigma)} = (-1)^{n_2(\tau)+n_2(\sigma)+\mathcal{N}(t)+\mathcal{N}(s)}.$$

If $\mathcal{N}(s) < \mathcal{N}(t) < \mathcal{N}(\sigma) < \mathcal{N}(\tau)$, then $n_1(s) = \mathcal{N}(s)$, $n_1(t) = \mathcal{N}(t)$, $n_2(\sigma) < \mathcal{N}(\sigma) - 2$, and $n_2(\tau) < \mathcal{N}(\tau) - 2$. It follows that

$$(-1)^{n_1(t)+n_1(s)+\mathcal{N}(\tau)+\mathcal{N}(\sigma)} = (-1)^{\mathcal{N}(t)+\mathcal{N}(s)+\mathcal{N}(\tau)+\mathcal{N}(\sigma)} = (-1)^{n_2(\tau)+n_2(\sigma)+\mathcal{N}(t)+\mathcal{N}(s)}.$$

If $\mathcal{N}(s) < \mathcal{N}(\sigma) < \mathcal{N}(t) < \mathcal{N}(\tau)$, then $n_1(s) = \mathcal{N}(s)$, $n_2(\sigma) < \mathcal{N}(\sigma)-1$, $n_1(t) < \mathcal{N}(t)-1$ and $n_2(\tau) < \mathcal{N}(\tau) - 2$. The result follows as above.

Finally, if $\mathcal{N}(s) < \mathcal{N}(\sigma) < \mathcal{N}(\tau) < \mathcal{N}(t)$, then $n_1(s) = \mathcal{N}(s)$, $n_2(\sigma) < \mathcal{N}(\sigma) - 1$, $n_2(\tau) < \mathcal{N}(\tau)$ and $n_1(t) = \mathcal{N}(t) - 2$. The result follows as above. Since $\bar{J}_1$ and $\bar{J}_2$ are chosen arbitrarily, all the other cases are contained in these. This proves that the contribution of a closed loop on a crossing change path is zero. The same proof works in the case where K_{++}, K_{-+}, K_{--} and K_{+-} are all links, $j = 0$ and e is all of Γ^d, and $\mathcal{V}_\gamma$ replaces all occurences of $\mathcal{I}_\gamma$ when evaluation is taken on K_{++}, K_{-+}, K_{--} or K_{+-}. □

We have shown that corresponding elementary components in BC's of related configurations satisfy the same skein relations and the same normalizations; therefore they must receive the same index. We have also shown that the index of an elementary component can be determined by any possible path to the principal component of the given cell. We conclude that $\mathcal{I}_\gamma(L^j)$ is independent of the unlinking path taken from L^j to the representative curve K^j and only depends on the family type of the diagrams. We have also shown that $\mathcal{V}_\gamma(L)$ is independent of the unlinking path taken from L to the unlink. As a result we can construct a reduced actuality table and obtain an invariant of finite type of order I. This concludes the proof of theorem 3.16. □

We have also now proved theorem 3.11.

What about d? Recall that d is the degree of the Fourier polynomials that constitute the coordinate functions of $p \in \Gamma^d$. Recall that all of our work has been for a fixed Γ^d. Thus we have implicitly worked with a family of spectral sequences $E_r^{p,q}(d)$ indexed by d. We will show that $E_r^{-I,I}(d)$ does not depend on d, as long as $I \leq \frac{3N+1}{5}$, where n is the number of components of the links in question and $N = n(2d+1)$. The result is that, as long as $d \geq \frac{3N+1}{5}$ to start with, cycles of $E_r^{-I,I}(d)$ stabilize when $d \to \infty$. By Alexander Duality, these cycles are dual to cocycles in

$$\lim_{d\to\infty} \tilde{H}^0(\Gamma^d - \Sigma)$$

and are link invariants, by the process we described in §2.4 and earlier in this section.

Theorem 3.21 *Fix an integer d and let $N = n(2d+1)$. Then $E_1^{p,q}(d) = 0$ when*
A) $p+q<0$,
B) $p \geq 1$ when $n=1$, and $p \geq 0$ for $n \geq 2$,
C) $p < -3N$.

Proof: Let J be an (A,B,C)-configuration of complexity $I = -p$. By theorem 3.4, no cells from complicated configurations enter any cycles in $\bar{H}_k(\Omega_p - \Omega_{p-1})$ with nonzero coefficient, as long as $k \geq 3N-2$, so assume that J is noncomplicated.

1. Suppose that $I \leq \frac{3N+1}{5}$. We have seen that no BC's from noncomplicated configurations have dimension greater than $3N-1$, as long as $p \leq \frac{3N+1}{5}$.

2. Suppose $\frac{3N+1}{5} < I \leq N$, and let

$$A_t = \{J' \mid J' \sim J \text{ and } M(\Gamma^d, J') \text{ has codimension } 3I - t, t \geq 1\}.$$

By lemma 1.6B, A_t has codimension $t(3N - 3I + t + 1)$ inside $\{J' \mid J' \sim J\}$, which has dimension $\leq 2I$. If J is noncomplicated, then

$$\bigcup_{J' \in A_t} M(\Gamma^d, J') \times S_J$$

has dimension no greater than $3N-1-t(3N-3I+t+1)+t = 3N-1-t^2+t-t(3N-3I)$, which is strictly less than $3N$.

3. Suppose $N < p \leq 3N$. Let

$$B_s = \{J' \mid J' \sim J \text{ and } M(\Gamma^d, J') \text{ has dimension } s \geq 0\}.$$

By lemma 1.6C,

$$\bigcup_{J' \in B_s} M(\Gamma^d, J') \times S_J$$

has dimension no greater than

$$3I - 1 + s - (s+1)(3I - 3N + s) = 3N - 1 - s^2 + s - s(3I - 3N)$$

which is strictly less than $3N$. By definition

$$E_1^{-I,q} = \bar{H}_{3n-1+I-q}(\Omega_I - \Omega_{I-1}).$$

We conclude that the statement of part A holds.

By definition $E_0^{p,q} = 0$ if p is positive. If $n = 1$, then $E_1^{-1,1} = 0$ and $E_0^{-1,q} = 0$ for $q > 1$. This proves part B.

Part C follows from the last sentence in lemma 1.6C, which says that $M(\Gamma^d, J) = \emptyset$ if $I \geq 3N$. □

As a corollary we get our second main result, which is a stablization result for cycles in $E_r^{-I,I}(d)$ for a suitable range of values for I. This range is exactly the range for I that has given us theorems 2.6 and 3.6.

Theorem 3.22 *Let d be a positive integer and let $I \leq \frac{3N+1}{5}$, $N = n(2d+1)$ and $\bar{d} > d$, Then*

$$E_r^{-I,I}(\bar{d}) \cong E_r^{-I,I}(d)$$

for $r = 0, 1, 2, 3, \ldots, \infty$. Moreover, if $\bar{\gamma} \in E_\infty^{-I,I}(\bar{d})$ corresponds, via this isomorphism, to $\gamma \in E_\infty^{-I,I}(d)$, then for each link L and each nice immersion L^j in $\Gamma^d \subset \Gamma^{\bar{d}}$ we have

$$\mathcal{V}_{\bar{\gamma}}(L) = \mathcal{V}_\gamma(L)$$

and

$$\mathcal{I}_{\bar{\gamma}}(L^j) = \mathcal{I}_\gamma(L^j).$$

Proof: The cell complex map taking each BC in $E_0^{-I,I}(\bar{d})$ to its counterpart in $E_0^{-I,I}(d)$ gives the result for $r = 0$. Both $E_r^{p,q}(\bar{d})$ and $E_r^{p,q}(d)$ are zero if $p + q < 0$, so no nontrivial differential maps into either $E_r^{-I,I}(\bar{d})$ or $E_r^{-I,I}(d)$; thus the process of calculating these groups, as given in detail, does not depend on d or $\bar{d} > d$. □

We define a *Vassiliev link invariant of order I* to be a cycle in $\bar{H}_{3N-1}(\Omega_I(d))$, where d is sufficiently large and where evaluation on a link uses the table T_γ as was described in §2.4. By construction it is a link invariant of n-component links of finite type and has order I.

4 Facets of the Invariants; Examples and Theorems

4.1 Introduction

In this chapter we begin, in 4.2, by presenting a potpourri of observations that concern the calculations involved in setting up an actuality table. Some of these observations lead to some interesting conclusions regarding invariants of link homotopy. In 4.3 we present several examples in depth. Our first example, in the case $n = 1$, is the simplest example that shows how to fill in the actuality indices in the rows of the actuality table that are below the top row. Our exposition covers the process of entering the actuality indices in the actuality table. We will also show examples of how the actuality tables are used. We end the section by presenting, diagramatically, a cycle in $E_\infty^{-3,3}$ for two component links that distinguishes the Whitehead link for the unlink and we give that calculation. Section 4.4 is devoted to presenting a generalization of Birman-Lin's result, theorem 4.1 in [BL], that establishes a connection between link invariants of finite type and a family of polynomial invariants.

4.2 Some Observations

First, let's investigate some of the different kinds of Vassiliev style invariants of links. Unless we state otherwise, let's assume that the integer n, which determines the number of components in the links under consideration, is fixed and $n > 1$.

DEFINITION: Let γ be a nonzero cycle in $E_\infty^{-I,I}$ and let T_γ be the actuality table for γ. If there is some subcollection $(S^1)_{i_1}, (S^1)_{i_2}, \ldots, (S^1)_{i_k}$ of $k < n$ circles in $\coprod S^1$ such that nonzero actuality indices only appear in boxes in T_γ that correspond to chord diagams that have chord pattern connecting these k circles, then we call γ a *k-connected Vassiliev link invariant.*

Obvious observation: if γ^k is an invariant of k-component links and $k < n$, then we can apply γ^k to some k components of any n-component link. Every k-connected invariant obtained by our process arises in this way and each invariant of k-component links gives one k-connected invariant of n-component links for each choice of k circles out of n. Without loss of generality, suppose that γ^k is a k-connected invariant of the first k components of n-component links. Then, when we evaluate $\mathcal{V}_{\gamma^k}$ on a link, we are allowing any crossing changes involving images of $(S^1)_j$, $k < j \leq n$, to be made free of charge. Thus the evaluation only measures, up to the stength of $\mathcal{V}_{\gamma^k}$, the degree to which the first k components of a given link are linked to each other and knotted to themselves.

Now let's investigate link invariants that are invariants of link homotopy. Recall that an *invariant of link homotopy* is any invariant of links which does not detect the knotting of any component of a link with itself. The simplest example of an invariant of link homotopy is the classical linking number between two components of some n-component link. In some sense the flavor of these invariants is similiar to k-connected ones, because when evaluating an invariant of link homotopy on a link, we get to change any crossing of any component of the link with itself free of charge. We should mention that Bar-Natan, in [B2], and Lin, in [L], have proved independently that Milnor's $\bar{\mu}$ invariants are invariants of finite type. Milnor's invariants are invariants of link homotopy that can be applied to *string links*. An n-component string link is a smooth embedding

$$\sigma : \coprod_{i=1}^{n} I_i \to D \times I,$$

where I and I_i, $1 \leq i \leq n$, denote the unit interval $[0,1]$, where D is the unit disk and where

$$\sigma|_{I_i}(\epsilon) = p_i \times \epsilon$$

for $\epsilon = 0, 1$ and $\{p_i\}_{i=1}^{n}$ is a fixed choice of n points in D.

Suppose that one of our link invariants γ has the property that the boxes of its reduced actuality table are assigned nonzero indices only when the associated $[j]$-configurations have empty A-configurations. It is easy to see that γ is an invariant of link homotopy. We will present examples of several such invariants in 4.3.

Vassiliev invariants of link homotopy have a natural division into two classes of invariants. Let's see how this occurs. Recall that X_I is the free abelian group generated by all of the BC's from $[I]$, $\langle I\rangle_A$ and $\langle I\rangle_C$ configurations, and recall that Y_I is the free abelian group generated by the BC's of all of the $[I-1]^*$, $\langle I\rangle_A^*$ and $\langle I\rangle_C^*$ configurations, as well as the FPC's from generators of X_I. By definition Y_I is equal to the cellular chain group $C_{3N-2}(\Omega_I - (\Omega_{I-1} \cup Z_I))$. If

$$\pi : C_{3N-2}(\Omega_I - \Omega_{I-1}) \to C_{3N-2}(\Omega_I - (\Omega_{I-1} \cup Z_I))$$

is projection, recall that $h_I : X_I \to Y_I$ is defined to be the composition $\pi \circ d_0$. Let

- X_{I1} be the free subgroup of X_I generated by the BC's from $[I]$-configurations that have nonempty A-configurations,
- X_{I2} be the free subgroup of X_I generated by all the BC's in X_I from $\langle I\rangle_A$ and $\langle I\rangle_C$ configurations, and
- X_{I3} be the free subgroup of X_I generated by the BC's of $[I]$-configurations that have empty A-configurations.

The following lemma is the generalization of lemma 3.3 in [BL1] and will help us to characterize link homotopy invariants. We can also exploit the lemma to help us understand some of the combinatorial structure of cycles in $E_1^{-I,I}$.

Lemma 4.1 *The restrictions $h_I|_{X_{I1}}$ and $h_I|_{X_{I2}}$ are one-to-one.*

Proof: We will prove the first statement first. Let $e_1, e_2, \ldots, e_r$ be a subset of the generating BC's of X_{I1} and let $J_1, J_2, \ldots, J_r$ be the corresponding $[I]$-configurations. Suppose that $c_1, \ldots, c_r$ are nonzero integers such that $h_I(\sum c_i e_i) = 0$. Recall that if s and t are points of some $[I]$-configuration and (s,t) form one of the groupings of cardinality two, then $n(s)$ and $n(t)$ refer to the numbers, in the ordering of the points of the configuration, of s and t. Define the *length of the chord* corresponding to the grouping (s,t) to be $|n(t) - n(s)|$.

For each i, $1 \le i \le r$, we define the length $\|e_i\|$ to be the minimum of all of the lengths of the chords of J_i. We know, by lemma 3.13, that no BC's from inadmissable configurations can be among the e_i's, thus the length of each e_i is at least two. Assume that the BC's are ordered such that

$$\|e_1\| \le \|e_2\| \le \cdots \le \|e_r\|.$$

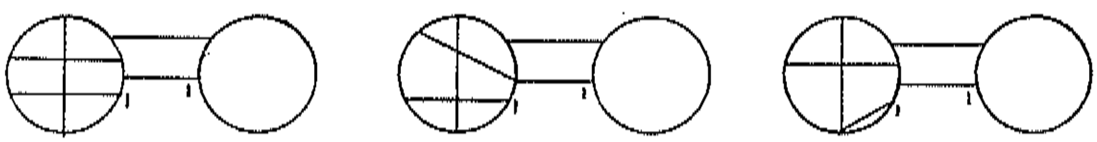
Now, for each i, order the chord pairs

$$(s_1, t_1), (s_2, t_2), \ldots, (s_I, t_I)$$

of J_i such that $n(s_k) < n(t_k)$, $1 \le k \le I$, and such that $n(s_1) < n(s_2) < \cdots < n(s_I)$.

Define $f(i, j)$ to be the BC in Y_I corresponding to the $\langle I \rangle_A^*$ or $\langle I \rangle_C^*$ configuration whose chord diagram is obtained from that of e_i by contracting the edge between the points numbered j and $j+1$.

Example with Figure 4.2 $n = 2$. *On the left is the diagram for a given* e_1 *of complexity* $I = 4$. *In the middle and on the right are diagrams for* $f(1,2)$ *and* $f(1,7)$.

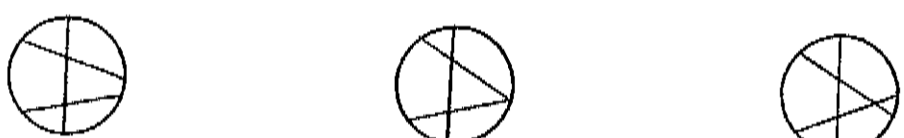


By definition, $h_I(e_i) = \sum \epsilon\zeta(-1)^{a+b} f(i, b-1)$, where the sum is taken over all legally contractible edges and the numbers ϵ, ζ, a and b are as in section 2.3. It follows that

$$h_I(\sum c_i e_i) = \sum c_i \sum \epsilon\zeta(-1)^{a+b} f(i, b-1) = 0.$$

Obviously, if $k \ne j$, then $f(i,k)$ cannot cancel $f(i,j)$. This is because the two diagrams cannot be the same. In the diagram for $f(i,j)$, the only point that connects to two chords is the point with number j, while in $f(i,k)$ it is the point with the number k. For example, in our last example $f(1,2)$ could not cancel $f(1,7)$. So, there must be integers p, q and r such that $e_p \ne e_r$, but $f(p,q) = f(r,q)$. It is easy to see that the chord diagrams of e_p and e_r must be identical except for a transposition of the two points which bound the q^{th} edge to be contracted.

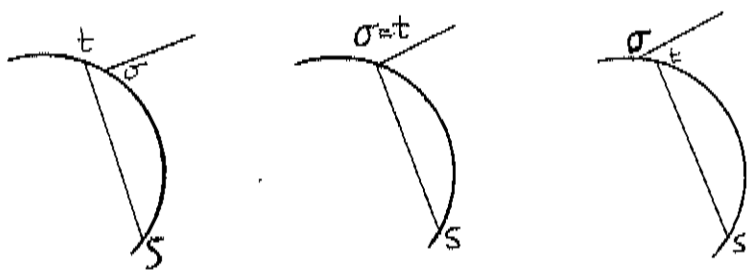
Example with Figure 4.3 *Pictured, left to right, are the chord diagrams for an example of the triple* e_p, $f(p,q) = f(r,q)$ *and* e_r. *Notice that the numbers of both of the points bounding the edge to be contracted in* e_p *are the same as the corresponding numbers for* e_r. *In this example, the pair of points bounding the contracting edge have numbers* 3 *and* 4. *The number of the edge, which is* q, *is* 3.



Suppose that $\|e_1\| = m$ and let (s_1, t_1) be the pair in J_1 whose chord is of length m. Suppose that (σ, τ) are a pair of points such that $n(s) < n(\sigma) = n(t) - 1 < n(t) < n(\tau)$.

Then the edge between σ and t contracts to form the chord diagram for $f(1, n(\sigma))$, which equals $f(r, n(\sigma))$ for some r. However, the chord diagram for e_r is obtained from that of e_1 by switching the points t and σ; thus $\|e_r\| = m-1$, which is a contradiction. Now, suppose that (σ, τ) is a pair of points such that $n(\sigma) < n(s) < n(\tau) = n(t) - 1$. Then the same thing happens with some e_r, where $f(1, n(\tau)) = f(r, n(\tau))$. If there is a pair (σ, τ) such that $n(s) < n(\sigma) < n(\tau) < n(t)$, then $|n(\tau) - n(\sigma)| < m$, which is again a contradiction. We know that $m > 1$ since J_1 is admissable, so there must be *some* pair of points (σ, τ) satisfying one of the above three possibilities. This shows that no length is possible for e_1 and proves the first part of the lemma.

Example with Figure 4.4 *Pictured below are segments of diagrams illustrating the case where $n(\sigma) = n(t) - 1$. On the right is the segment for e_1, in the middle the segment for $f(1, n(\sigma)) = f(r, n(\sigma))$ and on the left the segment for e_r.*

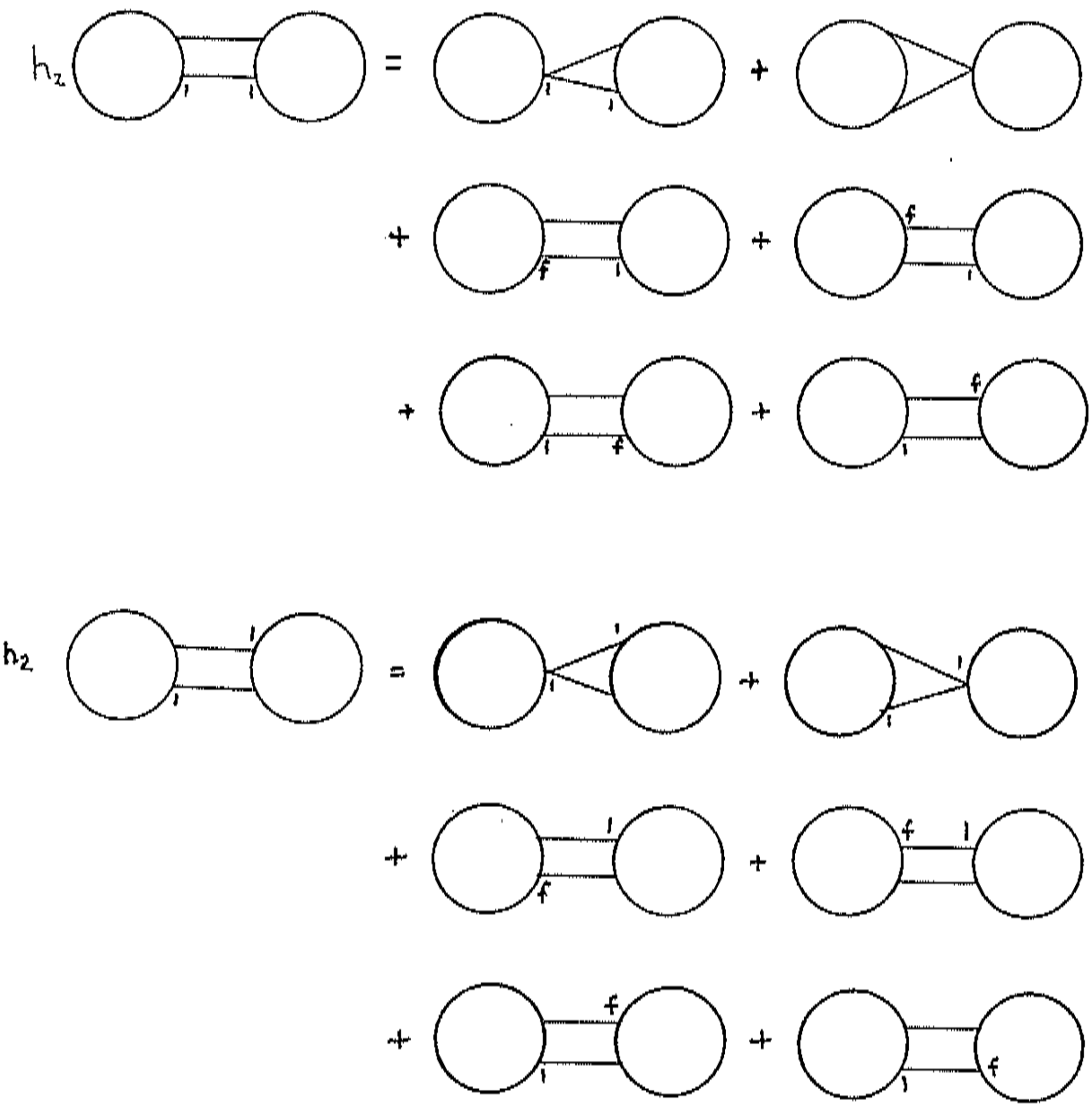


To see that $h_I|_{X_{I2}}$ is one-to-one, we observe that each BC e of an $\langle I\rangle_A^*$ or $\langle I\rangle_C^*$ configuration lies in the image under $h_I|_{X_{I2}}$ of the unique BC $\bar{e}$ from an $\langle I\rangle_A$ or $\langle I\rangle_C$ configuration, where the chord diagram of $\bar{e}$ can be obtained from that of e by simply adding the missing chord to the two chords that connect the three points of the grouping of cardinality three. This concludes the proof of the lemma. □

Example with Figure 4.5 *The chord diagram for e is pictured on the left. The diagram on the right, obtained by adding the missing chord in the grouping of three points, corresponds to the only BC $\bar{e}$ that lies in the $h_I|_{X_{I2}}$ pre-image of e.*

It is easy to find examples of chains in X_{I3} that go to zero under h_I.

Example with Figure 4.6 *In this example $n = 2$. Using diagrams, we have evaluated h_2 of the BC's of the only two generators of X_{23}. The two configurations are related and, normally, we would reorient one of the corresponding BC's, since the ROC of these two cells is -1, but we have not done that here. Recall that when a diagram is to stand for an FPC, we have put an f (to stand for fixed) next to whichever first point that has been fixed at zero.*



The lemma leads to the following observation about link homotopy invariants:

Proposition 4.7 *For each $n \geq 3$ and each $I \geq 2$, there are two classes of link homotopy invariants of order I. One class consists of all invariants of link homotopy in which the actuality indices of all $\langle I \rangle_C$-configurations in the extended actuality table are zero. The other class consists of those invariants in which at least one $\langle I \rangle_C$-configuration has nonzero index in the table.*

Remark: Of course, if $I = 1$, there is only the first type. If $n = 2$ there is only the first type, as any noncomplicated configuration with a grouping of three points must be an $\langle I \rangle_A$-configuration.

Proof: By the lemma, one class consists of cycles arising from chains in X_{I3}; the other class consists of all chains that have nonzero summands from X_{I2}. □

We will show that for any link homotopy invariant $\mathcal{V}_\gamma$ of order I for $n = 2$, there is a pair of polynomials p_γ^+ and p_γ^- such that the value of the invariant $\mathcal{V}_\gamma(L)$ is given by $p_\gamma^+(\ell(L))$ if $\ell(L) \geq 0$ and is given by $p_\gamma^-(\ell(L))$ if $\ell(L) < 0$, where $\ell(L)$ is the linking number of any two component link L. This is not surprising, since linking number is a complete invariant of two component links up to link homotopy. I suspect that, for all n, the first class of link homotopy invariants always arises from linking number, but I have not been able to prove it.

The lemma leads to another observation:

Proposition 4.8 *Suppose that $I \geq n$ and let $\gamma \in E_1^{-I,I}$ be an n-connected cycle that is not a link homotopy invariant. Then for every i, $1 \leq i \leq n$, and every m, $n-2 \leq m \leq I-1$, there is a BC, entering γ with nonzero coefficient, whose chord diagram is n-connected and contains $I - m$ chords that connect to $(S^1)_i$.*

Proof: This observation can be shown entirely diagramatically.

Pictured are five segments of chord diagrams, all of which stand for diagrams of BC's that enter γ with nonzero coefficient. In each figure, the vertical line segments represent segments of copies of S^1 and the diagonal lines represent chords. If two vertical line segments are not connected, then they are not to be part of the same circle. From figure to figure we will assume that corresponding vertical line segments represent fixed circles. For example, the unbroken vertical line segment that goes through the middle of each figure is to represent a segment of $(S^1)_i$, the same fixed i, in each of the figures. Descriptions of the five figures, top to bottom, now follow:

Figure 4.9 *The first figure is part of the diagram of an $[I]$-configuration J_1, which has $m > 1$ chords connecting to $(S^1)_i$. Let e_1 be the corresponding BC. Since we have assumed n-connectedness, there is at least one chord c connecting $(S^1)_i$ to another circle. Let's assume that c_1 is the chord which connects the middle vertical line segment to the one on the right, which is a segment of some $(S^1)_j$, $j \neq i$. Let c_2 be the chord connecting to $(S^1)_i$*

directly below the endpoint of c_1.

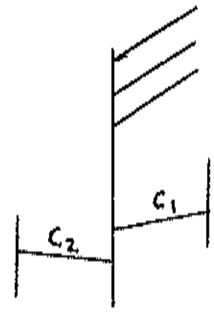


Figure 4.10 *The second figure is the diagram of the $\langle I\rangle_C^*$-configuration J_2 obtained by contracting the edge on $(S^1)_j$ between c_1 and c_2. Let e_2 be the corresponding BC. By the lemma, the contribution of e_2 in $h_I(e_1)$ must be cancelled by the contribution of e_2 in $h_I(e_3)$, where e_3 is the BC of the $\langle I\rangle_C$-configuration J_3 that lies in the pre-image $(h_I|_{X_{I2}})^{-1}(e_2)$.*

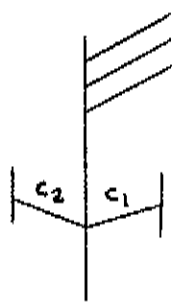


Figure 4.11 *The third figure is of the diagram for J_3, the configuration corresponding to e_3. Let c_3 be the chord that is added to the diagram of J_2 to get that of J_3.*

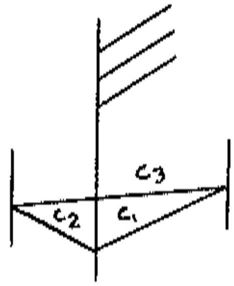


Figure 4.12 *Figure four is the diagram of an $\langle I\rangle_C{}^*$-configuration J_4. If e_4 is the BC of this configuration, then e_4 is one of the other two BC's that lies in the h_I image of e_2. The three point groupings of J_2, J_3 and J_4 are all the same, but the diagrams for J_2 and J_4 are*

missing chords c_3 and c_2, respectively. The diagram for J_3 has all three chords.

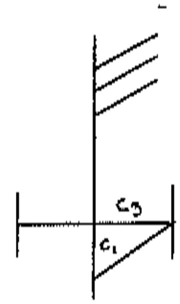
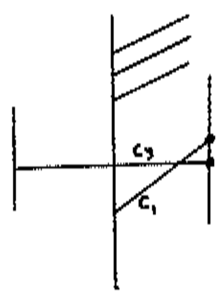


Figure 4.13 *The last figure is of the chord diagram for J_5, the configuration corresponding to e_5. The diagram has an edge that contracts to give the diagram for e_4.*



We claim that iterating the process of passing from J_1 to J_5 results in lowering the number of chords that have endpoints on $(S^1)_i$. If c_2 is a chord that has only one endpoint on $(S^1)_i$, then there must be a BC e_5, with nonzero coefficient in γ, that has fewer than m chords on $(S^1)_i$. If c_2 is a chord that terminates at both ends on $(S^1)_i$, then the diagram for e_5 has one less internal chord on $(S^1)_i$ than the diagram for e_1 does. If we iterate this process, then eventually we must obtain the situation where the diagram for e_1 has no internal chords connecting to $(S^1)_i$, and then the resulting diagram for e_5 will have fewer than m chords connecting to $(S^1)_i$. Obviously, this process works in reverse to show that there are diagrams with as many as $I - n + 2$ chords connecting to $(S^1)_i$. □

4.3 Examples

In this section we will explore several examples of actuality tables and will see both how to complete them and how to use them. Let's start with an example in the case $n = 1$. This example is the simplest example that shows the complexity of filling out the lower rows of the table, but it is small enough to be able to be presented in full detail. Next, we will present several examples of actuality tables for link homotopy invariants of two component links, and derive formulas for evaluating these invariants on link homotopy classes of two component links. Finally, we give an example of a cycle in $E_1^{-I,I}$ that illustrates the last observation of section 4.2, as well as being able to distinguish the Whitehead link from the unlink.

The first example is the construction of the actuality table for the only order three invariant of knots. Here's the plan: first, we will simply present the actuality table, complete with all actuality indices entered, representative curves for all but the top row. We need to say that, because of considerations of space, *top row* means all of the boxes containing chord diagrams that correspond to configurations of complexity equal to the order of the invariant. The *second row* means all of the boxes containing chord diagrams of complexity one less than the order of the invariant and so forth. Recall that if K^I is a curve realizing a configuration of complexity I, then $\mathcal{I}_\gamma(K^I)$ only depends on the configuration, so we do not need to assign curves to boxes in the top row of the table. Next, we will show how we arrived at these indices and, finally, we will evaluate the invariant on the left and right trefoils.

Figure 4.14 *Let* $\gamma \in E_1^{-3,3}$. *Here is the extended actuality table for* γ. *In order to be complete, we have included boxes for all configurations, even if we know that they must have actuality index zero. For example, any inadmissable* [3]*-configuration must receive index zero by lemma 2.14.*

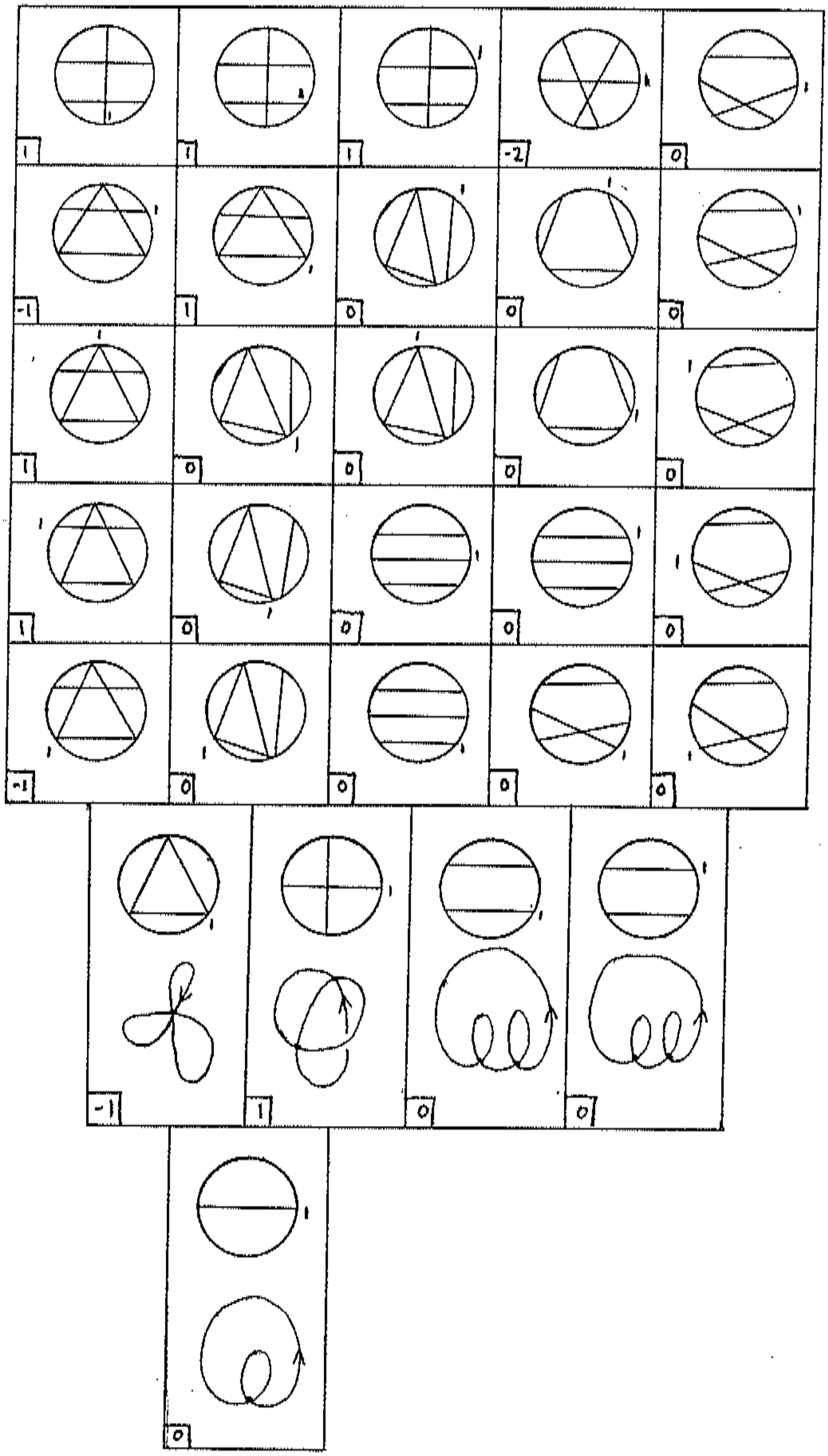


Figure 4.15 *Let's now use the diagrams to calculate h_I of all of the BC's of order three that do not automatically receive index zero:*

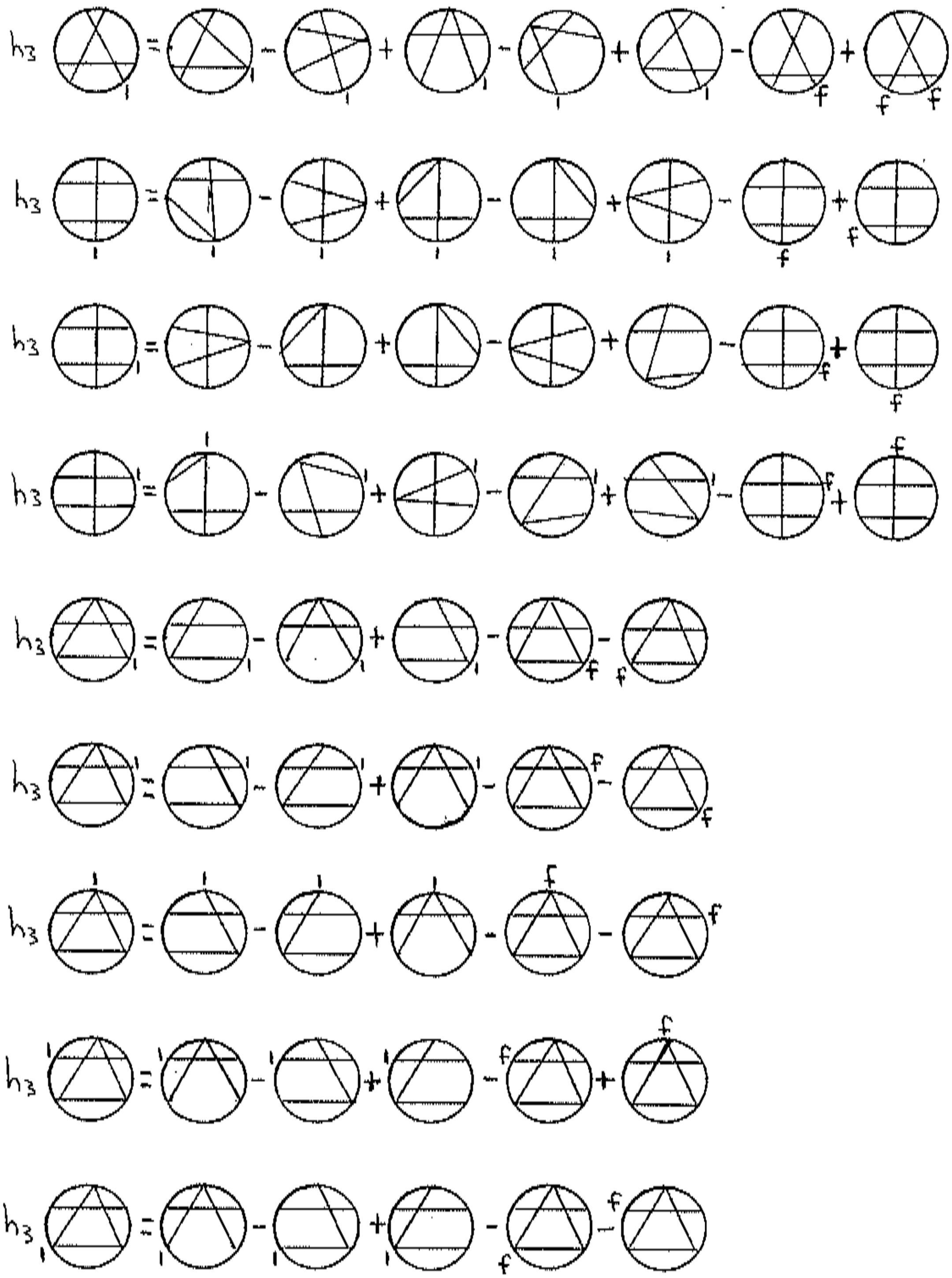
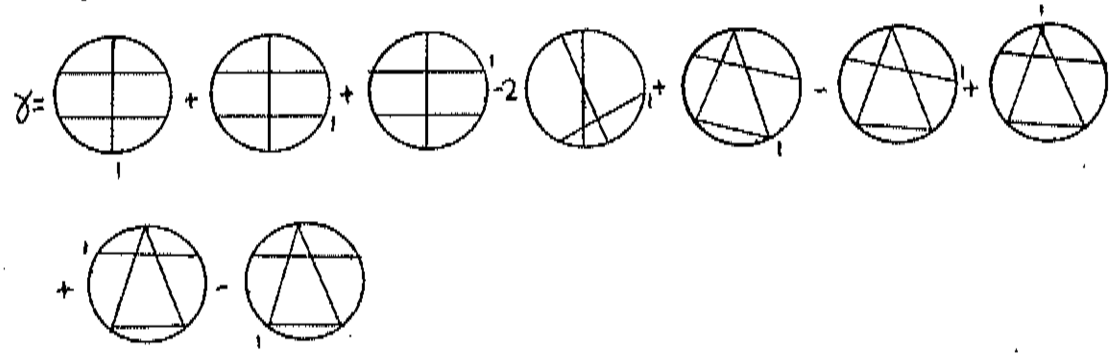


Figure 4.16 *We can see from the calculations above that the chain γ, defined below, is a cycle. In fact, $E_1^{-3,3}$ has only one generator. Recall that any cycle in $E_1^{-I,I}$ gives a candidate table. If it survives nontrivially to $E_\infty^{-I,I}$, then we get a full table.*



The coefficients, that precede the diagrams in the above expression, are entered as actuality indices in the appropriate boxes in the top row (in our case: the top five rows) of the extended actuality table. Zeroes are entered as indices in the box of any diagram that does not contribute to γ. Now comes the sticky part. Following our discussion in sections 2.6 and 3.4, we must fill in the next two rows of the table.

Figure 4.17 *Let J_1, J_2, J_3 and J_4 be the complexity two configurations and let e_1, e_2, e_3 and e_4 be the corresponding BC's, respectively, of the following four diagrams D_1, D_2, D_3, and D_4:*

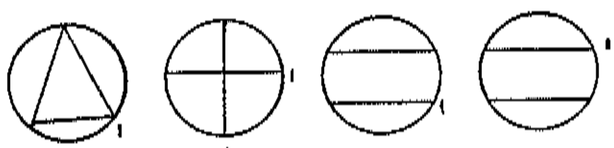


Figure 4.18 *Let L_1, L_2 and L_3 refer to the curves representing J_1, J_2 and both J_3 and J_4, respectively, in the table. L_1, L_2 and L_3 are shown below.*

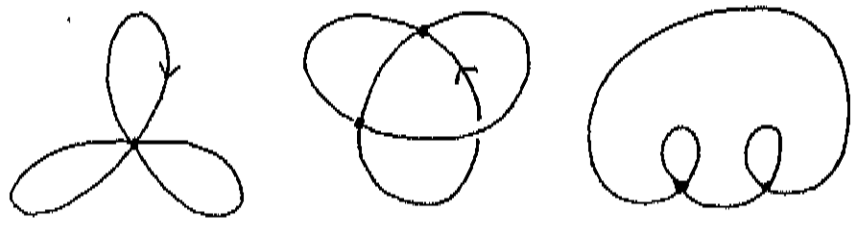
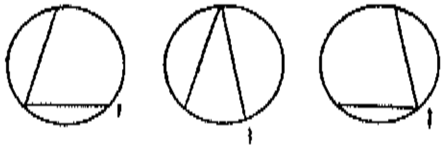
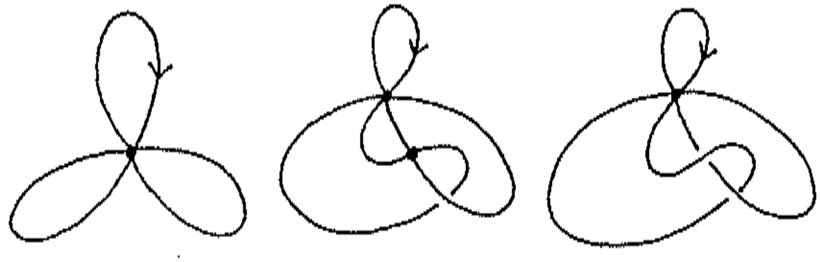


Figure 4.19 *Let J_{11}, J_{12} and J_{13} be the (only) three $\langle 2\rangle_A$-configurations that lie in the image under h_2 of J_1. Let e_{11}, e_{12} and e_{13} be their BC's. Their diagrams D_{11}, D_{12} and D_{13} are pictured below.*



Recall that a *main component* of e_i is the elementary component of the cell that contains the representative curve L_i. Similiarly, the main component of e_{1i} is defined to be the elementary component of e_{1i} containing the curve L_1. If we jump through Σ_3 to a neighboring elementary component, we obtain some other curve L_1'.

Example with Figure 4.20 *Shown below are* L_1*, the curve* $L \in \Sigma_3$ *through which we jump, and* L_1'*.*



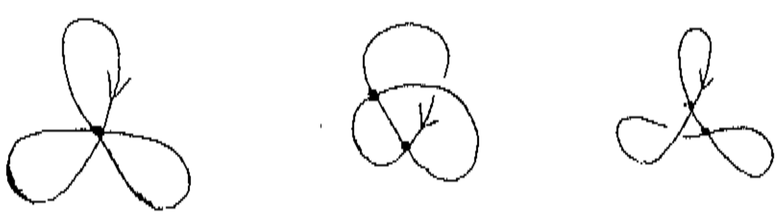
To fill in the complexity two row, we must calculate the coefficients with which e_{11}, e_{12} and e_{13} enter a certain chain ξ. This chain ξ is chosen to be *compatible* with $\partial_1\gamma$ and is chosen so that the main component of each cell enters it with coefficient zero (see sections 2.6 and 3.4).

The diagram D_{11} of e_{11} *spreads* or *resolves* into the diagrams D_2 and D_3. The cells e_2 and e_3 are the $h_2|_{X_{21}}$ preimage of e_{11} and, diagramatically, this process is realized by taking the point that is on D_{11} and is connected to two chords and spreading this point into two points, each connected to a single chord. There are two ways of doing this and D_2 and D_3 are the resulting diagrams.

Figure 4.21 D_{11}*,* D_2 *and* D_3 *are shown below.*

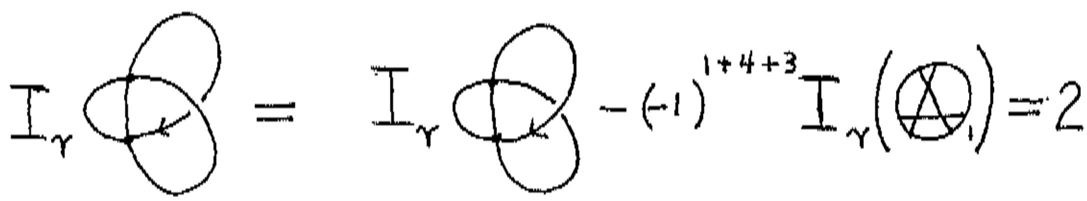
The resolution of D_{11} above corresponds to a resolution of the curve L_1 into new curves K_2 and K_3 realizing J_2 and J_3, respectively.

Figure 4.22 *In the illustrations of* L_1*,* K_2 *and* K_3 *below, we have tried to be consistent with our choice of placement of the orientation arrows to clarify the process.*



Our assumption that the main component of each cell enters ξ with coefficient zero is the same as saying that the index $\mathcal{I}_\gamma$ of the representative curves are zero. We can use the crossing change formulas to calculate all of the other indices. The curves K_3 and L_3 are equal up to isotopy, and we know that since the configuration J_3 is inadmissable it will receive actuality index zero. In any case $\mathcal{I}_\gamma(K_3) = 0$. Let's calculate $\mathcal{I}_\gamma(K_2)$.

Figure 4.23 *We calculate $\mathcal{I}_\gamma(K_2)$ using knot diagrams and the crossing change formulas.*



The incidence coefficient of e_{11} in both $h_2(e_2)$ and $h_2(e_3)$ is 1, so e_{11} enters $d_1\gamma$ with coefficient $1(0+2) = 2$. Now let's do the same for e_{12} and e_{13}. The diagram D_{12} for e_{12} resolves into the the diagrams D_2 and D_4, which correspond to the BC's e_2 and e_4 that make up $h_2^{-1}|_{X_{21}}(e_{12})$. The curve L_1 resolves into curves K_2' and K_4'. The resolution process on L_1 can be viewed as the resolution on the diagram e_1 (i.e. on D_1) restricted to the main component of e_1.

Figure 4.24 *D_{12}, D_2 and D_4:*

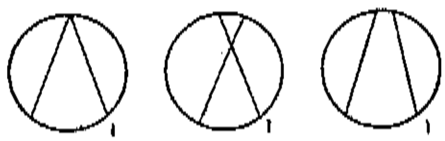
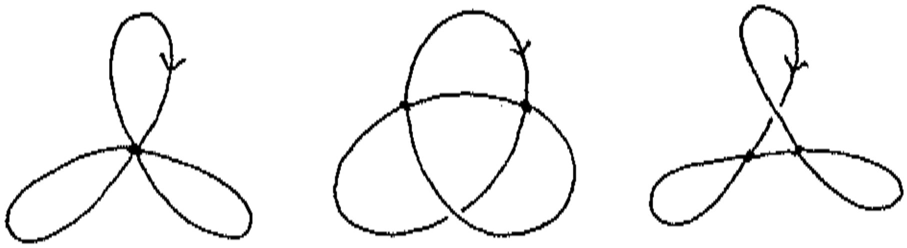


Figure 4.25 *L_1, K_2' and K_4':*



As before, K_4' is isotopic to L_3, and thus $\mathcal{I}_\gamma(K_4') = 0$. We also observe that K_2' and L_2 are isotopic, so we know that $\mathcal{I}_\gamma(K_2') = 2$. The incidence coefficient of e_{12} in $h_2(e_2)$ and $h_2(e_4)$ is -1. The required coefficient is therefore $-1(0+2) = -2$.

Finally, D_{13} resolves into D_2 and D_3, since $h_2^{-1}|_{X_{21}}(e_{13}) = \{e_2, e_3\}$. The corresponding resolution of curves spreads L_1 into curves K_2'' and K_3'' realizing J_2 and J_3, respectively.

Figure 4.26 *D_{13}, D_2 and D_3:*

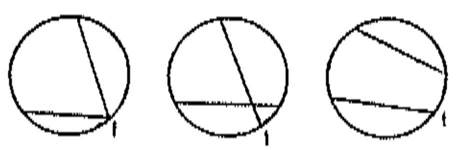
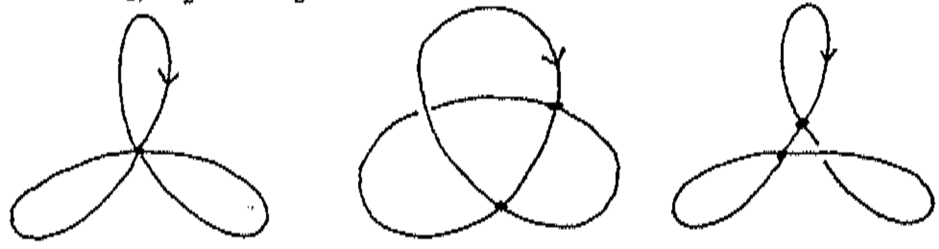


Figure 4.27 *L_1, K_2'' and K_3'':*



As above, K_3'' and L_3 are isotopic and therefore $\mathcal{I}_\gamma(K_3'') = 0$. We also have that K_2'' is isotopic to both K_2' and K_2; thus $\mathcal{I}_\gamma(K_2'') = 2$ and e_{13} enters ξ with coefficient $1(0+2) = 2$. The result is that $\tilde{d}_1\gamma = 2(e_{11} - e_{12} + e_{13})$. Now it remains to find a chain $\alpha \in X_2$ such that $h_2\alpha + \tilde{d}_1\gamma = 0$. Let $\alpha = e_2 - e_1$; then

$$h_2(e_2 - e_1) = -e_{11} + e_{12} - e_{13} - e_{11} + e_{12} - e_{13} = \tilde{d}_1\gamma.$$

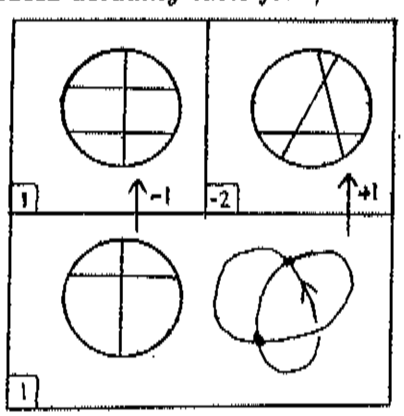
Notice that because there is only one BC in each of $\mathcal{B}_{J_1}$ and $\mathcal{B}_{J_2}$, there is only one FPC in each J-block and it cancels itself out in $h_2(e_1)$ and $h_2(e_2)$. We can now enter 1 and -1 into the extended actuality table as the indices for J_2 and J_1, respectively.

The configuration in the only box in the bottom row of the table is inadmissable, and so we can enter zero as its actuality index. We can now shift to the reduced actuality table, where there is only one box for each family of $[I]$-configurations, $1 \leq I \leq 3$. We will also delete any boxes that have actuality index zero. Recall that our skein relation, in the case of knots, takes the form:

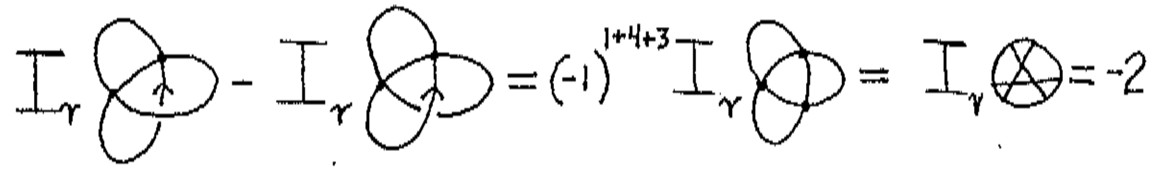
$$\mathcal{I}_\gamma(K_+^j) - \mathcal{I}_\gamma(K_-^j) = (-1)^{n(s)+n(t)+j+1}\mathcal{I}_\gamma(K_x^{j+1})$$

where K_+^j and K_-^j both respect some $[j]$-configuration J and vary by a jump across Σ_{j+1} at the curve $K_x^{j=1}$. Let $\bar{J}$ refer to the $[j+1]$-configuration that K_x^{j+1} respects. t and s are then the pair of points added to J to obtain $\bar{J}$ and $n(t)$ and $n(s)$ are the numbers of these points in $\bar{J}$. In this table we have indicated the value of the power of -1 that occurs on the right side of the skein equation. We have shown this coefficient as an arrow, and with it a number, between the two boxes of the table that are involved in the skein relation.

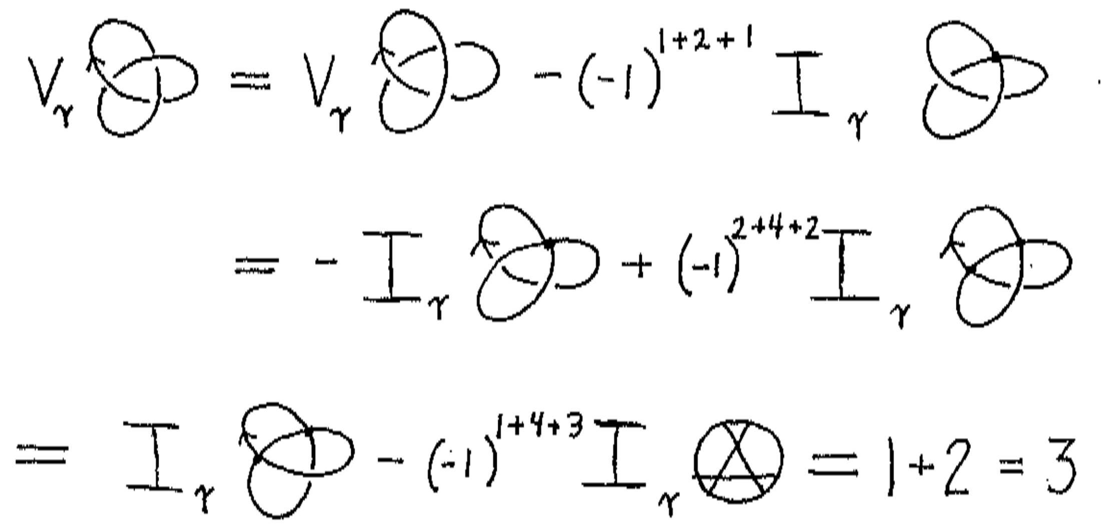
Figure 4.28 *Here is the reduced actuality table for γ:*

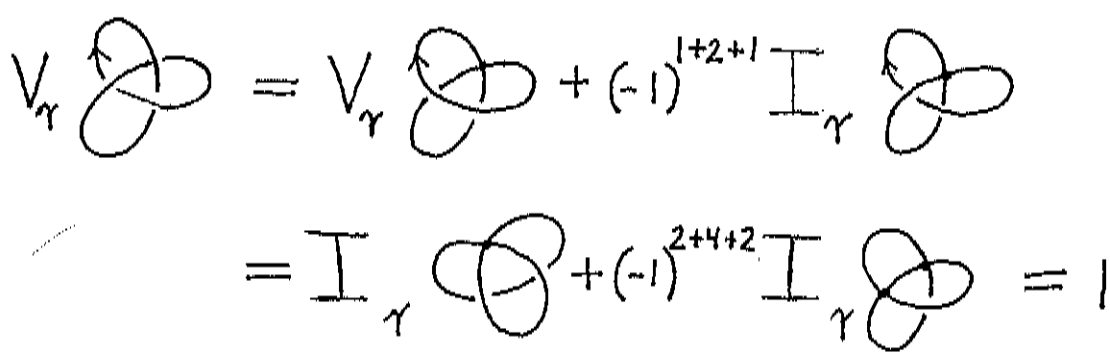


Example with Figure 4.29 *This is an example of curves* K_+^2 *and* K_-^2 *that respect the only admissable* [2]*-configuration* J*. The two curves vary by passage through a curve that respects the* [3]*-configuration* $\bar{J}$ *occupying the upper right hand box in the reduced table. By lemma 3.17, there is a choice of the power of* -1 *used in the skein equation, but this choice doesn't matter. The choice is a choice over where in* Σ_3 *to make the jump from* K_+^2 *to* K_-^2*. We have chosen here to use the same coefficient that we have used as the arrow in our table.*



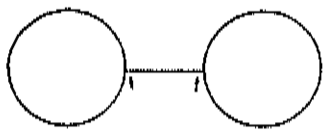
Example with Figure 4.30 *This is a paper on knot theory, so I am obliged to use our reduced table to evaluate the invariant* $\mathcal{V}_\gamma$ *on the left and right trefoils. Recall that if* K *is a knot that is isotopic to the unknot, then* $\mathcal{I}_\gamma(K) = 0$.





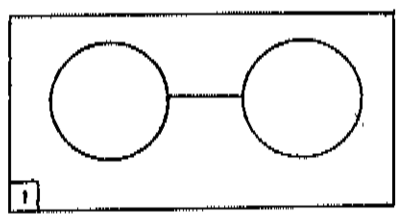
Our next examples are of Vassiliev link homotopy invariants of two component links. We will, therefore, only be considering $[I]$-configurations that have an empty A-configuration. The corresponding best cells are the generators of X_{I3}. It is easy to see that the single [1]-configuration whose BC is in X_{13} is itself a cycle, since it has no boundary. Let's call this configuration J_1, its BC e_1, and the corresponding chord diagram D_1.

Figure 4.31 *The diagram D_1:*

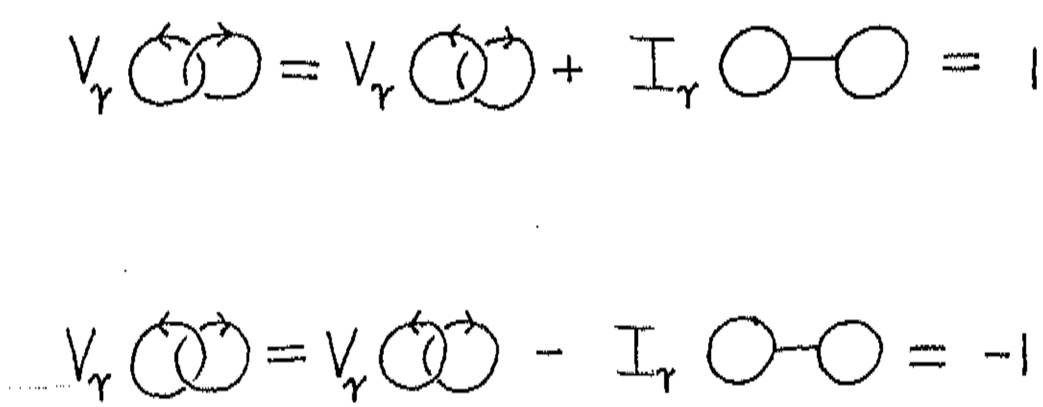


It is easy to see that the corresponding invariant, where e_1 is taken as the cycle, gives the linking number.

Figure 4.32 *This is the corresponding reduced actuality table:*

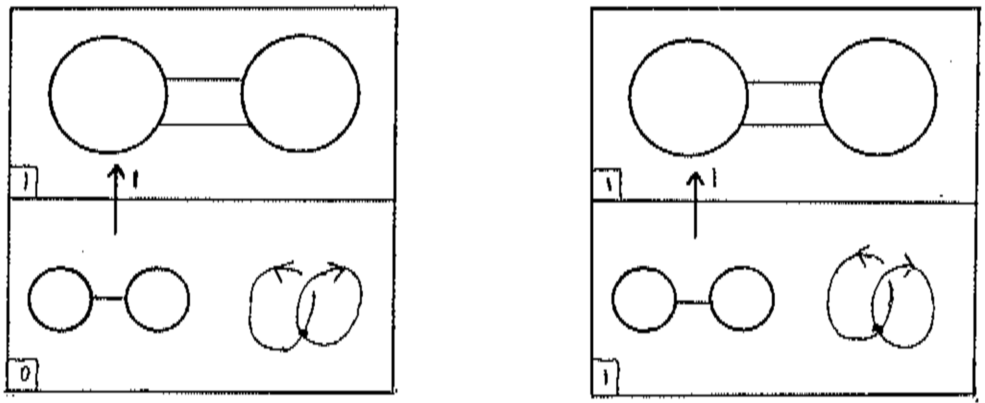


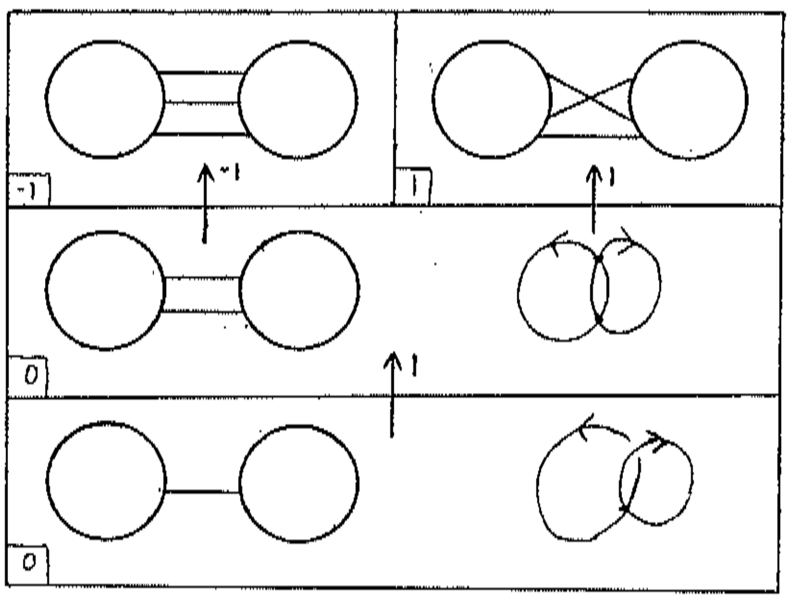
Example with Figure 4.33 *Some sample calculations:*



Let's consider three reduced actuality tables, all of invariants of link homotopy. The first two tables T_1 and T_2 both correspond to the same cycle $\gamma \in E_1^{-2,2}$, but differ by the choice of compatible chain used to fill out the bottom row. The last table T_3 is from a cycle in $E_\infty^{-3,3}$.

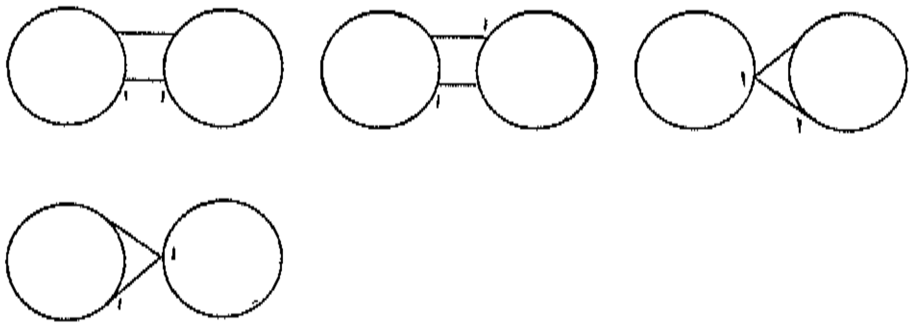
Figure 4.34 *T_1 and T_2 are pictured side by side; T_3 is below:*





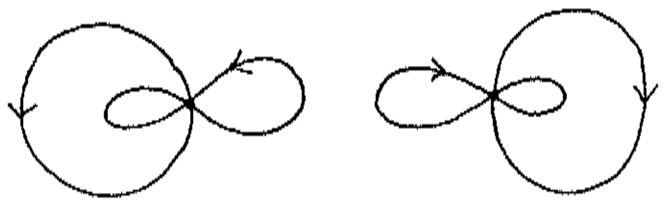
Let's examine T_1 and T_2 first. There are no $\langle 1\rangle_C$-configurations, thus $\tilde{d}_1\gamma = 0$. But, for both $\alpha = e_1$ and $\alpha = 0$, we have $\tilde{d}_1\gamma + h_1\alpha = 0$ and so either choice of α gives an acceptable choice of actuality table. Actually, we have the same choice in T_3 in both of the bottom two rows. Let J_{21} and J_{22} be the two related [2]-configurations with empty A-configurations and let e_{21} and e_{22} be their BC's. Let e_{23} and e_{24} be the BC's whose $h_2|_{X_{23}}$ preimage is $\{e_{21}, e_{22}\}$. Let J_{23} and J_{24} be the $\langle 2\rangle_C$-configurations corresponding to e_{23} and e_{24}.

Figure 4.35 *The chord diagrams for* e_{21}, e_{22}, e_{23} *and* e_{24}:



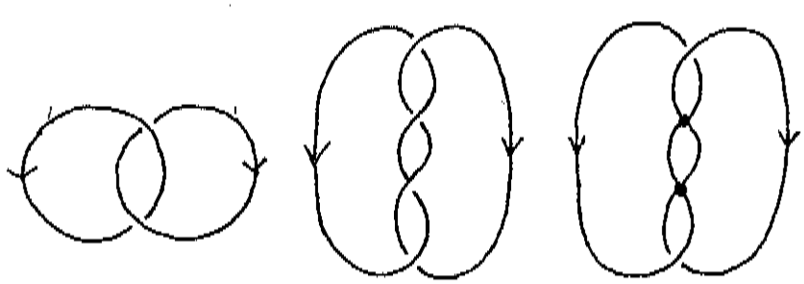
Let ε be the cycle in $E_1^{-3,3}$ that determines (the top row of) T_3. We can choose representative curves for J_{23} and J_{24} so that $\tilde{d}_1\varepsilon = 0$ and the same choices will result.

Figure 4.36 *These are representative curves for* J_{23} *and* J_{24} *that will make* $\tilde{d}_1\varepsilon = 0$.



For each positive integer k, let L_k refer to the two component link (diagram) that has $2k$ positively signed crossings between its two components and no other crossings. Let L_{-k} be the two component link that has $2k$ negative crossings between its two strands and no other knotting or linking. Considered up to link homotopy, the set $\{\ldots, L_{-1}, L_0, L_1, L_2, \ldots\}$ is the entire collection of links of two components. For each integer t, $1 \leq t \leq 2k$, let L_k^t refer to the linked graph of an immersion with $2k - t$ positive crossings and t self-intersections. Define L_{-k}^t similiarly.

Example with Figure 4.37 *Pictured are* L_{-1}, L_2 *and* L_2^2:



Let's use T_1 and evaluate the associated invariant $\mathcal{V}$ on L_1, L_2 and L_3:

$$\begin{aligned}\mathcal{V}(L_1) &= \mathcal{V}(L_0) + \mathcal{I}_\gamma(L_1^1)\\ &= 0\end{aligned}$$

$$\begin{aligned}\mathcal{V}(L_2) &= \mathcal{V}(L_1) + \mathcal{I}_\gamma(L_2^1)\\ &= \mathcal{I}_\gamma(L_1^1) + \mathcal{I}_\gamma(L_1^2)\\ &= 1\end{aligned}$$

$$\begin{aligned}\mathcal{V}(L_3) &= \mathcal{V}(L_2) + \mathcal{I}_\gamma(L_3^1)\\ &= 1 + \mathcal{I}_\gamma(L_2^1) + \mathcal{I}_\gamma(L_2^2)\\ &= 3.\end{aligned}$$

Now the pattern becomes clear:

$$\begin{aligned}\mathcal{V}(L_k) &= \mathcal{V}(L_{k-1}) + \mathcal{I}_\gamma(L_k^1)\\ &= \mathcal{V}(L_{k-1}) + \mathcal{I}_\gamma(L_{k-1}^1).\end{aligned}$$

It is easy to see that

$$\mathcal{I}_\gamma(L_k^1) = \mathcal{I}_\gamma(L_{k-1}^1) + 1 = \mathcal{I}_\gamma(L_{k-2}^1) + 2 = \cdots = \mathcal{I}_\gamma(L_1^1) + k - 1 = k - 1.$$

It follows that

$$\begin{aligned}\mathcal{V}(L_k) &= \mathcal{V}(l_{k-1}) + (k-1)\\ &= \mathcal{V}(L_{k-2}) + (k-1) + (k-2)\\ \vdots\ &\ \vdots\ \ \vdots\\ &= \mathcal{V}(L_1) + 1 + 2 + 3 + \cdots + (k-1)\\ &= \frac{k(k-1)}{2}.\end{aligned}$$

We can derive a formula for L_{-k} similiarly:

$$\begin{aligned}\mathcal{V}(L_{-1}) &= \mathcal{V}(L_0) - \mathcal{I}_\gamma(L_{-1}^1)\\ &= 0 - \mathcal{I}_\gamma(L_1^1) + \mathcal{I}_\gamma(L_{-1}^2)\\ &= 1\end{aligned}$$

$$\begin{aligned}\mathcal{V}(L_{-2}) &= \mathcal{V}(L_{-1}) - \mathcal{I}_\gamma(L_{-2}^1)\\ &= 1 - \mathcal{I}_\gamma(L_{-1}^1) + \mathcal{I}_\gamma(L_{-2}^2\\ &= 3\end{aligned}$$

$$\begin{aligned}\mathcal{V}(L_{-3}) &= \mathcal{V}(L_{-2}) - \mathcal{I}_\gamma(L_{-3}^1)\\ &= 3 - \mathcal{I}_\gamma(L_{-2}^1) + \mathcal{I}_\gamma(L_{-3}^2)\\ &= 6.\end{aligned}$$

The recursive equation is

$$\mathcal{V}(L_{-k}) = \mathcal{V}(L_{-k+1}) - \mathcal{I}_\gamma(L_{-k}^1)$$

where $\mathcal{I}_\gamma(L^1_{-k})$ is given by

$$\begin{aligned}\mathcal{I}_\gamma(L^1_{-k}) &= \mathcal{I}_\gamma(L^1_{-k+1}) - \mathcal{I}_\gamma(L^2_{-k}) \\ &= \mathcal{I}_\gamma(L^1_{-k+1}) - 1 \\ \vdots\ &\ \vdots\ \ \vdots \\ &= \mathcal{I}_\gamma(L^1_{-1}) - (k-1) \\ &= \mathcal{I}_\gamma(L^1_1) - k \\ &= -k.\end{aligned}$$

We see that the formulas for $\mathcal{V}(L_k)$ and $\mathcal{V}(L_{-k})$ are not symmetric, since the above work yields:

$$\begin{aligned}\mathcal{V}(L_{-k}) &= \mathcal{V}(L_{-k+1}) + k \\ \vdots\ &\ \vdots\ \ \vdots \\ &= \frac{k(k+1)}{2}.\end{aligned}$$

We observe that the invariant evaluated on a link gives a second degree polynomial of the linking number of the two component link; one for links that have positive linking number and a different one for links with negative linking number. It is easy to see, and harder to check, that the same will be true for arbitrarily high order link homotopy invariants of two component links. This is to be expected since linking number is a complete link homotopy invariant of two component links, but it is interesting that it is an example of a class of Vassiliev invariants in which the lowest order invariant is more sensitive than all of the higher order ones.

We conclude section 4.3 with an example, for two component links, of a reduced actuality table for a cycle $\gamma \in E_\infty^{-3,3}$ that involves BC's whose configurations include [3], $\langle 3\rangle_A$ and $\langle 3\rangle_C$ configurations, all but one family of which have nonempty A-configurations. We can observe the property discussed at the end of section 4.2; for each m between 1 and 3, and i equal to 1 or 2, there is a BC, with nonzero coefficient in γ, whose chord diagram has m chords that have at least one endpoint on $(S^1)_i$. This example will be used to show that our invariants distinguish the Whitehead link from the unlink.

Figure 4.38 *Here are all of the boxes of the reduced actuality table that receive nonzero indices in this cycle.*

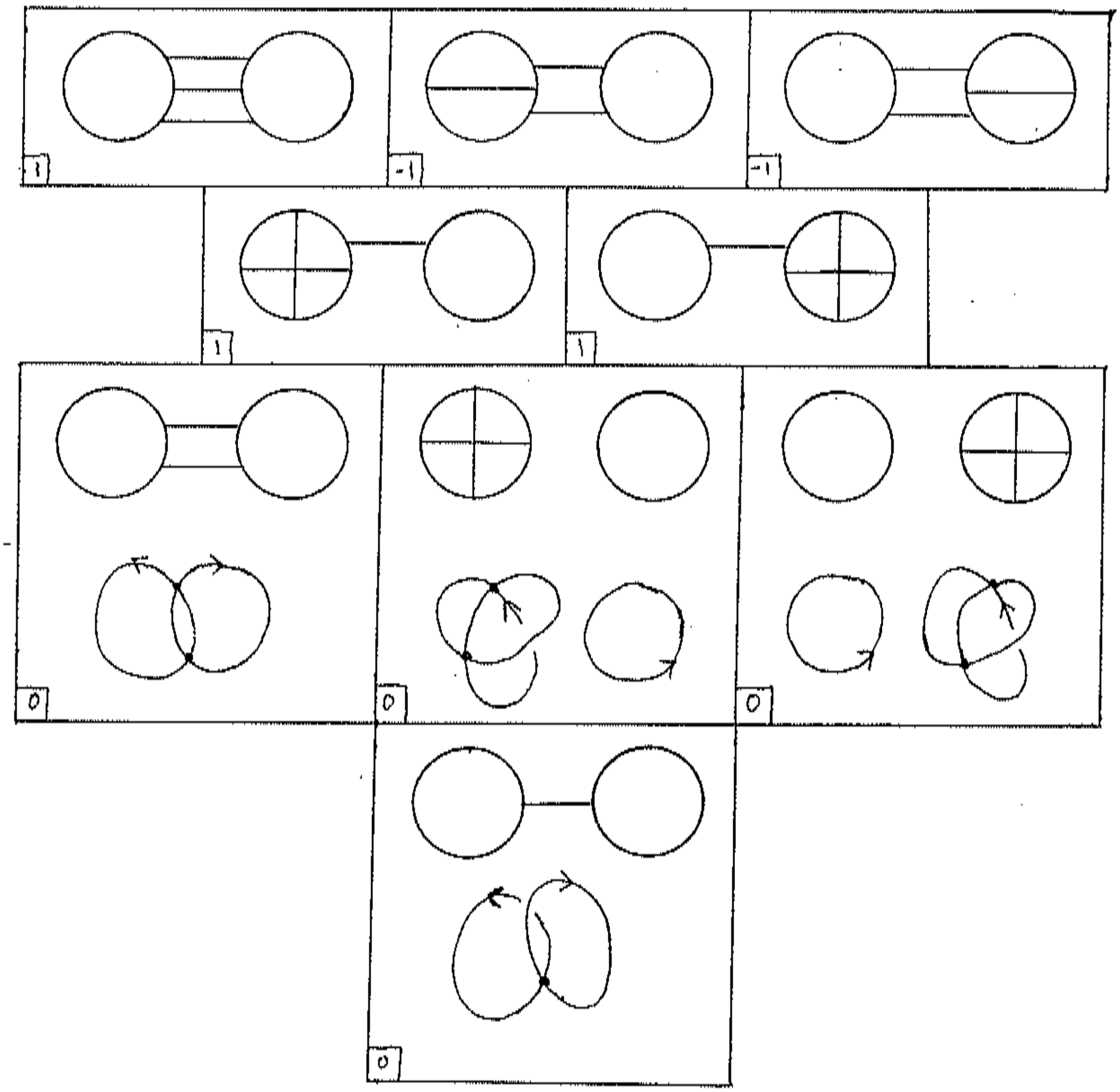
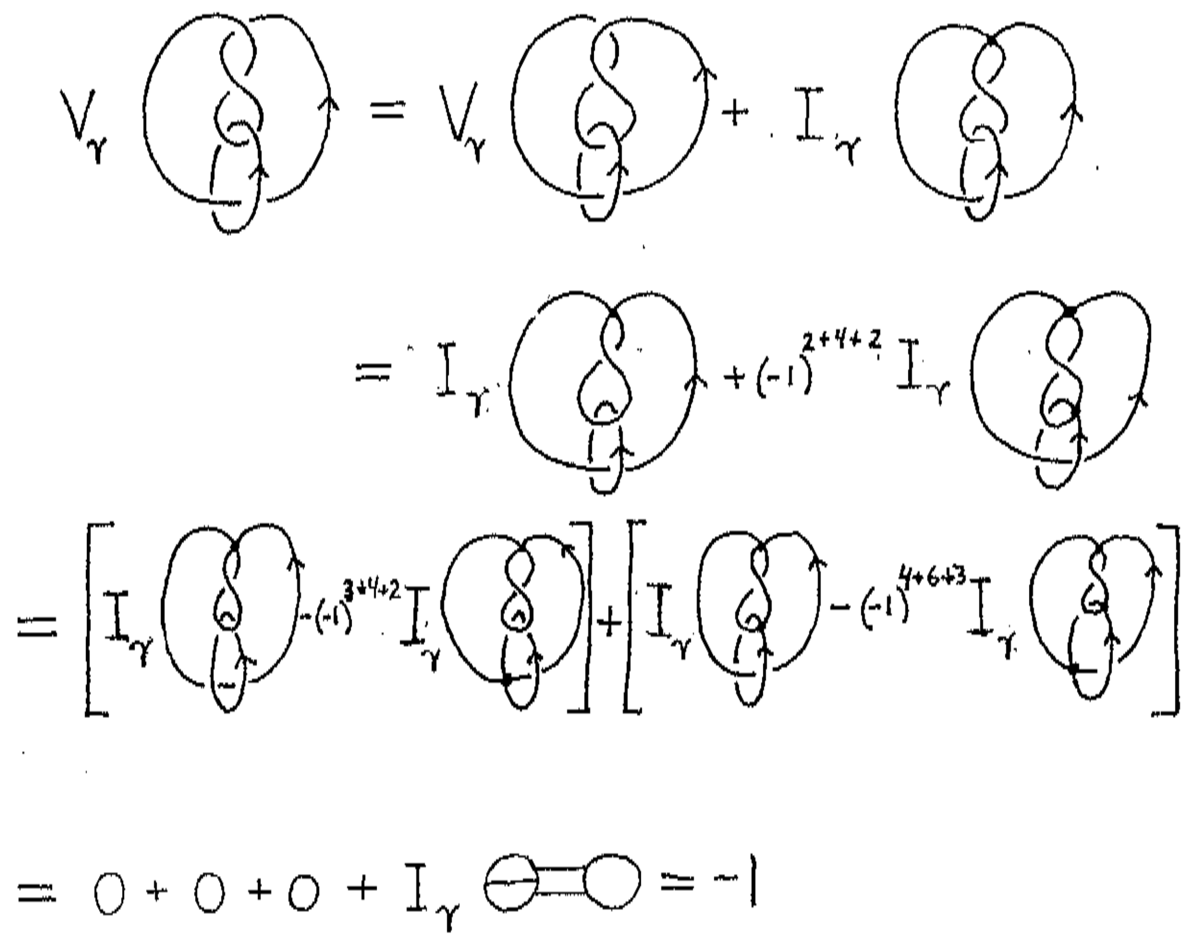


Figure 4.39 *Evaluation of the invariant on the Whitehead link.*



We should point out that to find an invariant that distinguished the Whitehead link from the unlink, we first worked out the calculation for the Whitehead link without actually knowing the value of $(\mathcal{V}, \mathcal{I})$ on the needed links and nice immersions. We then worked to find an invariant that used the chord diagrams that we knew we needed, so in a sense we worked backwards.

4.4 Vassiliev Invariants and Polynomial Invariants of Links

In this section we investigate the remarkable relationship between Vassiliev link invariants and some of the polynomial link invariants. This relationship was first explored in the case of knots by Birman and Lin in [BL] and is also discussed by Bar-Natan in [B]. Given a very general one variable form of the HOMFLY polynomial on a knot, Birman-Lin's result is that when we express the polynomial as a power series obtained by first replacing its variable t by the Taylor series expansion of e^y, the resulting series is seen to have coefficients that are knot invariants of finite type.

Recall the axioms V1, V2, V3 and V4 from §3.4. By construction, all of our invariants are finite type link invariants. In [BL] it is proved that all finite type knot invariants (with a slightly more pallatable axiom structure) come from invariants obtainable by Vassiliev's methods. The proof, which constitutes a large part of [BL], is essentially a (quite complex) tautology. This leads me to conjecture strongly that the invariants constructed in this paper also give all invariants satisfying these axioms. This belief is shared by Stanford, who has independently constructed finite type invariants for links using different methods in [S]. I believe that the proof of the equivalence of the two approaches should be straight-forward.

In its usual form, the HOMFLY polynomial, in [HOMFLY], is a two variable polynomial invariant of links. Jones showed in [J1] that there is a sequence of polynomials $H_{m,t}$ which determine both the two variable HOMFLY polynomial and the one variable Jones polynomial from [J]. Let L_+, L_- and L_0 be a triple of links for which some chosen set of planar link diagrams differ from each other only at the location of a single positively signed crossing in L_+.

Figure 4.40 *Illustrations of L_+, L_- and L_0 at the special location:*

Let U denote the unknot and let U_n denote the unlink of n components. The general form, where t is a variable and m is an integer, of this polynomial $H_{m,t}$ is completely determined by the following two axioms:

H1. $t^{(m+1)/2}H_{m,t}(L_+) - t^{-(m+1)/2}H_{m,t}(L_-) = (t^{1/2} - t^{-1/2})H_{m,t}(L_0)$,

H2. $H_{m,t}(U) = 1$

Axioms H1 and H2 imply a third axiom H3 which gives the value of $H_{m,t}$ on U_n. We list H3 because we will need it to prove the next theorem.

H3. $H_{m,t}(U_n) = \left(\frac{t^{(m+1)/2} - t^{-(m+1)/2}}{t^{1/2} - t^{-1/2}}\right)^{n-1}$.

The following theorem generalizes theorem 4.1 in [BL] to our setting:

Theorem 4.41 *Let L be an n-component link and let $H_{m,t}(L)$ be its HOMFLY polynomial. Let $W_{m,y}(L)$ be obtained by replacing t by e^y. Suppose that*

$$W_{m,y}(L) = \sum_{i=0}^{\infty} w_{m,i}(L) y^i$$

is the power series expansion of $W_{m,y}$ that results from replacing e^y by its Taylor series about $y = 0$. Let $c_{n,i}$ be the i^{th} coefficient of the power series expansion of

$$\left(\frac{e^{y(m+1)/2} - e^{-y(m+1)/2}}{e^{y/2} - e^{-y/2}}\right)^{n-1}$$

where the series is obtained by replacing e^y by its Taylor series about $y = 0$. Then

$$u_{m,i} = w_{m,i}(L) - c_i$$

is an n-component link invariant of finite type of order i.

Proof: This proof follows that of Birman-Lin closely, proving that each $u_{m,i}$ satisfies the axioms we presented for invariants of finite type. The first step is to define $H_{m,t}(L^j)$ for any immersion L^j of n circles that has j transverse double points, i.e. realizes a $[j]$-configuration. We can do this recursively. For any immersion L^1_x, let L_+ and L_- be the links obtained by the two ways of perturbing the double point into a nearby passing of strands.

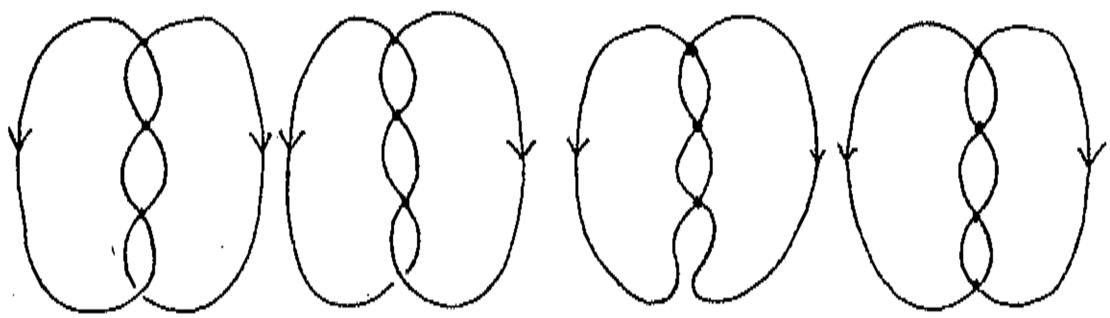
Example with Figure 4.42 *One such triple L_+, L_- and L^1_x:*

Now define $H_{m,t}(L^1_x)$ by

H4A. $H_{m,t}(L_+) - H_{m,t}(L_-) = H_{m,t}(L^1_x)$.

Let L^j_+, L^j_-, L^j_0 and L^{j+1}_x be a set of four immersions that vary as in the example below. In the same way as the crossing of L^j_+ is called a positive crossing, the lack of a crossing in that location in L^j_0 is called a *smoothing*.

Example with Figure 4.43 *One set of L^3_+, L^3_-, L^3_0 and L^4_x:*



We can recursively define $H(L_x^{j+1})$ by

H4B. $H_{m,t}(L_+^j) - H_{m,t}(L_-^j) = \alpha H_{m,t}(L_x^{j+1})$,

where $\alpha = (-1)^{n(s)+n(t)+j+1}$ for the usual meanings of s, t, $n(s)$ and $n(t)$. This equation is simply the extension of H4A to immersions with more double points. The axiom H1 gives two more equivalent skein relations, which are its generalizations to linked graphs:

H5A. $t^{(m+1)/2}H_{m,t}(L_+^{j-1}) - t^{-(m+1)/2}H_{m,t}(L_-^{j-1}) = (t^{1/2} - t^{-1/2})H_{m,t}(L_0^j)$,

or, equivalently, solving for $H_{m,t}(L_+^{j-1})$:

H5B. $H_{m,t}(L_+^{j-1}) = t^{-(m+1)}H_{m,t}(L_-^{j-1}) + (t^{-m/2} - t^{-(m+2)/2})H_{m,t}(L_0^j)$.

Combine H4B and H5B to obtain

$$\begin{aligned}\alpha H_{m,t}(L_x^j) + H_{m,t}(L_+^{j-1}) &= H_{m,t}(L_+^{j-1}) - H_{m,t}(L_-^{j-1}) + t^{-(m+1)}H_{m,t}(L_-^{j-1}) \\ &\quad + (t^{-m/2} - t^{-(m+2)/2})H_{m,t}(L_0^{j-1}),\end{aligned}$$

which gives

$$H_{m,t}(L_x^j) = \alpha\left((t^{-(m+1)} - 1)H_{m,t}(L_-^{j-1}) + (t^{-m/2} - t^{-(m+2)/2})H_{m,t}(L_0^{j-1})\right).$$

We have expressed the value of the polynomial on an immersion L^j by a sum of polynomials of other immersions with fewer crossings and perhaps more components. Number the double points of L^j by $1, 2, \ldots, j$. Each of these crossings can be split, or *resolved*, into a negatively signed crossing and a smoothing by the above equation. Now we can express the HOMFLY polynomial as a sum of HOMFLY polynomials of 2^j links as follows:

1. Let $\mathcal{A}$ be the collection of sequences of j letters that have either an M_i or an S_i in the i^{th} entry, $1 \le i \le j$.

2. Write L^j as $L(x_1, x_2, \ldots, x_j)$, where each x_i stands for a self crossing. If we resolve the first crossing, let $L(S_1, x_2, \ldots, x_j)$ refer to the immersion with $j-1$ self intersections and a smoothing at the location of the first self intersection of $L(x_1, \ldots, x_j)$. Let $L(M_1, x_2, \ldots, x_j)$ refer to the immersion with $j-1$ self intersections and a negatively signed crossing at the location of the first self intersection of $L(x_1, \ldots, x_j)$.

3. Now resolve the second crossing x_2 in both $L(M_1, x_2, \ldots)$ and $L(S_1, x_2, \ldots)$ in the second entry. Continue this process to obtain a collection of links (with no self intersections)
$$\{L(a) \mid a \in \mathcal{A}\}.$$

4. The polynomial $H_{m,t}(L(x_1, \ldots, x_j))$ can be written
$$H_{m,t}(L(x_1, \ldots, x_j)) = \alpha(\textstyle\sum_{a\in\mathcal{A}}(t^{-(m+1)} - 1)^{\mathcal{M}_a}(t^{-m/2} - t^{-(m=2)/2})^{S_a}H_{m,t}(L(a)),$$
where $\mathcal{M}_a$ is the total number of M's, regardless of subscripts, in a and $\mathcal{S}_a$ is the total number of S's in a.

We will rewrite $H_{m,t}(L^j)$, replacing the variable t by e^y. Let $W_{m,y}(L^j)$ be this new expression; then

$$W_{m,y}(L^j) = \alpha(\sum_{a\in\mathcal{A}}(e^{-y(m+1)/2} - 1)^{\mathcal{M}_a}(e^{-ym/2} - e^{-y(m+2)/2})^{\mathcal{S}_a}\, W_{m,y}(L(a)).$$

Jones showed in [J] that if we set $t = 1$, then $H_{m,1}(L(a)) = (m+1)^{\mathcal{N}-1}$, where $\mathcal{N}$ is the number of components of the link $L(a)$. Setting $t = 1$ is the same as setting $y = 0$, which gives the constant term of $W_{m,y}(L(a))$ when we expand $W_{m,y}(L(a))$ as a power series. The result is that every summand $W_{m,y}(L(a))$ in the expression for $W_{m,y}(L^j)$ has a nonzero constant term.

Replace e^y by its Taylor series expansion about $y = 0$. When we expand the term

$$(e^{-y(m=1)/2} - 1)^{\mathcal{M}_a},$$

the first nonzero term in the series is $(-y(m+1))^{\mathcal{M}_a}$, since

$$e^{-y(m+1)/2} = (1 - y(m+1) + \frac{y^2(m+1)^2}{2} - \cdots) - 1.$$

Similiarly, the first nonzero term of the series of $(e^{-ym/2} - e^{-y(m+2)/2})^{\mathcal{S}_a}$ is $y^{\mathcal{S}_a}$. We know that $\mathcal{M}_a + \mathcal{S}_a = j$, so the series expansion of $W_{m,y}(L^j)$ is divisible by y^j. Therefore the series expansion has the form

$$W_{m,y}(L^j) = \sum_{i=0}^{\infty} w_{m,i}(L^j)y^i = \sum_{i\geq j} w_{m,i}(L^j)y^i,$$

since $w_{m,i}(L^j) = 0$ if $i < j$. We see now that $w_{m,i}$ satisfies V3, and it satisfies V1 simply by using term by term series addition and H4A and H4B. We know by H3 that $w_{m,i}(U_n) = c_{n,i}$, where U_n is the unlink of n components. Recall that for evaluations on n component links we defined $u_{m,i} = w_{m,i} - c_{n,i}$. Since each $c_{n,i}$ is a constant, we see that $u_{m,i}$ satisfies both V1 and V3 and by construction $u_{m,i}(U_n) = 0$, which is V2. The axiom V4 falls out of the definition of H4A and H4B. To see that we can get a legitimate reduced actuality table from each of the $u_{i,m}$, simply evaluate some $u_{m,I}$ on one immersion L^I realizing each $[I]$-configuration, with the provision that if the $[I]$-configuration is inadmissable, then L^I is chosen to be a good model (see section 2.5). The evaluations give numbers which can be entered into the appropriate boxes in the top row of a reduced actuality table. It can be shown that these indices satisfy the so called 4T relations; see [B]. For $i \in \{1,2,3\}$, and $j \in \{1,2\}$, the 4T relations are relations between the indices of diagrams D_{ij}, pictured below, from $[I]$-configurations and a certain diagram D, also pictured below, from some J that is either an $\langle I\rangle_A$-configuration or an $\langle I\rangle_C$-configuration. Using our notation, the relations are

$$\begin{aligned}
\mathcal{I}_I(D) &= (-1)^{I+1+n(t_1)}(\mathcal{I}_I(D_{11}) + \mathcal{I}_I(D_{12}))\\
&= (-1)^{I+n(t_2)}(\mathcal{I}_I(D_{21}) + \mathcal{I}_I(D_{22}))\\
&= (-1)^{I+1+n(t_3)}(\mathcal{I}_I(D_{31}) + \mathcal{I}_I(D_{32}))
\end{aligned}$$

where $\{t_1, t_2, t_3\}$ is the grouping of three points in J and $n(t_k)$ is the number of t_k in the standard ordering of the points of J.

Figure 4.44 *These are the relevant segments of the diagrams D_{11}, D_{12}, D_{21}, D_{22}, D_{31}, D_{32} and D. Let's assume that all seven pictured diagrams are identical outside the locations we are showing. Curved segments indicate parts of circles and straight segments indicate chords.*

The fact that our diagrams satisfy the 4T relations is immediate from our construction. The relations are just another way of saying that, in a cycle $\gamma \in E_1^{-I,I}$, certain cells in the boundary of BC's from the diagrams D_{ij} must cancel the contribution of the BC whose diagram is D. Of course, this holds even if the BC corresponding to D enters γ with coefficient zero (which may happen if the BC's pictured are all from X_{I3}). It is also immediate that our construction provides all possible diagramatic solutions to these relations. We can conclude that the tables of the invariants of finite type that come from the coefficients of the expanded HOMFLY polynomials have a top row that is obtainable using our methods, since the top row of the table is given by some diagramatic solution to the 4T relations.

Now, for each $j < I$, pick legal representative curves L^j for each $[j]$-configuration, and evaluate $u_{m,I}(L^j)$. The numbers obtained by these evaluations are entered into the appropriate boxes in the j^{th} row of the reduced actuality table for $u_{m,I}$. This table does give a link invariant, since it is constructed to give one of the coefficients of the HOMFLY polynomial. The proof that this table is obtainable by our methods could be completed by proving that the the lower levels of the table of any finite type invariant correspond to the lower levels of a table obtained by our method in Chapter three, which would essentially be generalization to links of lemma 3.6 and theorem 3.7 in [BL]. As I mentioned, I conjecture that this can be done.

5 The Proofs of Lemma 1.6, Theorem 3.4, and Theorem 3.6

5.1 Introduction

We will now present the proofs of the above mentioned lemma and theorems. I believe that these proofs do not provide any significant geometric insight into our process and that is the reason to delay these proofs until now. The proofs of lemma 1.6 and theorem 3.6 are both straight forward applications using linear algebra. The proof of theorem 3.4 is a very straight forward generalization of theorem 3.1.2 in [V1].

5.2 The Proof of Lemma 1.6

Proof: Our choice of perturbation of Γ^d insures that the zero map is not contained in $\Gamma^d \subset \Gamma^{2d}$, as the zero map is too singular for our purposes. Therefore Γ^d, which is $3n(2d+1)$ dimensional, can be viewed as lying in some $3N+1$ dimensional vector subspace of $\Gamma^{2d} \cong \mathrm{R}^{6N}$, where $N = n(2d+1)$. Let J be an (A, B, C)-configuration of complexity I. Recall that $p \in M(\Gamma^d, J)$ if and only if p is a solution to the matrix equation $\Phi_J(\vec{x})$, where $\Phi_J \in \hom_{\mathrm{R}}(\mathrm{R}^{3N+1}, \mathrm{R}^{3I})$.

As an illustration, let s and t be the first and second points in the first grouping of the A-configuration of J. The last $3N$ entries of the first row of such a matrix Φ_J would look like:

$$1 \;\; \cos t - \cos s \;\; \sin t - \sin s \;\; \cos 2t - \cos 2s \;\; \cdots \;\; \cos dt - \cos ds \;\; 0 \;\; 0 \;\; 0 \;\; \cdots \;\; 0,$$

where, without loss of generality, we assume that Γ^d takes up the last $3N$ coordinates of a $3N+1$ dimensional subspace of Γ^{2d}. Let Φ be the map which takes any J', where J' is equivalent to J, to the associated $\Phi_{J'}$, i.e.

$$\Phi : \{J' \mid J' \sim J\} \to \hom_{\mathrm{R}}(\mathrm{R}^{3N+1}, \mathrm{R}^{3I}).$$

Let

$$L_r = \{M \in \hom_{\mathrm{R}}(\mathrm{R}^{3N+1}, \mathrm{R}^{3I}) \mid M \text{ has rank } r \leq m = \min\{3N+1, 3I\}\ \}.$$

We know, by the Products of Coranks theorem in [AVG], that L_r is a smooth submanifold of $\hom_{\mathrm{R}}(\mathrm{R}^{3N+1}, \mathrm{R}^{3I})$ of codimension $(3N+1-r)(3I-r)$ and is empty if this product is negative.

It is a consequence of the Weak Transversality theorem [AVG] that we can perturb Φ an arbitrarily small amount in order to insure that Φ is transverse to each of the L_r, for $r < m$. Let's assume from now on that Φ is transverse to $\cup_{r<m} L_r$. The result is that $\Phi^{-1}(L_{m-t})$ has codimension $(3N+1-m+t)(3I-m+t)$ for $t \geq 1$.

To prove part A of lemma 1.6 we need a little lemma:

Lemma 5.1 *If $t \geq 1$, then $\Phi^{-1}(L_{m-t})$ has measure zero in $\{J' \mid J' \sim J\}$.*

Proof of the lemma: We can cover $\Phi^{-1}(L_{m-t})$ by a countable collection $\{B_k\}_{k=1}^{\infty}$ of open balls that have all have rational radius. The intersection $B_k \cap \Phi^{-1}(L_{m-t})$ has measure zero inside B_k, thus $\Phi^{-1}(L_{m-t})$ has measure zero. □

The consequence is that for almost all $J' \sim J$, the map $\Phi_{J'}$ has maximal rank m. Equivalently, almost all $J' \sim J$ have the property that $M(\Gamma^d, J')$ has codimension exactly $3I$ inside Γ^d provided that $I \leq N$.

If $I > N$ and $\Phi_{J'}$ has maximal rank $3N$, then $\Phi_{J'}$ is injective and has trivial kernel. Since Γ^d is affine and doesn't contain zero, the set $M(\Gamma^d, J')$ is empty. This concludes the proof of part A of lemma 1.6.

Let's prove part B: If $I \leq N$, then

$$\Phi^{-1}(L_{3I-t}) = \{J' \sim J \mid M(\Gamma^d, J') \text{ has codimension } 3I - t, t \geq 1\}$$

and this set has codimension at least

$$(3N + 1 - 3I + t)(3I - 3I + t) = t(3N - 3I + t + 1)$$

as desired. If $I \leq \frac{3N+1}{5}$, then $2I \leq 3N - 3I + 1$ and it follows that

$$\dim\{J' \mid J' \sim J\} \leq 2I \leq 3N - 3I + 1 \leq t(3N - 3I + t + 1)$$

for all $t \geq 1$. Therefore $\Phi^{-1}(L_{3I-t})$ is empty for each $t \geq 1$ and every $M(\Gamma^d, J')$ has codimension exactly $3I$. Part B is now proved.

Finally, let's prove part C: Suppose that $I > N$ and $\dim_{\mathbb{R}} M(\Gamma^d, J') = s \geq 0$. Then

$$\dim_{\mathbb{R}} \ker \Phi_{J'} = s + 1,$$

and the set of such J' have codimension at least $(s + 1)(3I - 3N + s)$ inside the set of configurations equivalent to J. If

$$3I - 3N \geq 2I \geq \dim\{J' \mid J' \sim J\},$$

then the set of J' for which $M(\Gamma^d, J') = \emptyset$ is itself empty. If $I \geq 3N$, then $M(\Gamma^d, J') = \emptyset$ for every J' equivalent to J. This proves part C, and concludes the proof of lemma 1.6.

5.3 The Proof of Theorem 3.4

This proof is a generalization of the proof of Vassiliev's theorem 3.1.2 in [V] and [V1], and follows Vassiliev's proof closely. Recall that $\rho(J)$ refers to the number of geometrically distinct points of an (A, B, C)-configuration J. Recall that $|A|$ and $|C|$ are the number of points in the A and C configurations of J, respectively, and that $\sharp A$ and $\sharp C$ are the number of groupings in the A and C configurations of J, respectively. Also recall that b refers to the cardinality of the B-configuration of J. Let $F_a = \bigcup_{\rho(J) \leq a} B_J$. We can filter $\Omega_I - \Omega_{I-1}$ by

$$\emptyset \subset F_{\frac{I}{2}} \subset F_{\frac{I}{2}+1} \subset \cdots F_{2I} = \Omega_I - \Omega_{I-1}.$$

Recall that Z_I refers to the union of all J-blocks from complicated J.

Lemma 5.2 $Z_I = F_{2I-2}$.

Proof: Let J be a configuration of complexity I. First, we will show that if J is noncomplicated, then $\rho(J) = 2I$ or $2I-1$; thus showing that the J-blocks of noncomplicated configuratiuons are contained in $F_{2I} - F_{2I-1}$. If J is an $[I]$-configuration, then $|A| = 2\sharp A$, $|C| = 2\sharp C$, and $b = 0$, and so $|A| + |C| = 2I = \rho(J)$.

If J is an $\langle I\rangle_A$-configurationor an $\langle I\rangle_C$-configuration, then $b = 0$ and $2(\sharp A + \sharp C) + 1 = |A| + |C|$. It follows that $|A| + |C| = 2I - 1 = \rho(J)$.

If J is an $[I-1]^*$-configuration, then $|A| = 2\sharp A$, $|C| = 2\sharp C$, and $b = 1$. It follows that

$$\begin{aligned} & \sharp A + \sharp C + 1 = I \\ \Rightarrow\ & |A| + |C| = 2(I-1) \\ \Rightarrow\ & \rho(J) = |A| + |C| + 1 = 2I - 1. \end{aligned}$$

Next, we will show that if $\rho(J) = 2I$ or $2I - 1$, then J is noncomplicated. We will use the fact ($\star$) that $|A| + |C| \geq 2(\sharp A + \sharp C)$. If $\rho(J) = 2I$ and $b = 0$, then $|A| + |C| = 2I$ by ($\star$), and J is noncomplicated.

Suppose that $\rho(J) = 2I$ and $b = 1$. If the point of the B-configuration of J is coincident with some point in the A or C configuration of J, then $\rho(J) = |A| + |C|$. This implies that $\sharp A + \sharp C = I + 1$, which cannot happen by ($\star$). If the point of the B-configuration is noncoincident, then $\rho(J) = |A| + |C| + 1 = 2I$. This implies that $\sharp A + \sharp C = I$, which contradicts ($\star$).

Suppose that $\rho(J) = 2I$, and $b > 1$. We know that $|A| + |C| + b \geq 2I$. So, $\sharp A + \sharp C \geq I$, and thus $|A| + |C| \geq 2I$ by ($\star$). This contradicts the hypothesis.

Now, suppose that $\rho(J) = 2I - 1$. If b=0, then $|A| + |C| = 2I - 1$, which tells us that $\sharp A + \sharp C = I - 1$. It is immediate that J must be noncomplicated.

If $b = 1$ and the distinguished point is coincident with a point of the A or C configurations of J, then $|A| + |C| = 2I - 1$. It follows that $\sharp A + \sharp C = I$, which contradicts ($\star$). If the distinguished point is not coincident, then $|A| + |C| = 2I - 2$. This gives $\sharp A + \sharp C = I - 1$, which implies again that J is noncomplicated.

Finally, if $b > 1$, then $|A| + |C| + b \geq 2I - 1$. This gives $\sharp A + \sharp C \geq I - 1$; thus $|A| + |C| \geq 2I - 2$ by ($\star$). This contradicts the supposition that $\rho(J) = 2I - 1$. □.

Following Vassiliev, let $\mathcal{E}^c_{a,b}$ be the homology spectral sequence converging to $\bar{H}_*(\Omega_I - \Omega_{I-1})$, induced by the filtration given above. By definition $\mathcal{E}^1_{a,b} = \bar{H}_{a+b}(F_a - F_{a-1})$.

Lemma 5.3 *The connected components of $F_a - F_{a-1}$ are in one-to-one correspondence with the J-blocks where J has complexity I, and $\rho(J) = a$.*

Proof: By lemma 3.2, two J-blocks from two nonrelated configurations do not intersect, i.e. the space $F_a - F_{a-1}$ is the disjoint union of a finite number of J-blocks. It remains to show that each J-block is closed in $F_a - F_{a-1}$, and this fact follows from the discussion in sections 2.3 and 3.3. Briefly, any cell (of any dimension, not just codimension one) that lies in the boundary, in $\Omega_I - \Omega_{I-1}$, of a given J-block B_J is either a subset of B_J or arises diagramatically from an edge contraction. Suppose that e is any cell in B_J, and that the diagram of a cell f is obtained from the diagram of e by an edge contraction. Reguardless of whether e is a FPC or a BC, the diagram of e has $a = \rho(J)$ points and that of f has $\rho(J) - 1$ points. Thus f is not in the space $F_a - F_{a-1}$. □

As a result of this lemma, $\mathcal{E}^1_{a,b}$ splits into a direct sum of groups $\bar{H}_{a+b}(B_J)$ over all (A, B, C)-configurations J of complexity I and such that $\rho(J) = a$. We must now show that if J is complicated and $k \geq 3N - 2$, then $\bar{H}_k(B_J) = 0$. Let's view B_J as a fibration with base $\{J' \mid J' \sim J\} \times M(\Gamma^d, J)$ and fiber S_J. If J is complicated, then $\rho(J) \leq 2I - 2$ and so the base of this fibration has dimension less than or equal to $\rho(J) + 3N - 3I$, with equality for $d \leq \frac{3N+1}{5}$ (see the proof of theorem 3.21). So, we need to show that the one point compactification of the fiber has trivial homology groups in dimensions greater than or equal to $I + 1$. Recall that we are taking closed homology, so we need to consider the homology of the one point compactification of the fiber.

Let M be a grouping of points of cardinality μ in the A or C configuration of J, and suppose that S is a simplex spanned by $\frac{\mu(\mu+1)}{2}$ vertices corresponding to all unordered pairs of points of M. A face of S is called *generating* if for every s and t in M, there is a chain of pairs that are vertices of S joining s and t.

Example 5.4 *Suppose that $M = \{t_1, t_2, t_3, t_4, t_5\}$. The generating complex $K(M)$ is the simplex spanned by the $\frac{5(5-1)}{2}$ pairs of unordered points of M. The face, spanned by the pairs*

$$(t_1, t_2), (t_2, t_3), (t_3, t_4) \text{ and } (t_4, t_5),$$

is generating, while the face spanned by

$$(t_1, t_2), (t_1, t_3), (t_2, t_3) \text{ and } (t_4, t_5)$$

is not.

Define $K(M)$ to be the cell complex obtained by taking the quotient of S by the subcomplex consisting of the nongenerating faces of S. For each (A, B, C)-configuration J, define $K(J)$ to be the join of generating complexes $K(M)$, over all groupings M of the A and C configurations of J. We will call $K(J)$ the *generating complex* of J. It is immediate, and is lemma 3.6.3 in [V], that the closed homology of the fiber of B_J, viewed as above, is isomorphic to the homology group of the generating complex $K(J)$. The next example demonstrates this.

Example 5.5 *Let $n = 2$ and let J be a $\langle 3 \rangle_C$-configuration. Suppose there is a pair (s_1, s_2) in the A-configuration of J and that (t_1, t_2, t_3) is the grouping of three points in the C-configuration of J. Recall that the standard simplex S_J associated to this configuration is the union in R^M of the interiors of four simplices. These simplices are spanned by the following four collections of points in R^M, where the map λ and the integer M are as in lemma 1.9 and the discussion following it.*

This collection corresponds to the maximal generating collection of J, in the sense of section 1.5:

$$\{\lambda(s_1, s_2), \lambda(t_1, t_2), \lambda(t_1, t_3), \lambda(t_2, t_3)\}.$$

The next three collections correspond to the three submaximal generating collections of J:

$$\{\lambda(s_1, s_2), \lambda(t_1, t_2), \lambda(t_1, t_3)\},$$
$$\{\lambda(s_1, s_2), \lambda(t_1, t_2), \lambda(t_2, t_3)\},$$
$$\{\lambda(s_1, s_2), \lambda(t_1, t_3), \lambda(t_2, t_3)\}.$$

Let g_1 and g_2 be the groupings of two and three points, respectively. $K(g_1)$ is a point and the chain group $C(K(g-2))$ is the quotient of a free abelian group $\mathcal{G}$ and a certain subgroup $\mathcal{H}$. The group $\mathcal{G}$ is generated by all of the faces of $K(g_2)$. For convenience of notation, let

- $s_{12} = \lambda(s_1, s_2)$,
- $t_{ij} = \lambda(t_i, t_j)$, *where* $1 \leq i < j \leq 3$,
- (t_{ij}) *refer to the (zero dimensional) simplex spanned by the point* t_{ij},
- (t_{ij}, t_{pq}) *refer to the simplex spanned by* t_{ij} *and* t_{pq}, *and*
- (t_{12}, t_{13}, t_{23}) *refer to the simplex spanned by these three points.*

Using this notation, $\mathcal{G}$ is generated by

$$(t_{12}, t_{13}, t_{23}), (t_{12}, t_{13}), (t_{12}, t_{23}), (t_{13}, t_{23}), (t_{12}), (t_{13}) \text{ and } (t_{23})$$

and $\mathcal{H}$ is generated by $(t_{12}), (t_{13})$, and (t_{23}). The complex $K(J)$ is defined to be the join of $K(g_2)$ with $K(g_1)$, which is a zero dimensional simplex (s_{12}) in the above notation. The (graded) group $\mathcal{G}/\mathcal{H}$ is isomorphic to the free abelian group generated by

$$(t_{12}, t_{13}, t_{23}), (t_{12}, t_{13}), (t_{12}, t_{23}) \text{ and } (t_{13}, t_{23}),$$

and so the join $K(J)$ has chain group isomorphic to one generated by

$$(s_{12}, t_{12}, t_{13}, t_{23}), (s_{12}, t_{12}, t_{13}), (s_{12}, t_{12}, t_{23}) \text{ and } (s_{12}, t_{13}, t_{23}).$$

It is immediate that $\bar{H}_*(S_J) \cong H_*(K(J))$.

Now it remains to prove that if J is complicated and of complexity I, then $\bar{H}_k(K(J)) = 0$ when $k \geq I - 1$. Following Vassiliev, we can reduce to the following theorem, using the Künneth formula, and the homology of joins (see appendix 1 in [V1]). The theorem is theorem 3.6.3 in both [V] and in [V1].

Theorem 5.6 *For any set M with μ elements, $\bar{H}_k(K(M)) = 0$ for $k \neq \mu - 2$, and $\bar{H}_{\mu-2}(K(M)) = G^{(\mu-1)!}$, where G is the given coefficient group.*

Proof: This is proved as theorem 3.6.3 in [V1]. □

This completes the proof of theorem 3.4.

5.4 The Proof of Theorem 3.6

We now turn to proving that B_J is a trivial bundle, when viewed as a bundle with base $\{J^r \mid J^r \leftrightarrow J\}$ and fiber $M(\Gamma^d, J) \times S_J$. Let J be an (A, B, C)-configuration and, without loss of generality, assume that J has points on all n copies of S^1 in $\coprod_{i=1}^n S^1$. The configuration space $C = \{J^r \mid J^r \leftrightarrow J\}$ is homeomorphic to $(S^1)^n \times (0,1)^{\rho(J)-n}$. We can think of C as being covered by as many open sets as there are BC's in B_J, where each BC is enlarged slightly so that the resulting set in the open cover overlaps all of the neighboring sets. Fix some J_1 in C and some ι between 1 and n. Consider the open sets corresponds to to a class of configurations whose chord diagram differs from that of J_1 by the placement of the first point on $(S^1)_\iota$. Let μ be the number of these open sets and pick representatives $J_2, J_3, \ldots, J_\mu$ of these classes so that the first point of J_k on $(S^1)_\iota$ corresponds to the second point of J_{k-1} on $(S^1)_\iota$, i.e. the first point rotates counterclockwise as we pass from J_{k-1} to J_k. Let $M(\Gamma^d, J) \times S_J$ refer to a generic fiber and let o_k be the open set corresponding to (the equivalence class of) J_k, $1 \le k \le \mu$. As we pass from o_k to o_{k+1}, there is an induced twisting map $T_k : M(\Gamma^d, J) \times S_J \to M(\Gamma^d, J) \times S_J$. We use T_μ to refer to the twisting map of the transition from o_μ to o_1. To prove the theorem it is sufficient to show that the composition of the twisting maps, indicating the passage from o_1 to o_2, then o_2 to o_3, and so on, finally around to o_1 again, is the identity on the fiber.

Recall that, for any $J^r \leftrightarrow J$, we can view $M(\Gamma^d, J^r)$ as the solution space in Γ^d to the matrix equation $\Phi_{J^r}\vec{x} = 0$, where $\Phi_{J^r} \in \mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^{3N+1}, \mathbb{R}^{3I})$. Let $[J_j]$ refer to the equivalence class $\{J_j' \mid J_j' \sim J_j\}$. Let's point out that we will abuse this terminology slightly; when we say that we are going to pass from $[J_j]$ to $[J_{j+1}]$, we are really saying that we are moving through the open set o_j, which contains $[J_j]$, through the overlap $o_j \cap o_{j+1}$ and into $[J_{j+1}] \subset o_{j+1}$. Given this, as we pass from $[J_1]$ to $[J_2]$, we can view the zero point $0 \in (S^1)_\iota$ as travelling from some location between the last and first points of $[J_1]$ on $(S^1)_\iota$ to a point between the first and second points of $[J_1]$ on $(S^1)_\iota$. We are assuming that all of the points of $[J_1]$ on $(S^1)_\iota$, and indeed on all other circles in $\coprod S^1$, are fixed relative to each other. We can make this assumption because the relative positions of the points of $[J_1]$ on $(S^1)_\iota$ is determined by the $(0,1)^{\rho_\iota(J_1)}$ coordinates in the bundle. Our passage actually places us in the overlap $o_1 \cap o_2 \cap [J_2]$. In general, we will view the passage from $[J_j]$ to $[J_{j+1}]$, for $1 \le j \le \mu - 1$, and from $[J_\mu]$ to $[J_1]$ the same way.

For any $J^r \leftrightarrow J$, let $p_1(J^r), p_2(J^r), \ldots, p_{\rho(J)}(J^r)$ be the parameters of the points of J^r written in the order in which the points appear in the standard ordering of the points of J^r and let $p_m(J^r), p_{m+1}(J^r), \ldots, p_{m+\rho_\iota(J)}(J^r)$ be the parameters of the points of $J^r \cap (S^1)_\iota$. The transition from o_1 to o_2 induces a map F_1 taking the (parameters of) points of J_1 to the (parameters of) points of J_2. The map F_1 simply changes the order in which some of these points are written:

$$\begin{aligned} F_1(p_k(J_1)) &= p_k(J_2),\ \forall k \notin \{m, m+1, \ldots, m + rho_\iota(J)\}, \\ F_1(p_k(J_1)) &= p_{k-1}(J_2),\ \forall k \in \{m+1, m=2, \ldots, m_{\rho_\iota(J)}\}, \\ F_1(p_m(J_1)) &= p_{\rho_\iota(J)}(J_2). \end{aligned}$$

The transitions from o_j to o_{j+1}, $2 \le j \le \mu - 1$, and from o_μ to o_1 induce similiar maps $F_2, F_3, \ldots, F_\mu$.

Let's fix an explicit format for the matrix Φ_{J_1}. One way of obtaining an explicit expression for Φ_{J_1} uses the $3I$-form w_{J_1} (see section 2.2). Recall that w_{J_1} is the wedge product of I 3-forms, each corresponding to either a pair of points from a grouping in the A or C configurations of J_1 or to a single point in the B-configuration of J_1. We can construct Φ_{J_1} by letting each of these three forms correspond to a block of three rows in Φ_{J_1}. We order these blocks of three rows from top to bottom in Φ_{J_1} in the same order that the corresponding 3-forms wedged together on w_{J_1}. Let g_1 be any grouping in the A or C configuration of J_1 and let $s_1, s_2, \ldots, s_\alpha$ be the (parameters of the) points of g_1, written in the order they appear in the ordering of the points of J_1. The grouping g_1 contributes a $(3(\alpha-1)\times(3N+1))$ block to Φ_{J_1} that can be represented in shorthand form by

$$\begin{pmatrix} (s_2 - s_1) \\ (s_3 - s_2) \\ (s_4 - s_3) \\ \vdots \\ (s_{\alpha-1} - s_{\alpha-2}) \\ (s_\alpha - s_{\alpha-1}) \end{pmatrix},$$

where our shorthand follows that of section 2.2 and the $(3\times(3N+1))$-block $(s_k - s_{k-1})$ corresponds, in shorthand, to the 3-form $d(s_k - s_{k-1})$. Similiarly, if β is a point in the B-configuration of J_1, then β contributes a $(3 \times 3N+1)$-block to Φ_{J_1}. This block can be represented in shorthand by $(\partial\beta)$, which corresponds to the 3-form, in shorthand, $d(\partial\beta)$. We will refer to this procedure for obtaining Φ_{J_1} from w_{J_1} as $\mathrm{P1}(J_1)$.

We could apply P1 to $J_2, J_3, \ldots, J_\mu$, but it is easier not to. Let's create a second procedure P2 to obtain the other desired matrices. We can get an expression for Φ_{J_2} by simply replacing the parameter of each point $p_k(J_1)$ by the parameter $F_1(p_k(J_1))$, $1 \le k \le \rho(J)$. The procedure P2 is defined recursively. We obtain $\Phi_{J_{j+1}}$ from Φ_{J_j} by replacing each $p_k(J_j)$ by $F_j(p_k(J_j))$.

We assert that the solution space to Φ_{J_j}, obtained by P2, is the same as it would be if Φ_{J_j} was obtained using P1. Without loss of generality we only need to prove the assertion for Φ_{J_2}. The assertion follows from the fact that the two procedures yield matrices which are row equivalent. Let Φ_{J_2} refer to the desired matrix obtained by P2 and let $\bar{\Phi}_{J_2}$ refer to the matrix obtained by P1. Let g_1 and $\{s_k\}$ be as above, assume that g_1 has nonempty intersection with $(S^1)_\iota$ (otherwise there is nothing to prove) and let $s_m, s_{m+1}, \ldots, s_{m+\delta}$ be the points of $g_1 \cap (S^1)_\iota$. Let g_2 be the corresponding grouping in J_2. The grouping g_2 consists of points $F_1(s_k)$, $1 \le k \le \alpha$. Let $t_1, t_2, \ldots, t_\alpha$ be these points written in the order that they appear in the standard ordering of the points of J_2. The subcollection $t_m, t_{m+1}, \ldots, t_{m+\delta}$ are the points of $g_2 \cap (S^1)_\iota$. First suppose that s_m is not the first point of J_1 on $(S^1)_\iota$. Then, for each k, $F_1(s_k) = t_k$ and the $(3(\alpha-1)\times(3N+1))$-block contributed by g_2 to Φ_{J_2} has the shorthand expression

$$\begin{pmatrix} (t_2 - t_1) \\ (t_3 - t_2) \\ \vdots \\ (t_\alpha - t_{\alpha-1}) \end{pmatrix}$$

which is exactly the one it would have as a block in $\bar{\Phi}_{J-2}$.

If s_m is the first point of J_1 on $(S^1)_\iota$, then

$$F_1(s_k) = t_k, \forall k \notin \{m, m+1, \ldots, m+\delta\}$$
$$F_1(s_{m+k}) = t_{m+k-1}, 1 \leq k \leq \delta$$
$$F_1(s_m) = t_{m+\delta}.$$

The resulting block in Φ_{J_2} has the form

$$\begin{pmatrix} (t_2 - t_1) \\ (t_3 - t_2) \\ \vdots \\ (t_{m-1} - t_{m-2}) \\ (t_{m+\delta} - t_{m-1}) \\ (t_m - t_{m+\delta}) \\ (t_{m+1} - t_m) \\ \vdots \\ (t_{m+\delta-1} - t_{m+\delta-2}) \\ (t_{m+\delta+1} - t_{m+\delta-1}) \\ \vdots \\ (t_\alpha - t_{\alpha-1}) \end{pmatrix}$$

which is row equivalent to the block of rows in $\bar{\Phi}_{J_2}$.

Now suppose that $\beta_1, \beta_2, \ldots, \beta_b$ are the points of the B-configurationof J_1 written in order and suppose that $\beta_m, \ldots, \beta_{m+\delta}$ are the points of the intersection of the B-configuration of J_1 with $(S^1)_\iota$. Let $\gamma_1, \ldots, \gamma_b$ be the corrseponding points in J_2. Then $\gamma_m, \ldots, \gamma_{m+\delta}$ are the points of the B-configuration of J_2 that lie on $(S^1)_\iota$. If β_m is not the first point of J_1 on $(S^1)_\iota$, then $F_1(\beta_{m+k}) = \gamma_{m+k}$, for $1 \leq k \leq \delta$. The resulting $(3b \times (3N+1))$-block in Φ_{J_2} has the shorthand form

$$\begin{pmatrix} \partial\gamma_1 \\ \partial\gamma_2 \\ \vdots \\ \partial\gamma_b \end{pmatrix}$$

which is exactly the same as the corresponding block in $\bar{\Phi}_{J_2}$.

Finally, if γ_m is the first point of J_1 on $(S^1)_\iota$, then

$$\begin{aligned} F_1(\beta_k) &= \gamma_k, \forall k \notin \{m, m+1, \ldots, m+\delta\} \\ F_1(\beta_{m+k}) &= \gamma_{m+k-1}, 1 \leq k \leq \delta \\ F_1(\beta_m) &= \gamma_{m+\delta}. \end{aligned}$$

The contributed $(3b \times (3N+1))$-block is then

$$\begin{pmatrix} \partial\gamma_1 \\ \partial\gamma_2 \\ \vdots \\ \partial\gamma_{m-1} \\ \partial\gamma_{m+\delta} \\ \partial\gamma_m \\ \vdots \\ \partial\gamma_{m+\delta-1} \\ \partial\gamma_{m+\delta+1} \\ \vdots \\ \partial\gamma_b \end{pmatrix}$$

which is row equivalent to the corresponding block in $\bar{\Phi}_{J_2}$. This proves the assertion.

Now fix expressions for Φ_{J_j}, $2 \le j \le \mu$, using P2. Then the passage, from o_1 to o_2 to o_3, and so on, finishing with o_μ to o_1, induces a smooth loop $\Phi_{J(\theta)}$, $0 \le \theta \le 2\pi$, in $\{\Phi_{J^r} \mid J^r \leftrightarrow J\}$. This loop passes through all of the Φ_{J_j}'s, only depends on the coordinates in $(S^1)_\iota$ of the points of $J_j \cap (S^1)_\iota$ and has initial and terminating points at Φ_{J_1}, i.e. $\Phi_{J(0)} = \Phi_{J_1} = \Phi_{J(2\pi)}$. All we've done in this passage is rename the parameters p_k every time we hit the overlap of two of our covering sets. That is to say that while the parameters themselves are renamed, their coordinates vary smoothly as we move the zero on $(S^1)_\iota$ around the circle. We will now show that the induced twisting map on the $M(\Gamma^d, J)$ component of the fiber is the identity.

Next, fix an algorithm for row reducing each $\Phi_{J(\theta)}$ to echelon form. Now, $M(\Gamma^d, J(\theta))$ is the solution space to the matrix equation $\Phi_{J(\theta)}\vec{x} = 0$, where $\vec{x} = (x_1, x_2, \ldots, x_{3N+1})$. We know that the kernel of $\Phi_{J(\theta)}$ has dimension $3N - 3I + 1$, so we can assume that we have chosen $x_1, \ldots, x_{3N-3I+1}$ to be free variables. Let $v_k(\theta)$ be the vector obtained by setting $x_k = 1$ and $x_i = 0$, for $1 \le i \le 3N-3I+1$. The set $\{v_k\}$ is a basis for $\ker \Phi_{J(\theta)}$, and, for each k, v_k depends only on the entries of the matrix $\Phi_{J(\theta)}$. These entries are all either constants or linear combinations of trigonometric functions whose domains are the parameters of the points of $J(\theta)$. The result is that $v_k(0) = v_k(2\pi)$, for $1 \le k \le 3N - 3I + 1$. This is to say that the vectors $v_k(\theta)$ are sections from the base to the $M(\Gamma^d, J)$ component of the fiber that are defined over the entire ι^{th} circle, hence the composition $T_\mu \circ T_{\mu-1} \circ \cdots \circ T_1$ of twisting maps is trivial on the $M(\Gamma^d, J)$ component of the fiber. Since ι was arbitrary, we have shown that the bundle is trivial on the $M(\Gamma^d, J)$ component of the fiber.

To see that the bundle is trivial on the S_J component of the fiber, recall that the closure of each S_{J_j} is spanned by the image under the map λ (see lemma 1.9) of unordered pairs from the maximal generating collection of J_j. The orientation of S_{J_j} is determined by the standard ordering of the pairs of the maximal generating collection of J_j. The action of any twisting map on the fiber induced by the reordering of these pairs, given by functions G_j to be defined shortly. If g_1 is a grouping of J_1 as above, then the ordered collection of pairs

$$(s_1, s_2), (s_1, s_2), \ldots, (s_1, s_\alpha), (s_2, s_3), \ldots, (s_\alpha, s_{\alpha-1}),$$

taken in the (dictionary) order in which the are written, contribute an oriented face to S_{J_1}. Let the points $t_1, \ldots, t_\alpha$, the grouping g_2, the configuration J_2 and F_1 all be as above. The transition from o_1 to o_2 induces the map G_1 on these pairs. G_1 is defined by

$$G_1(s_p, s_q) = (F_1(s_p), F_1(s_q)).$$

Similiarly, define

$$G_j(s_p, s_q) = (F_j(s_p), F_j(s_q)).$$

for $2 \leq j \leq \mu$, where (s_p, s_q) is a pair from a grouping g_j of J_j and F_j is as above. Now, the composition $F_\mu \circ F_{\mu-1} \circ \cdots \circ F_1$ is the identity on the points of g_1 and so it follows that $G_\mu \circ \cdots \circ G_1$ is the identity function on the above set of pairs. This proves that the twisting map is trivial on every face of S_{J_1} contributed by a grouping. It is clear that the same argument works when applied to the set of pairs

$$(\beta_1, \beta_1), \ldots, (\beta_b, \beta_b)$$

that come from the points $\beta_1, \ldots, \beta_b$ of the B-configuration of J_1. We have now shown that, over the $(S^1)_\iota$ component of the base, the bundle is trivial on the S_J component of the fiber . Since ι was arbitrary, we have shown that the bundle is trivial on the S_J component of the fiber. Now we are done.

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