BOSONIC-FERMIONIC DIAGONAL COINVARIANTS AND THETA OPERATORS

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Abstract. We present a number of conjectures and open problems involving combinatorial models for graded characters of the symmetric group associated to Macdonald polynomial operators and diagonal coinvariant rings.

Contents

1. Introduction

Given a polynomial $f(x_1,\dots,x_n,y_1,\dots,y_n) \in \mathbb{C}[x_1,\dots x_n,y_1,\dots,y_n]$, the symmetric group S_n acts diagonally by permuting the $\boldsymbol{x}_n := (x_1,\dots,x_n)$ and $\boldsymbol{y}_n := (y_1,\dots,y_n)$ variables identically, i.e.,

$$\sigma f(x_1,\dots,x_n,y_1,\dots,y_n) = f(x_{\sigma_1},\dots,x_{\sigma_n},y_{\sigma_1},\dots,y_{\sigma_n}) \quad \text{for all } \sigma \in S_n. \tag{1.1}$$

Let $\boldsymbol{C}_{2;n}$ denote the *diagonal coinvariant ring* in two sets of variables, defined as the quotient

$$\boldsymbol{C}_{2;n} := \mathbb{C}[\boldsymbol{x}_n,\boldsymbol{y}_n]/I_n(\boldsymbol{x}_n,\boldsymbol{y}_n), \tag{1.2}$$

where $I_n(\boldsymbol{x}_n,\boldsymbol{y}_n)$ is the ideal generated by all S_n-invariant polynomials in $\mathbb{C}[\boldsymbol{x}_n,\boldsymbol{y}_n]$ without constant term.

The module $\boldsymbol{C}_{2;n}$ can be decomposed into components bigraded by homogeneous $\boldsymbol{x}_n$ and $\boldsymbol{y}_n$ degree (i.e. $\boldsymbol{C}_{2;n} = \sum_{i,j} \boldsymbol{C}_{2;n}^{(i,j)}$), and the action respects the bigrading. The Frobenius characteristic of $\boldsymbol{C}_{2;n}$ is defined as

$$\boldsymbol{C}_{2;n}(q,t;\boldsymbol{z}) := \sum_{i,j\geq 0} q^i t^j \sum_{\lambda\vdash n} s_\lambda(\boldsymbol{z})\,\mathrm{Mult}(\lambda,\boldsymbol{C}_{2;n}^{(i,j)}), \tag{1.3}$$

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where the inner sum is over all partitions λ of n, s_λ is the Schur function, and $\mathrm{Mult}(\lambda, \boldsymbol{C}_{2;n}^{(i,j)})$ is the multiplicity of the irreducible S_n module corresponding to λ in the decomposition of $\boldsymbol{C}_{2;n}^{(i,j)}$ into irreducible submodules.

Let ∇ denote the Bergeron-Garsia nabla operator [BG99], defined on the modified Macdonald basis $\widetilde{H}_\mu(q,t;\boldsymbol{z})$ by

$$\nabla \widetilde{H}_\mu(q,t;\boldsymbol{z}) = T_\mu \widetilde{H}_\mu(q,t;\boldsymbol{z}), \tag{1.4}$$

where $T_\mu = T_\mu(q,t) = \prod_{(i,j)\in\mu} q^i t^j$. If $e_n = e_n(\boldsymbol{z})$ is the n^{th} elementary symmetric function, a famous theorem of Haiman [Hai01] says

$$\nabla(e_n(\boldsymbol{z})) = \boldsymbol{C}_{2;n}(q,t;\boldsymbol{z}), \tag{1.5}$$

which forms the bridge between the study of coinvariant rings and Macdonald polynomial operators. In this article we will present a number of open problems associated to the combinatorial structure of ∇e_n and related functions.

A central result in this direction is the Shuffle Theorem of Carlsson and Mellit [CM18]. First conjectured in [HHL+05], this result expresses ∇e_n as weighted sum over combinatorial objects known as parking functions. The parking function model has been very successful in describing the monomial expansion of a growing list of Macdonald polynomial operators applied to various symmetric functions. Many of these expansions are known or conjectured to be the Frobenius characteristic of a bigraded S_n module associated to a coinvariant ring or other object of geometric interest.

One important example involves the Delta operators. For a given cell $c = (i,j)$ in the Ferrers diagram of μ, with (i,j) the Cartesian coordinates of the bottom-left corner of the cell, we let the $\mathrm{arm}(c)$ and $\mathrm{leg}(c)$ denote the distances from c to the right and top border of μ, respectively, as in Figure 1.

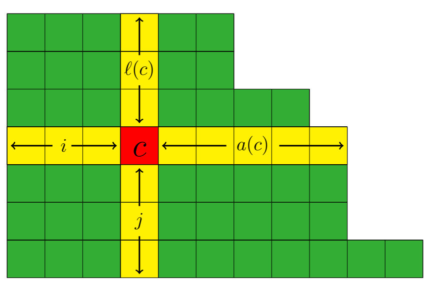

FIGURE 1. The arm $a(c)$ and leg $l(c)$ of a cell (i,j).

Define $B_\mu(q,t)$ as

$$B_\mu(q,t) = \sum_{(i,j)\in\mu} q^i t^j. \tag{1.6}$$

For example, $B_{221}(q,t) = 1+q+t+qt+t^2$. Now for a symmetric function $f(\boldsymbol{z})$, define linear operators Δ_f and Δ'_f via their values on the modified Macdonald basis as follows:

$$\Delta_f \widetilde{H}_\mu = f[B_\mu(q,t)]\widetilde{H}_\mu, \tag{1.7}$$

$$\Delta'_f \widetilde{H}_\mu = f[B_\mu(q,t)-1]\widetilde{H}_\mu. \tag{1.8}$$

For example, $\Delta_{e_2}\widetilde{H}_{221} = (q+t+qt+t^2+qt+q^2t+qt^2+t^2q+t^3+qt^3)\widetilde{H}_{221}$ and $\Delta'_{e_2}\widetilde{H}_{221} = (qt+q^2t+qt^2+t^2q+t^3+qt^3)\widetilde{H}_{221}$.

The Delta Conjecture from [HRW18] says that

$$\sum_{k=0}^{n-1} u^{n-1-k}\Delta'_{e_k} e_n(\boldsymbol{z}) = \mathrm{Rise}(n) \tag{1.9}$$

$$= \mathrm{Valley}(n), \tag{1.10}$$

where Rise(n) and Valley(n) are two different weighted sums over parking functions, involving parameters q,t,u in addition to a monomial term in the $\boldsymbol{z}$ variables. The equality of $\sum_{k=0}^{n-1} u^{n-1-k}\Delta'_{e_k} e_n$ with Rise(n) and Valley(n) are known as the Rise version and Valley version of the Delta Conjecture, respectively. Two different proofs of the Rise version have recently been found. The first, due to D'Adderio and Mellit [DM22], proves a refinement of the Rise version called the compositional Delta Conjecture. This refinement involves more technical operators called Theta operators, which have found many applications and are the subject of Section 9. The second proof is due to Blasiak, Haiman, Morse, Pun, and Seelinger [BHM+23]. They give a combinatorial model for $\Delta_{h_j}\Delta'_{e_k} e_n$ (the Extended Delta Conjecture from [HRW18]), which reduces to the Rise version when $j=0$. Here, h_j is the complete homogeneous symmetric function. Their proof uses properties of the Elliptic Hall algebra. Neither the Extended Delta Conjecture or compositional Delta Conjecture imply the other. The Valley version is still open; in Section 2 we describe the Delta Conjecture in detail and highlight open problems associated to it. We mention that $\nabla e_n = \Delta'_{e_{n-1}} e_n$, and the $u=0$ case of either the Rise or Valley version reduces to the Shuffle Theorem.

Mike Zabrocki (see [Zab19]) has introduced an exciting conjecture linking Equation 1.9 to coinvariant algebras. On top of the above two sets $\boldsymbol{x}_n, \boldsymbol{y}_n$ of commuting (i.e. bosonic) variables, let $\boldsymbol{v}_n$ be an n-set of anti-commuting (i.e. fermionic) variables, all of which commute with the x_i's and y_i's. Zabrocki conjectures that

$$\sum_{k=0}^{n-1} u^{n-1-k}\Delta'_{e_k} e_n(\boldsymbol{z}) \tag{1.11}$$

equals the tri-graded Frobenius characteristic of the super diagonal coinvariant ring $\boldsymbol{C}_{2,1;n}$ under the diagonal action of S_n. Here $\boldsymbol{C}_{2,1;n}$ is defined as

$$\boldsymbol{C}_{2,1;n} := \mathbb{C}[\boldsymbol{x}_n, \boldsymbol{y}_n, \boldsymbol{v}_n]/I_n(\boldsymbol{x}_n, \boldsymbol{y}_n, \boldsymbol{v}_n), \tag{1.12}$$

where $I_n(\boldsymbol{x}_n, \boldsymbol{y}_n, \boldsymbol{v}_n)$ is the ideal generated by all S_n-invariant polynomials in $\mathbb{C}[\boldsymbol{x}_n, \boldsymbol{y}_n, \boldsymbol{v}_n]$, without constant term. Then,

$$\boldsymbol{C}_{2,1;n}(q,t;u;\boldsymbol{z}) := \sum_{i,j,k\geq 0} q^i t^j u^k \sum_{\lambda\vdash n} s_\lambda(\boldsymbol{z})\,\mathrm{Mult}(\lambda, \boldsymbol{C}_{2,1;n}^{(i,j,k)}), \tag{1.13}$$

with $\boldsymbol{C}_{2,1;n}^{(i,j,k)}$ the homogeneous component of $\boldsymbol{C}_{2,1;n}$ in degree i in $\boldsymbol{x}_n$, degree j in $\boldsymbol{y}_n$ and degree k in $\boldsymbol{v}_n$. A conjectured symmetric function expression, involving Theta operators, for the quad-graded Frobenius characteristic of the diagonal coinvariant ring involving two sets of bosonic variables and two sets of fermionic variables was introduced in [DIVW21], and is described in Section 5.

Let $\boldsymbol{v}, \boldsymbol{z}$ be two sets of variables. In a recent preprint on the arXiv [BHIR23], the authors introduce a new linear operator, called the *super nabla operator*, denoted $\nabla_{\boldsymbol{v}}$, which is defined as

$$\nabla_{\boldsymbol{v}} \widetilde{H}_\mu(q,t;\boldsymbol{z}) = \widetilde{H}_\mu(q,t;\boldsymbol{v})\widetilde{H}_\mu(q,t;\boldsymbol{z}). \tag{1.14}$$

It is well-known that $\langle \widetilde{H}_\mu(q,t;\boldsymbol{z}), s_{n-k,1^k}\rangle = e_k[B_\mu(q,t)-1]$, $0 \leq k \leq n-1$, which implies that the coefficient of $s_{n-k,1^k}(\boldsymbol{v})$ in $\nabla_{\boldsymbol{v}} e_n(\boldsymbol{z})$ equals $\Delta'_{e_k} e_n(\boldsymbol{z})$. Hence $\nabla_{\boldsymbol{v}}$ contains the Δ'_{e_k} operators as specializations. In [BHIR23] we conjecture that for sets of variables $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_j$, the symmetric function $\nabla_{\boldsymbol{v}_1} \cdots \nabla_{\boldsymbol{v}_j} e_n(\boldsymbol{z})$ is Schur positive in all sets of variables. In Section 8 we give a few combinatorial models for the monomial expansion of $\nabla_{\boldsymbol{v}_1} \cdots \nabla_{\boldsymbol{v}_j} e_n(\boldsymbol{z})$ when $t = 1$, and discuss several associated open problems, such as finding a t-statistic to incorporate into one or more of our models to give $\nabla_{\boldsymbol{v}_1} \cdots \nabla_{\boldsymbol{v}_j} e_n(\boldsymbol{z})$ for general t.

As the first author has shown [Ber13b], [Ber], [Ber22] there is a GL_n action on $\boldsymbol{C}_{2;n}$ which allows one to view the coefficient of a given Schur function $s_\lambda(\boldsymbol{z})$ in $\nabla e_n(\boldsymbol{z})$ as a positive sum of Schur functions in the variable set $\{q,t\}$. This property extends to multiple bosonic and fermionic sets of variables, and allows one to express formulas and conjectures involving these coefficients compactly. Sections 3, 4, 5 and 6 contain a discussion of open problems involving this general setup. One intriguing conjecture in Section 4 we might highlight here is that the Frobenius characteristic for the k-bosonic, k-fermionic set of variables diagonal coinvariant module can be obtained from the Frobenius characteristic for the k-bosonic case by a simple plethystic substitutiion.

2. The Delta Conjecture

A Dyck path is a lattice path from $(0,0)$ to (n,n) consisting of unit North and East steps which never go below the line $y = x$. A parking function P is a Dyck path π together with a placement of "cars"(the integers 1 through n) just to the right of the north steps of π, with strict decrease down columns, as in Figure 2. Here the first column of numbers to the right of the figure are the row lengths; a_i is the number of complete squares to the right of the path and left of the main diagonal (the line $y = x$) in row_i (the ith row from the bottom). The d_i values are more complicated. We call a pair (i,j) of rows, where $1 \leq i < j \leq n$, an *inversion pair* if either

1) $a_i = a_j$ and the car in row_i is less than the car in row_j, or

2) $a_i = a_j + 1$, and the car in row_i is greater than the car in row_j.

For example, for P as in Figure 2, the inversion pairs are

$$\{(3,7),(4,5),(4,7),(5,7),(6,7)\}.$$

We let $d_i(P)$ denote the number of inversion pairs involving row_i and rows above it, and set $\text{area}(P) = \text{area}(\pi) = \sum_i a_i$ and $\text{dinv}(P) = \sum_i d_i$.

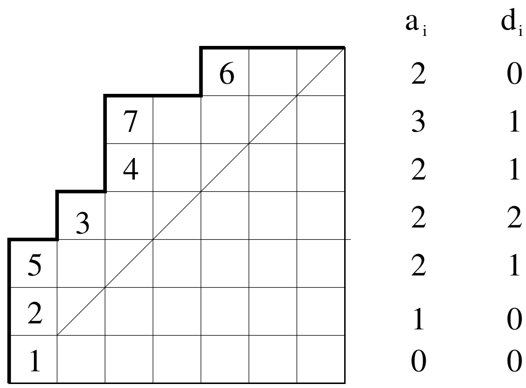

FIGURE 2. A parking function P with its a_i and d_i values. Here $\text{area}(P) = 12$ and $\text{dinv}(P) = 5$

Next define the *reading order* of the rows of π to be the order in which the rows are listed by decreasing value of a_i, where if two rows have the same length, the row above is listed first. For the path π above, the reading order is

$$\text{row}_6, \text{row}_7, \text{row}_5, \text{row}_4, \text{row}_3, \text{row}_2, \text{row}_1. \tag{2.1}$$

For a given P, let the reading word of P, denoted $\text{read}(P)$, be the list of cars, as they appear in reading order. For P as in Figure 2, the reading word is 7643521. Furthermore, if for i in the range $1 < i \leq n$ row_i contains a car at the bottom of a column, call row_i a *moveable valley* if either $a_i < a_{i-1}$ or $a_i = a_{i-1}$ and the car in row_i is greater than the car in row_{i-1}. Geometrically, row_i is a moveable valley if we can shift the car in row_i one square to the left and still have a parking function.

Conjecture 1 (The Delta Conjecture [HRW18])**.** *For any integer k, $0 \leq k \leq n-1$,*

$$\Delta'_{e_k} e_n = \sum_{\pi} \sum_{P \in \text{PF}(\pi)} t^{\text{area}(\pi)} q^{\text{dinv}(P)} F_{\text{Des}(\text{read}(P)^{-1})} \prod_{\substack{a_i > a_{i-1} \\ 1 < i \leq n}} (1 + u/t^{a_i}) \Big|_{u^{n-1-k}} \tag{2.2}$$

$$= \sum_{\pi} \sum_{P \in \text{PF}(\pi)} t^{\text{area}(\pi)} q^{\text{dinv}(P)} F_{\text{Des}(\text{read}(P)^{-1})} \prod_{\text{moveable valleys for } P} (1 + u/t^{d_i+1}) \Big|_{u^{n-1-k}}, \tag{2.3}$$

where $\text{PF}(\pi)$ is the set of parking functions for the path π, $\text{Des}(\sigma)$ is the descent set of a permutation σ, and F is Gessel's fundamental quasisymmetric function associated to the set $\text{Des}(\text{read}(P)^{-1})$. (Readers unfamiliar with Gessel's F's can consult [Hag08, Chapter 6] for examples and definitions.)

The case $k = 0$ of (2.2) is the Shuffle Theorem of Carlsson and Mellit [CM18]. Eq. (2.2) is known as the Rise version, and (2.3) the Valley version. As mentioned in Section 1, the Rise version has been proved, but the Valley version is still open. In fact, there is still no proof that the right-hand-side of (2.3) is even a symmetric function.

One might ask whether or not proving the Valley version is important, given the truth of the Rise version, but there are some definite advantages to working with the Valley version. For example, the second author and E. Sergel have shown the Valley version satisfies a "schedules formula" [HS21], which generalizes the well-known schedules formula for ∇e_n [HL05]. This formula was used by Carlsson and Oblomkov in obtaining their beautiful monomial basis for $\boldsymbol{C}_{2;n}$ [CO18]. The paper [HS21] contains a candidate basis for the super case $\boldsymbol{C}_{2,1;n}$, modeled on the Carlsson-Oblomkov $\boldsymbol{C}_{2;n}$ basis and corresponding schedules formula for the Valley version. In addition, the Valley version is naturally connected to one of our combinatorial models for the monomial expansion of $\nabla_{\boldsymbol{v}} e_n(\boldsymbol{z})$ (see Figure 5 in Section 8).

3. General setup

As mentioned in the Introduction, much interesting work has been done recently on diagonal coinvariant spaces in both commuting and anticommuting variables. See for instance [OZ20, SW21, Wal19]. The purpose of this short note is to present a general conjecture expressing the fact that one can simply calculate all cases of multivariate diagonal coinvariant modules in k sets of n commuting variables (bosons), and j sets of n anticommuting variables (fermions), just from the generic case of multivariate diagonal coinvariant spaces.

Let $\boldsymbol{B} = (\beta_{ab})$ and $\boldsymbol{F} = (\varphi_{cd})$ be matrices of variables of respective dimensions $k \times n$ and $j \times n$. One may even assume that k and j are infinite. The (bosonic) variables in $\boldsymbol{B}$ commute with all variables (both those in $\boldsymbol{B}$ and $\boldsymbol{F}$), whereas the (fermionic or grassmanian) variables $\boldsymbol{F}$ are anticommuting among themselves, *i.e.* for φ and φ' in $\boldsymbol{F}$ one has $\varphi\varphi' = -\varphi'\varphi$. We consider that the ring of polynomials $\mathcal{R}_n = \mathcal{R}_{k,j;n} := \mathbb{Q}[\boldsymbol{B}; \boldsymbol{F}]$ is equipped with the group action (expressed here with matrix multiplication)

$$f(\boldsymbol{B}; \boldsymbol{F}) \longmapsto f(P\,\boldsymbol{B}\,\sigma; Q\,\boldsymbol{F}\,\sigma),$$

with P and Q lying respectively in GL_k and GL_j, whilst elements σ of $\mathbb{S}_n$ are considered as $n \times n$ permutation matrices. One says that σ acts diagonally on the $\boldsymbol{B}$ and $\boldsymbol{F}$ variables, and the three actions commute.

Definition 1. *Denoting by $\mathcal{R}_n^{\mathbb{S}_n}$ the subring (of $\mathcal{R}_n$) consisting of $\mathbb{S}_n$-invariants of $\mathcal{R}_n$, the general* **boson-fermion diagonal coinvariant module** *is defined to be the quotient*

$$\boldsymbol{C}_{k,j;n} := \mathcal{R}_{k,j;n}/\langle \mathcal{R}_+^{\mathbb{S}_n}\rangle,$$

where $\mathcal{R}_+^{\mathbb{S}_n}$ stands for the constant term free portion of $\mathcal{R}_n^{\mathbb{S}_n}$. Since $\mathcal{R}_+^{\mathbb{S}_n}$ is globally invariant under the action of $\mathcal{G} = \mathrm{GL}_k \times \mathrm{GL}_j \times \mathbb{S}_n$, there is an induced action of $\mathcal{G}$ on $\boldsymbol{C}_{k,j;n}$.

Consider any $\mathcal{V}$ which is a $\mathcal{G}$-submodule (or stable quotient module) of $\mathcal{R}_{k,j;n}$. As is well known, the decomposition of $\mathcal{V}$ into irreducibles is entirely encoded in the symmetric function expression[1]

$$\mathcal{V}(\boldsymbol{q}; \boldsymbol{u}; \boldsymbol{z}) := \sum_{\mu \vdash n} \Big(\sum_{\lambda,\rho} v_\mu^{\lambda\rho} s_\lambda(\boldsymbol{q}) s_\rho(\boldsymbol{u}) \Big) s_\mu(\boldsymbol{z}), \tag{3.1}$$

where $s_\lambda(\boldsymbol{q})$ and $s_\rho(\boldsymbol{u})$ are respectively characters for (polynomial) irreducible representations of GL_k and GL_j expressed as functions of the "parameters" $\boldsymbol{q} = q_1, \ldots, q_k$ and $\boldsymbol{u} = u_1, \ldots, u_j$;

[1]Observe that various sets of variables are separated by semi-colons.

and $s_\mu(\boldsymbol{z})$ is the Frobenius transform of an $\mathbb{S}_n$-irreducible, in the variables $\boldsymbol{z} = z_1, z_2, \ldots$. The **graded hilbert series** (or $(\mathrm{GL}_k \times \mathrm{GL}_j)$**-character**) of $\mathcal{V}$ is:

$$\begin{aligned} \mathcal{V}(\boldsymbol{q};\boldsymbol{u}) &:= \langle \mathcal{V}(\boldsymbol{q};\boldsymbol{u};\boldsymbol{z}), p_1(\boldsymbol{z})^n \rangle \\ &= \sum_{\mu\vdash n} \Big(\sum_{\lambda,\rho} v_\mu^{\lambda\rho}\, s_\lambda(\boldsymbol{q}) s_\rho(\boldsymbol{u}) \Big) f^\mu, \end{aligned} \tag{3.2}$$

where $f^\mu = \langle s_\mu, p_1^n \rangle$ is the dimension of the irreducible associated to s_μ, which is well known to be equal to the number of standard tableaux of shape μ. For the modules that we consider, the coefficients $v_\mu^{\lambda\rho}$ do not depend on k and j. The dependence on k and j is rather reflected in the fact that some of the functions $s_\lambda(\boldsymbol{q})$ and $s_\rho(\boldsymbol{u})$ when the number of variables is too small, *i.e.* k (resp. j) is less than the number of parts of λ (resp. ρ). In other words, the stable expression for $\mathcal{V}(\boldsymbol{q};\boldsymbol{u};\boldsymbol{z})$ is obtained whenever k and j become large enough[2]. Such modules are said to be **coefficient stable**.

When this is the case, it is often useful to write Equation 3.1 in the form of a "variable free" expression:

$$\mathcal{V} := \sum_{\mu\vdash n} \Big(\sum_{\lambda,\rho} v_\mu^{\lambda\rho}\, s_\lambda \otimes s_\rho \Big) \otimes s_\mu.$$

For the module $\boldsymbol{C}_n = \boldsymbol{C}_{k,j;n}$, the analogous coefficients are denoted $c_\mu^{\lambda\rho}$, and thus

$$\boldsymbol{C}_n(\boldsymbol{q};\boldsymbol{u};\boldsymbol{z}) = \sum_{\mu\vdash n} \Big(\sum_{\lambda,\rho} c_\mu^{\lambda\rho} s_\lambda(\boldsymbol{q}) s_\rho(\boldsymbol{u}) \Big) s_\mu(\boldsymbol{z}), \tag{3.3}$$

$$= \sum_{\mu\vdash n} \boldsymbol{c}_\mu(\boldsymbol{q};\boldsymbol{u}) s_\mu(\boldsymbol{z}), \tag{3.4}$$

so that the $(\mathrm{GL}_k \times \mathrm{GL}_j)$-characters $\boldsymbol{c}_\mu(\boldsymbol{q};\boldsymbol{u})$ are the weighted multiplicities of $\mathbb{S}_n$-irreducibles in $\boldsymbol{C}_n$.

3.1. **Structure of the boson-fermion ring of polynomials.** Using plethystic notation[3], and the notations

$$\Omega(\boldsymbol{q}) := \prod_{i=1}^{k} \frac{1}{1-q_i} = \sum_{n\geq 0} h_n(\boldsymbol{q}) \qquad \text{and} \qquad \Omega'(\boldsymbol{u}) := \prod_{i=1}^{j} 1 + u_i = \sum_{n\geq 0} e_n(\boldsymbol{u}),$$

we have the following.

Proposition 1. *For all n, the Frobenius characteristic of the ring of polynomials in bosonic and fermionic variables is given by the formula*

$$\mathcal{R}_n(\boldsymbol{q};\boldsymbol{u};\boldsymbol{z}) = h_n[\Omega(\boldsymbol{q})\,\Omega'(\boldsymbol{u})\,\boldsymbol{z}]. \tag{3.5}$$

In preparation for what follows, we recall the plethystic identity

$$\Omega[\boldsymbol{q} - \varepsilon\,\boldsymbol{u}] = \sum_{n\geq 0} \sum_{k+j=n} h_k(\boldsymbol{q}) e_j(\boldsymbol{u}) \tag{3.6}$$

$$= \Omega(\boldsymbol{q})\,\Omega'(\boldsymbol{u}). \tag{3.7}$$

[2]It is sufficient to take k larger or equal to n, since this holds for the whole space of polynomials.
[3]See [Ber09] for notions not described here.

By definition, ε is such that $p_i[\varepsilon] = (-1)^i$, so that $p_i[-\varepsilon\, \boldsymbol{u}] = \omega\, p_i(\boldsymbol{u})$. In particular, the classical summation formula for Schur functions gives

$$s_\lambda[\boldsymbol{q} - \varepsilon\, \boldsymbol{u}] = \sum_{\nu\subseteq\lambda} s_\nu(\boldsymbol{q}) s_{\lambda'/\nu'}(\boldsymbol{u}).$$

Exploiting the usual Hall scalar product on symmetric function in $\boldsymbol{z}$, we may consider the **coefficient** $\boldsymbol{r}_\mu(\boldsymbol{q};\boldsymbol{u}) := \langle \mathcal{R}_n(\boldsymbol{q};\boldsymbol{u};\boldsymbol{z}), s_\mu(\boldsymbol{z})\rangle$ of s_μ in $\mathcal{R}_n$. In other terms, the $\boldsymbol{r}_\mu(\boldsymbol{q};\boldsymbol{u})$'s encode the **weighted multiplicities**[4] of the various $\mathbb{S}_n$-isotypic components of $\mathcal{R}_n$. Again as a variable free expression, we write

$$\boldsymbol{r}_\mu := \langle \mathcal{R}_n, s_\mu\rangle = \sum_{\lambda,\rho} r_\mu^{\lambda\rho}\, s_\lambda \otimes s_\rho,$$

and likewise for any coefficient stable $\mathcal{G}$-module $\mathcal{V}$:

$$\boldsymbol{v}_\mu := \langle \mathcal{V}, s_\mu\rangle = \sum_{\lambda,\rho} v_\mu^{\lambda\rho}\, s_\lambda \otimes s_\rho,$$

It is clear that the $\boldsymbol{r}_\mu$ form an upper bound for all $\boldsymbol{v}_\mu$, so that

$$0 \le v_\mu^{\lambda\rho} \le r_\mu^{\lambda\rho}, \qquad \text{for all } \lambda, \rho, \text{ and } \mu.$$

Thus it is interesting to observe that the Cauchy kernel formula implies that

Corollary 3.1. *When $j = 0$, the coefficient $\boldsymbol{r}_\mu(\boldsymbol{q})$ of $s_\mu(\boldsymbol{z})$ in $\mathcal{R}_n(\boldsymbol{q};0;\boldsymbol{z})$ is given by the formula*

$$\begin{aligned}\boldsymbol{r}_\mu(\boldsymbol{q}) &= s_\mu[\Omega(\boldsymbol{q})]\\ &= s_\mu[\textstyle\sum_{i\ge 0} h_i(\boldsymbol{q})].\end{aligned} \tag{3.8}$$

Since $\boldsymbol{r}_\mu(\boldsymbol{q})$ is a symmetric function in $\boldsymbol{q}$, we have a Schur expansion

$$\boldsymbol{r}_\mu(\boldsymbol{q}) = \sum_\lambda c_\lambda^\mu s_\lambda(\boldsymbol{q}).$$

One interesting question associated to this expansion is the so-called restriction problem: View the GL_k-character $s_\mu(\boldsymbol{q})$ as a character of $\mathbb{S}_k$ by restricting to the set of permutation matrices in GL_k. Then c_λ^μ is the multiplicity of the $\mathbb{S}_k$-irreducible character indexed by λ. Therefore, one could solve the problem of describing the restriction of GL_k characters to $\mathbb{S}_k$ by producing a combinatorial description for the Schur expansion of $\boldsymbol{r}_\mu(\boldsymbol{q})$.

4. From boson to boson-fermion

As shown in [Ber13a], there exist a coefficient stable expression for the Frobenius characteristic of the "pure bosonic" (commuting variables) multivariate coinvariant module, which we denote by

$$\mathcal{E}_n = \sum_{\mu\vdash n} \boldsymbol{a}_\mu \otimes s_\mu, \qquad \text{with} \qquad \boldsymbol{a}_\mu := \sum_\lambda a_\mu^\lambda\, s_\lambda.$$

[4]With the q_i and u_j respectively keeping track of homogeneous components in bosonic and fermionic variable ssets.

The integers a_μ^λ are non-vanishing only for partitions λ of size at most $\binom{n}{2} - \eta(\mu')$, and having at most $n - \mu_1$ parts. Recall that $\eta(\mu) := \sum_i \mu_i\,(i-1)$. Expressed in terms of variables, the above expression takes the form

$$\begin{aligned}\mathcal{E}_n(\boldsymbol{q};\boldsymbol{z}) &= \sum_{\mu\vdash n}\Big(\sum_\lambda a_\mu^\lambda\, s_\lambda(\boldsymbol{q})\Big) s_\mu(\boldsymbol{z}) \\ &= \sum_{\mu\vdash n}\Big(\sum_\lambda a_\mu^\lambda\, s_\lambda\Big) \otimes s_\mu.\end{aligned} \tag{4.1}$$

Our main conjecture is that

Conjecture 2 (Boson-Fermion Frobenius)**.** *The multigraded Frobenius characteristic of the boson-fermion diagonal module may be directly calculated from the generic Frobenius characteristic for bosons modules via the universal formula*

$$\begin{aligned}\boldsymbol{C}_n(\boldsymbol{q};\boldsymbol{u};\boldsymbol{z}) &= \mathcal{E}_n(\boldsymbol{q} - \varepsilon\,\boldsymbol{u};\boldsymbol{z}) \\ &= \sum_{\mu\vdash n} \boldsymbol{a}_\mu[\boldsymbol{q} - \varepsilon\,\boldsymbol{u}]\, s_\mu(\boldsymbol{z}) \\ &= \sum_{\mu\vdash n}\Big(\sum_\lambda a_\mu^\lambda\, s_\lambda[\boldsymbol{q} - \varepsilon\,\boldsymbol{u}]\Big) s_\mu(\boldsymbol{z}).\end{aligned} \tag{4.2}$$

In other words $\boldsymbol{c}_\mu(\boldsymbol{q};\boldsymbol{u}) = \boldsymbol{a}_\mu[\boldsymbol{q} - \varepsilon\,\boldsymbol{u}]$.

Thus, the (k,j)-multi-degree enumeration (or $\mathrm{GL}_k \times \mathrm{GL}_j$-character) of the $\mathbb{S}_n$-irreducible component of type μ in $\boldsymbol{C}_n$ is obtained as

$$\begin{aligned}\boldsymbol{a}_\mu[\boldsymbol{q} - \varepsilon\boldsymbol{u}] &= \sum_\lambda a_\mu^\lambda\, s_\lambda[\boldsymbol{q} - \varepsilon\,\boldsymbol{u}] \\ &= \sum_\lambda a_\mu^\lambda \sum_{\nu\subseteq\lambda} s_\nu(q_1,\ldots,q_k) s_{\lambda'/\nu'}(u_1,\ldots,u_j).\end{aligned} \tag{4.3}$$

Observe that the specification of k and j in $\boldsymbol{C}_{k,j;n}(\boldsymbol{q};\boldsymbol{u};\boldsymbol{z})$ is redundant once the parameters $\boldsymbol{q} = q_1,\ldots,q_k$ and $\boldsymbol{u} = u_1,\ldots,u_j$ are specified. We may thus omit them and present the **generic diagonal Boson-Fermion Frobenius** in the form

$$\boldsymbol{C}_n = \sum_{\mu\vdash n}\sum_\lambda a_\mu^\lambda \Big(\sum_{\nu\subseteq\lambda} s_\nu \otimes s_{\lambda'/\nu'}\Big) \otimes s_\mu. \tag{4.4}$$

Clearly, we have

$$\boldsymbol{C}_n(q;0;\boldsymbol{z}) = \widehat{h}_n(\boldsymbol{z})/\widehat{h}_n(1), \qquad \text{with} \qquad \widehat{f}(\boldsymbol{z}) := f[\boldsymbol{z}/(1-q)], \tag{4.5}$$

$$\boldsymbol{C}_n(0;u;\boldsymbol{z}) = \sum_{a=0}^{n-1} u^a\, s_{(n-a,1^a)}(\boldsymbol{z}), \tag{4.6}$$

$$\boldsymbol{C}_n(q+t;0;\boldsymbol{z}) = \nabla(e_n)(q,t;\boldsymbol{z}), \tag{4.7}$$

$$\boldsymbol{C}_n(q_1,\ldots,q_k;0;\boldsymbol{z}) = \mathcal{E}_n(q_1,\ldots,q_k;\boldsymbol{z}), \tag{4.8}$$

$$\boldsymbol{C}_n(0;u_1,\ldots,u_j;\boldsymbol{z}) = \sum_{\mu\vdash n}\Big(\sum_\lambda a_\mu^\lambda\, s_{\lambda'}(u_1,\ldots,u_j)\Big) s_\mu(\boldsymbol{z}), \tag{4.9}$$

where the ∇ operator occurring in Equation 4.7 is the Macdonald "eigenoperator" introduced in Section 1.

5. Special cases

It has been conjectured[5] in [BPR12] that

$$\boldsymbol{C}_n(1+q+t;0;\boldsymbol{z}) = \sum_{\alpha\preceq\beta} q^{\mathrm{dist}(\alpha,\beta)}\,\mathbb{L}_\beta(t;\boldsymbol{z}), \tag{5.1}$$

where the sum is over pairs of elements of the Tamari lattice, and $\mathrm{dist}(\alpha,\beta)$ is the length of the longest chain going from α to β. Here, $\mathbb{L}_\beta(t;\boldsymbol{z})$ stands for the LLT-polynomial associated to the Dyck-path β (see [GLk20] for more details). Furthermore, it has been conjectured by N. Bergeron-Machacek-Zabrocki[6] that

$$\boldsymbol{C}_n(q;u;\boldsymbol{z}) = \sum_{k=0}^{n-1}\sum_{\lambda\vdash n}\sum_{\tau\in\mathrm{SYT}(\lambda)} q^{\alpha(\tau)} \begin{bmatrix}\mathrm{des}(\tau)\\ k\end{bmatrix}_q u^k\, s_\lambda(\boldsymbol{z}), \tag{5.2}$$

where, for a standard tableau τ of shape λ, one sets

$$\alpha(\tau) := \mathrm{maj}(\tau) - k\,\mathrm{des}(\tau) + \binom{k}{2}.$$

In [KR22], Kim and Rhoades show that

$$\boldsymbol{C}_n(0;u+v;\boldsymbol{z}) = \sum_{a+b\,\leq\, n-1} u^a\, v^b\,\Big(s_{(n-a,1^a)}\star s_{(n-b,1^b)} - s_{(n-(a-1),1^{a-1})}\star s_{(n-(b-1),1^{b-1})}\Big)(\boldsymbol{z}), \tag{5.3}$$

with "$\star$" standing for the Kronecker product. Denoting by $g^\mu_{\alpha,\beta}$ the **Kronecker coefficients**:

$$g^\mu_{\alpha,\beta} := \langle s_\alpha\star s_\beta, s_\mu\rangle,$$

one may reformulate the above as

$$\boldsymbol{C}_n(0;u+v;\boldsymbol{z}) = \sum_{\mu\vdash n}\left(\sum_{b+d\,\leq\, n-1} u^b\, v^d\,(g^\mu_{(a\,|\,b),(c\,|\,d)} - g^\mu_{(a+1\,|\,b-1),(c+1\,|\,d-1)})\right) s_\mu(\boldsymbol{z}), \tag{5.4}$$

using the Frobenius notation $(a\,|\,b) = (a+1,1^b)$ for hook-shaped partitions. For each term in the inner sum above, we assume that $a+b=n-1$ (likewise for c and d). The differences are know to be positive (see [Rem89]). The various results (see [SS16]) on the stability of Kronecker coefficients certainly have a bearing here, since they imply corresponding stabilities for the coefficients of the s_μ.

In Section 1 we mentioned that Zabrocki has conjectured

$$\boldsymbol{C}_n(q+t;u;\boldsymbol{z}) = \sum_{a=0}^{n-1} u^a \Delta'_{e_{n-a-1}}(e_n(\boldsymbol{z})). \tag{5.5}$$

The parameters q and t arise from the application of the operators Δ'_{e_k}. It follows that Equation 5.2 may also be written as

$$\boldsymbol{C}_n(q;u;\boldsymbol{z}) = \sum_{a=0}^{n-1} u^a \Delta'_{e_{n-a-1}}(e_n(\boldsymbol{z}))\Big|_{t=0}. \tag{5.6}$$

[5]Notice that one of the parameters is equal to 1. This is because the lacking "statistic" on Dyck-path pairs (α,β) is not yet known.

[6]In fact, this follows from Equation 5.5, via a formula of Haglund, Rhoades and Shimozono (see [HRS18]).

Finally, we have

$$\boldsymbol{C}_n(1;2;\boldsymbol{z}) = \frac{1}{2}\sum_{\mu\vdash n} 2^{\ell(\mu)}(-1)^{n-\ell(\mu)}\binom{\ell(\mu)}{d_1,\dots,d_n}\, p_\mu(\boldsymbol{z}), \tag{5.7}$$

where $d_i = d_i(\mu)$ stands for the number of parts of of size i in μ. Equivalently, in terms of the elementary basis, one has the following: Denote the set of all compositions whose parts rearrange to μ by by $R(\mu)$. Then

$$\boldsymbol{C}_n(1;2;\boldsymbol{z}) = \sum_{\mu\vdash n} e_\mu(\boldsymbol{z}) \sum_{\alpha\in R(\mu)} \alpha_1(2\alpha_2-1)\cdots(2\alpha_{\ell(\alpha)}-1). \tag{5.8}$$

Finally, D'Adderio-Iraci-Wyngaerd conjecture in [DIVW21, Conj. 8.2.] the more inclusive identity:

$$\begin{aligned}\boldsymbol{C}_n(q+t;u+v;\boldsymbol{z}) &= \sum_{k=1}^{n-1}\sum_{i+j=k} u^i v^j \Theta_{e_i e_j}\nabla(e_{n-k}),\\ &= \sum_{k=0}^{n-1}\Theta_{e_k[(u+v)\boldsymbol{z}]}(\nabla(e_{n-k}))\end{aligned} \tag{5.9}$$

where, for any symmetric functions g and f, $\Theta_g f$ is defined as

$$\Theta_g f(\boldsymbol{z}) := \Pi\, g^*\, \Pi^{-1} f(\boldsymbol{z}), \qquad \text{setting} \qquad g^*(\boldsymbol{z}) := g[\boldsymbol{z}/(1-t)(1-q)].$$

Here, Π stands for the Macdonald eigenoperator having as eigenvalues for $\widetilde{H}_\mu$ the product $\prod_{(i,j)\in\mu/(1)}(1-q^it^j)$, for (i,j) running over cartesian coordinates of cells in μ (omitting the cell $(0,0)$).

$$\Pi\,\widetilde{H}_\mu(q+t;\boldsymbol{z}) := \prod_{(i,j)\in\mu/(1)}(1-q^it^j)\;\widetilde{H}_\mu(q+t;\boldsymbol{z}).$$

It should be apparent that Equation 5.9 specializes at $u=v=0$ to give ∇e_n, the diagonal coinvariant case mentioned in Section 1, proved by Mark Haiman [Hai01]. When $q=t=0$, it was shown in [IRR22] that Equation 5.9 specializes to Equation 5.3. However, at this time, it seems no other specialization is known to hold.

Table 1 summarizes the overall situation[7]. Conjecture 2 essentially states that all entries

$k \setminus j$	0	1	2	$\cdots$	j
0	1	(4.6)	(5.3)	$\cdots$	(4.9)
1	(4.5)	(5.2)	(5.7)	$\cdots$	
2	(4.7)	(5.5)	(5.9)	$\cdots$	
3	(5.1)			$\cdots$	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
k	(4.8)			$\cdots$	(4.4)

Table 1. Overall situation for the various formulas.

of Table 1 may be obtained from Equation 4.8 (or equivalently from Equation 4.9). It is interesting to observe that one obtains polynomial expressions in k and j, when setting all

[7]With k standing for the numbers of sets of commuting variables, and j for those that are anticommuting.

parameters $q_i = 1$ and $u_j = 1$. More precisely, writing $\boldsymbol{C}_n(k; j; \boldsymbol{z})$ for the resulting expression, we have the following.

Proposition 2. *The coefficients of each $s_\mu(\boldsymbol{z})$, in the Schur expansion of $\boldsymbol{C}_n(k; j; \boldsymbol{z})$, is a polynomial in k and j, with coefficients in $\mathbb{Q}$. Hence, this is also the case for the associated dimension*[8] *$\boldsymbol{C}_n(k; j)$.*

6. Links with the main conjecture

The conjecture of Equation 5.5 directly led to our main conjecture, in view of an elegant link (first stated in 2017, but only recently published) between the generic expression for $\mathcal{E}_n$ and the effect of the Δ'_{e_k} operators on e_n. The precise relevant statement (see [GLk20, Conj. 1]) says that

Conjecture 3 (Delta via skew)**.** *For all a,*

$$\begin{aligned}\Big((e_a^\perp \otimes \mathrm{Id})\, \mathcal{E}_n\Big)\,(q+t; \boldsymbol{z}) &= \sum_{\mu \vdash n} (e_a^\perp \boldsymbol{a}_\mu)(q+t)\, s_\mu(\boldsymbol{z}) \\ &= \Delta'_{e_{n-a-1}}(e_n(\boldsymbol{z})). \end{aligned} \tag{6.1}$$

In other words, we get $\Delta'_{e_{n-a-1}}(e_n(\boldsymbol{z}))$ from $\mathcal{E}_n$, first by applying the skew operator $e_a^\perp$ to the various s_λ, and then by evaluation of the resulting expression in q, t. To see how this relates to our general conjecture, we recall that the effect on a symmetric function $f(\boldsymbol{q}) = f(q_1, \ldots, q_k)$ of the operator $\sum_{a=0}^{n-1} u^a e_a^\perp$ may be simply expressed in plethystic notation as

$$\sum_{a=0}^{n-1} u^a e_a^\perp f(\boldsymbol{q}) = f[\boldsymbol{q} - \varepsilon\, u].$$

Thus, assuming that Conjecture 3 holds, we see that Equation 5.5 may be coined as

$$\boldsymbol{C}_n(q+t; u; \boldsymbol{z}) = \mathcal{E}_n[q+t-\varepsilon\, u; \boldsymbol{z}].$$

This immediately[9] suggested that the more general formula of Conjecture 2 should hold. All experiments confirmed this. Moreover, it agrees with all known or conjectured formulas (due to various researchers) for the dimensions of $\boldsymbol{C}_n(k; j)$ as functions of n. These are displayed in Table 2. Here, $\left\{ {n \atop k} \right\}$ stands for the Stirling numbers of the second kind. Currently known or conjectured formulas for the multiplicities of alternating component in $\boldsymbol{C}_n$ appear in Table 3, where $\mathfrak{s}(n) = \frac{1}{n}\sum_{i=0}^{n-1} \binom{n}{i}\binom{n}{i+1} 2^i$ denotes the n^{th} small Schröder number, and F_n stands for the n^{th} Fibonacci number.

Formulas for low degree components of the corresponding general expressions have also been conjectured to hold. The first of these, see [Ber13a], states that

$$\boldsymbol{C}_n(\boldsymbol{q}; 0; \boldsymbol{z}) =_{(n)} \frac{h_n[\Omega(\boldsymbol{q})\, \boldsymbol{z}]}{h_n[\Omega(\boldsymbol{q})]}, \tag{6.2}$$

where $\Omega(\boldsymbol{q}) = 1 + h_1(\boldsymbol{q}) + h_2(\boldsymbol{q}) + \ldots$; with equality holding for terms of degree at most n in the $\boldsymbol{q}$ variables. It may be worth recalling here that, when $\boldsymbol{q}$ consists of only one variable,

[8] Obtained by replacing each $s_\mu(\boldsymbol{z})$ by the number, f^μ, of standard tableaux of shape μ.

[9] During the January 2019 Banff meeting where Mike Zabrocki presented his conjecture for the first time.

$k \setminus j$	0	1	2
0	1	2^{n-1}	$\binom{2n-1}{n}$
1	$n!$	$\sum_{i=1}^{n} i! \left\{ {n \atop i} \right\}$	$2^{n-1}\, n!$
2	$(n+1)^{n-1}$	$\sum_{i=0}^{n+1} \binom{n+1}{i} \frac{i^n}{2(n+1)}$	?
3	$2^n (n+1)^{n-2}$	?	?

TABLE 2. Dimensions of $\boldsymbol{C}_n(k; j)$.

$k \setminus j$	0	1	2	3
0	0	1	n	$n^2 - n + 1$
1	1	2^{n-1}	3^{n-1}	$2^{-1} F_{3n-1}$
2	$\frac{1}{n+1}\binom{2n}{n}$	$\mathfrak{s}(n)$	$\frac{2^{n-1}}{n+1}\binom{2n}{n}$	?
3	$\frac{2}{n(n+1)}\binom{4\,n+1}{n-1}$	?	?	?

TABLE 3. Coefficients of $s_{1^n}(\boldsymbol{z})$ in $\boldsymbol{C}_n(k; j; \boldsymbol{z})$.

the above is well known to be an equality. Indeed, this the symmetric group case of the Chevalley-Shephard-Todd theorem.

A slightly stronger form of a conjecture stated in [DIVW21], is that the difference

$$\delta_n(q + t, \boldsymbol{u}; \boldsymbol{z}) := \boldsymbol{C}_n(q + t; \boldsymbol{u}; \boldsymbol{z}) - \sum_{k=0}^{n-1} \Theta_{e_k[\boldsymbol{u}\boldsymbol{z}]}(\nabla(e_{n-k})) \tag{6.3}$$

is Schur positive in all three sets of variables: $\boldsymbol{u} = \{u_1, u_2, \ldots, u_r\}$ (for any r), $\boldsymbol{z}$, and $\{q, t\}$. Furthermore, the expression $\sum_{k=0}^{n-1} \Theta_{e_k[\boldsymbol{u}\boldsymbol{z}]}(\nabla(e_{n-k}))$ is itself Schur positive in all three sets of variables. For small values of n, we have

$$\begin{aligned}
&\delta_2(q + t; \boldsymbol{u}; \boldsymbol{z}) = 0,\\
&\delta_3(q + t; \boldsymbol{u}; \boldsymbol{z}) = s_{111}(\boldsymbol{u})\, s_{111}(\boldsymbol{z}),\\
&\delta_4(q + t; \boldsymbol{u}; \boldsymbol{z}) = s_{1111}(\boldsymbol{u})\, s_{22}(\boldsymbol{z})\\
&\qquad + (s_{11111}(\boldsymbol{u}) + (q + t + 1)\,(s_{111}(\boldsymbol{u}) + s_{1111}(\boldsymbol{u})) + s_{211}(\boldsymbol{u}))\, s_{211}(\boldsymbol{z})\\
&\qquad + (s_{111111}(\boldsymbol{u}) + (q + t)\, s_{11111}(\boldsymbol{u}) + s_{2111}(\boldsymbol{u}) + (q + t + 1)\, s_{211}(\boldsymbol{u})\\
&\qquad\qquad + \left(q^2 + qt + t^2 + q + t\right)(s_{111}(\boldsymbol{u}) + s_{1111}(\boldsymbol{u})))\, s_{1111}(\boldsymbol{z}).
\end{aligned}$$

It is interesting to observe, in view of our main conjecture, that a similar approximation Θ-formula may be given for the purely commuting variables context. More precisely, the corresponding statement is that the difference

$$\delta_n'(q+t+\boldsymbol{q};\boldsymbol{z}) := \mathcal{E}_n[q+t+\boldsymbol{q};\boldsymbol{z}] - \sum_{k=0}^{n-1} \Theta_{h_k[\boldsymbol{q}\,\boldsymbol{z}]}(\nabla(e_{n-k})) \tag{6.4}$$

is always Schur positive. For small values of n, we have

$$\begin{aligned}
\delta_2'(q+t+\boldsymbol{q};\boldsymbol{z}) &= 0,\\
\delta_3'(q+t+\boldsymbol{q};\boldsymbol{z}) &= s_3(\boldsymbol{q})\, s_{111}(\boldsymbol{z}),\\
\delta_4'(q+t+\boldsymbol{q};\boldsymbol{z}) &= s_4(\boldsymbol{q})\, s_{22}(\boldsymbol{z})\\
&\qquad + (s_5(\boldsymbol{q}) + (q+t+1)\,(s_3(q)+s_4(\boldsymbol{q})) + s_{31}(\boldsymbol{q}))\, s_{211}(\boldsymbol{z})\\
&\qquad + (s_6(\boldsymbol{q}) + (q+t)\, s_5(\boldsymbol{q}) + s_{41}(\boldsymbol{q}) + (q+t+1)\, s_{31}(\boldsymbol{q})\\
&\qquad\qquad + \left(q^2+qt+t^2+q+t\right)(s_3(\boldsymbol{q})+s_4(\boldsymbol{q})))\, s_{1111}(\boldsymbol{z}).
\end{aligned}$$

Clearly, one goes from δ_n to δ_n' by replacing the $s_\mu(\boldsymbol{u})$'s by corresponding $s_{\mu'}(\boldsymbol{q})$. We also get the "approximation"

$$\mathcal{T}(\boldsymbol{u};\boldsymbol{z}) := \sum_{k=0}^{n-1} \Theta_{e_k[\boldsymbol{u}\boldsymbol{z}]}(\nabla(e_{n-k})(\boldsymbol{z}))\Big|_{q=t=0}, \tag{6.5}$$

of $\boldsymbol{C}_n(0;\boldsymbol{u};\boldsymbol{z})$, by setting $q=t=0$ in Equation 6.3; as well as an equivalent approximation for $\boldsymbol{C}_n(\boldsymbol{q};0;\boldsymbol{z})$, simply by replacing all $s_\mu(\boldsymbol{u})$'s by $s_{\mu'}(\boldsymbol{q})$.

7. An explicit example

With $n=3$, we have

$$\mathcal{E}_3 = 1\otimes s_3 + (s_1+s_2)\otimes s_{21} + (s_{11}+s_3)\otimes s_{111},$$

hence

$$\begin{aligned}
\boldsymbol{C}_3 = (1\otimes 1)\otimes s_3 &+ ((s_1+s_2)\otimes 1 + s_1\otimes s_1 + 1\otimes(s_1+s_{11}))\otimes s_{21}\\
&+ ((s_{11}+s_3)\otimes 1 + (s_1+s_2)\otimes s_1 + s_1\otimes s_{11} + 1\otimes(s_2+s_{111}))\otimes s_{111}.
\end{aligned}$$

The above general formula specializes as:

$$\begin{aligned}
\boldsymbol{C}_3(q+t;u+v;\boldsymbol{z}) = s_3(\boldsymbol{z}) &+ \Big((q^2+qt+t^2) + (q+t)(u+v) + (u+v+uv)\Big)\, s_{21}(\boldsymbol{z})\\
&+ \Big((q^3+q^2t+qt^2+t^3+qt) + (q^2+qt+t^2+q+t)(u+v)\\
&\qquad + (q+t)uv + (u^2+uv+v^2+0)\Big)\, s_{111}(\boldsymbol{z}).
\end{aligned}$$

Observe in this last expression that there is no contribution for the term s_{111}, as it is evaluated in only two variables. Observe also that the two special cases

$$\boldsymbol{C}_3(\boldsymbol{q};0;\boldsymbol{z}) = s_3(\boldsymbol{z}) + (s_1(\boldsymbol{q})+s_2(\boldsymbol{q}))\, s_{21}(\boldsymbol{z}) + (s_{11}(\boldsymbol{q})+s_3(\boldsymbol{q}))\, s_{111}(\boldsymbol{z}),$$

$$\boldsymbol{C}_3(0;\boldsymbol{u};\boldsymbol{z}) = s_3(\boldsymbol{z}) + (s_1(\boldsymbol{u})+s_{11}(\boldsymbol{u}))\, s_{21}(\boldsymbol{z}) + (s_2(\boldsymbol{u})+s_{111}(\boldsymbol{u}))\, s_{111}(\boldsymbol{z}),$$

may be deduced from each other. In other words, the general boson world knows the fermion one; and vice-versa.

The polynomial expressions of the (k, j)-dimension[10] and the Frobenius characteristic for $\boldsymbol{C}_3$ are, respectively,

$$\boldsymbol{C}_3(k; j) = \frac{1}{6}(k + j + 1)\,(k^2 + 2\,k\,j + j^2 + 11\,k + 5\,j + 6) \qquad \text{and}$$

$$\boldsymbol{C}_3(k; j; \boldsymbol{z}) = s_3(\boldsymbol{z}) + \frac{1}{2}(k^2 + 2\,k\,j + j^2 + 3\,k + j)\,s_{21}(\boldsymbol{z}) + \frac{1}{6}(k^3 + 3\,k^2 j + 3\,k\,j^2 + j^3 + 6\,k^2 + 6\,k\,j - k + 5\,j)\,s_{111}(\boldsymbol{z}).$$

Explicit dimensions values for $n = 3, 4$, and 5:

$n=3$	0	1	2
0	1	4	10
1	6	13	23
2	16	28	45
3	32	50	74

$n=4$	0	1	2
0	1	8	35
1	24	75	192
2	125	288	597
3	400	785	1440

$n=5$	0	1	2
0	1	16	126
1	120	541	1920
2	1296	3936	10541
3	6912	17072	38912

8. Multilabeled paths

We start by describing monomial expansions for super nabla operators.

A k^n-Dyck path $D \in D_{k^n}$ is a sequence of North (N) and East (E) unit steps from $(0, 0)$ to (kn, n) which stay weakly above the main diagonal line $y = kx$. Alternatively, D is a path which only touches the path $E(E^k N)^n$ at the beginning and end. Denote by $\mathrm{area}(D)$ the number of complete lattice cells between D and the main diagonal. A multilabeled Dyck path $P \in \mathrm{MD}_{k^n}$ is a pair (D, w) where $D \in D_{k^n}$ and w is a word $w = w^1 w^2 \cdots w^n$, whose letters are themselves words $w^i = (w^i_1, \ldots, w^i_{k+1}) \in \mathbb{N}_+^{k+1}$ of length $k + 1$ satisfying the following rule:

$$\text{(the number of East steps in } D \text{ along the line } y = i) \geq \#\{j : w^i_j \geq w^{i+1}_j\} \tag{8.1}$$

We view w^i as the labels associated to the i-th North step in P. For instance, Figure 8 gives a multilabeled Dyck path $P \in \mathrm{MD}_{3^5}$.

To every $P = (D, w) \in \mathrm{MD}_{k^n}$ we associate the monomial weight

$$\mathcal{X}^P = \prod_{i=1}^{n} v_{1,w^i_1} \cdots v_{k,w^i_k} z_{w^i_{k+1}};$$

and the area of P is given by $\mathrm{area}(P) = \mathrm{area}(D)$.

[10] Setting $\boldsymbol{q} = \underbrace{(1, 1, \ldots, 1)}_{k \text{ copies}}$ and $\boldsymbol{u} = \underbrace{(1, 1, \ldots, 1)}_{j \text{ copies}}$.

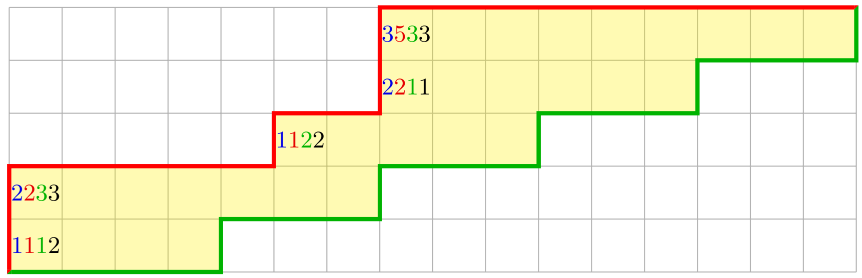

Figure 3. A multilabeled 3^5-Dyck path with area 11 and monomial weight $y_{1,1}^2 y_{1,2}^2 y_{1,3} y_{2,1}^2 y_{2,2}^2 y_{2,5} y_{3,1}^2 y_{3,2} y_{3,3}^2 z_1 z_2^2 z_3^2$.

Proposition 3 ([BHIR23])**.** *For any n and k,*

$$\nabla_{\boldsymbol{v}_1} \cdots \nabla_{\boldsymbol{v}_k} e_n(\boldsymbol{z})\Big|_{t=1} = \sum_{P \in \mathrm{MD}_{k^n}} q^{\mathrm{area}(P)} \mathcal{X}^P. \tag{8.2}$$

This leads to our first open problem regarding multilabeled paths:

Conjecture 4. *There exists a statistic $d : \mathrm{MD}_{k^n} \to \mathbb{N}$ which gives*

$$\nabla_{\boldsymbol{v}_1} \cdots \nabla_{\boldsymbol{v}_k} e_n(\boldsymbol{z}) = \sum_{P \in \mathrm{MD}_{k^n}} q^{\mathrm{area}(P)} t^{d(P)} \mathcal{X}^P. \tag{8.3}$$

A path $EE^{\alpha_1}NE^{\alpha_2}N \cdots E^{\alpha_n}N$ is called a γ, k^n-staircase if $(\alpha_1 - k, \ldots, \alpha_n - k)$ is a rearrangement of $\gamma, 0^{n-\ell(\gamma)}$. A pair of lattice paths (D_1, D_2) is a γ, k^n-Dyck path if D_2 is a γ, k^n-staircase, and D_1 touches D_2 only at the beginning and end. The area of (D_1, D_2) is given by the number of lattice cells between the two paths that never touch the bottom path, D_2. A labeled γ, k^n-Dyck path is a pair $P = ((D_1, D_2), w) \in \mathrm{MD}_{\gamma,k^n}$ where (D_1, D_2) is a γ, k^n-Dyck path, and w is a word satisfying the same multilabeling rule stated in Equation 8.1. The area of P, $\mathrm{area}(P)$, is the area of (D_1, D_2). The return of P, $\mathrm{ret}(P)$, is the first i for which D_1 and D_2 are one unit apart along the line $y = i$. Let $\mathrm{RMD}_{\gamma,k^n}$ be the set of pairs (P, r) with $P \in \mathrm{MD}_{\gamma,k^n}$ and $1 \le r \le \mathrm{ret}(P)$. This is the set of multilabeled γ, k^n-Dyck paths with a marked row before the return of P. For instance, Figure 8 gives a multilabeled γ, k^n-Dyck path with $\gamma = (2,1)$, $k = 2$, and return 4. The circle in row 3 represents the choice of $r = 3$ for the pair $(P, 3) \in \mathrm{RMD}_{(2,1),2^6}$.

Proposition 4 ([BHIR23])**.** *For any n, k, and nonempty partition γ,*

$$\Delta_{m_\gamma} \nabla_{\boldsymbol{v}_1} \cdots \nabla_{\boldsymbol{v}_k} (-1)^{n-1} p_n(\boldsymbol{z})\Big|_{t=1} = \sum_{P \in \mathrm{RMD}_{\gamma,k^n}} q^{\mathrm{area}(P)} \mathcal{X}^P.$$

This leads to another open question.

Conjecture 5. *There exists a statistic $d : \mathrm{RMD}_{\gamma,k^n} \to \mathbb{N}$ which gives*

$$\nabla \nabla_{\boldsymbol{v}_1} \cdots \nabla_{\boldsymbol{v}_k} (-1)^{n-1} p_n(\boldsymbol{z}) = \sum_{P \in \mathrm{RMD}_{1^n,k^n}} q^{\mathrm{area}(P)} t^{d(P)} \mathcal{X}^P.$$

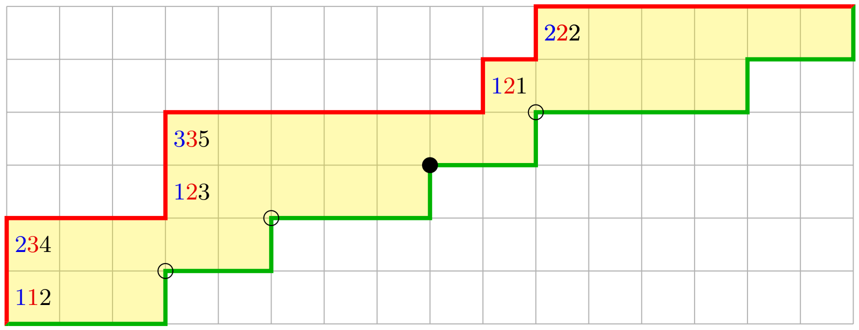

FIGURE 4. A multilabeled $(2,1), 2^6$-Dyck path with area 10 and monomial weight $v_{1,1}^3 v_{1,2}^2 v_{1,3} v_{2,1} v_{2,2}^3 v_{2,3}^2 z_1 z_2^2 z_3 z_4 z_5$.The unfilled circles represent other potential choices for a mark.

8.1. **Schur expansions.** Given a standard tableau $T \in \mathrm{SYT}(\lambda)$ of shape $\lambda \vdash n$, we can construct a lattice word $r = (r_1, \ldots, r_n)$ by letting r_i be the row of λ in which i appears. In such a case, we would write $\lambda(r) = \lambda$. For instance, $\lambda(1,2,1,3,2,1) = (3,2,1)$. All lattice words are attained in this way. Given a labeling $w = w^1 \cdots w^n$, where $w^i = (w_1^i, \ldots, w_{k+1}^i) \in \mathbb{N}_+^{k+1}$, we say that w is lattice if for every j, $(w_j^1, w_j^2, \ldots, w_j^n)$ is a lattice word. A lattice multilabeled γ, k^n-Dyck path is a multilabeled γ, k^n-Dyck path whose labeling is lattice; the collection of all such multilabeled paths is denoted by $\mathrm{LMD}_{\gamma,k^n}$. Given a lattice multilabeled γ, k^n-Dyck path P with labelling w, we will denote $\lambda(w_j^1, w_j^2, \ldots, w_j^n)$ by $\lambda^j(P)$. We then have the following results.

Proposition 5 ([BHIR23])**.** *For any n and k,*

$$\nabla_{\boldsymbol{v}_1} \cdots \nabla_{\boldsymbol{v}_k} e_n(\boldsymbol{z})\Big|_{t=1} = \sum_{P \in \mathrm{LMD}_{k^n}} q^{\mathrm{area}(P)} s_{\lambda^1(P)}(\boldsymbol{v}_1) \cdots s_{\lambda^k(P)}(\boldsymbol{v}_k) s_{\lambda^{k+1}(P)}(\boldsymbol{z}).$$

For any nonempty partition γ,

$$\Delta_{m_\gamma} \nabla_{\boldsymbol{v}_1} \cdots \nabla_{\boldsymbol{v}_k} (-1)^{n-1} p_n(\boldsymbol{z})\Big|_{t=1} = \sum_{P \in \mathrm{RLMD}_{\gamma,k^n}} q^{\mathrm{area}(P)} s_{\lambda^1(P)}(\boldsymbol{v}_1) \cdots s_{\lambda^k(P)}(\boldsymbol{v}_k) s_{\lambda^{k+1}(P)}(\boldsymbol{z}).$$

This again, leads to the following open question.

Conjecture 6. *The symmetric functions $\nabla_{\boldsymbol{v}_1} \cdots \nabla_{\boldsymbol{v}_k} e_n(\boldsymbol{z})$ and $\Delta_{s_\nu} \nabla_{\boldsymbol{v}_1} \cdots \nabla_{\boldsymbol{v}_k} (-1)^{n-1} p_n(\boldsymbol{z})$ (for nonempty partitions ν) are simultaneously Schur positive in each set of variables. Furthermore, this would imply that there exists statistics d, d_R such that*

$$\nabla_{\boldsymbol{v}_1} \cdots \nabla_{\boldsymbol{v}_k} e_n(\boldsymbol{z}) = \sum_{P \in \mathrm{LMD}_{k^n}} q^{\mathrm{area}(P)} t^{d(P)} s_{\lambda^1(P)}(\boldsymbol{v}_1) \cdots s_{\lambda^k(P)}(\boldsymbol{v}_k) s_{\lambda^{k+1}(P)}(\boldsymbol{z}).$$

and

$$\begin{aligned} &\Delta_{s_\nu} \nabla_{\boldsymbol{v}_1} \cdots \nabla_{\boldsymbol{v}_k} (-1)^{n-1} p_n(\boldsymbol{z}) \\ &\quad = \sum_{\gamma \vdash |\nu|} \sum_{T \in \mathrm{SSYT}(\nu,\gamma)} \sum_{P \in \mathrm{RLMD}_{k^n}} q^{\mathrm{area}(P)} t^{d_R(P,T)} s_{\lambda^1(P)}(\boldsymbol{v}_1) \cdots s_{\lambda^k(P)}(\boldsymbol{v}_k) s_{\lambda^{k+1}(P)}(\boldsymbol{z}). \end{aligned}$$

In the second summation, $T \in \mathrm{SSYT}(\nu, \gamma)$ ranges over all semistandard tableaux T of shape ν with content $1^{\gamma_1}2^{\gamma_2}\cdots$.

8.2. **e-Positivties when $q = 1 + u$.** One of the remarkable positivities regarding these operators is an e-positivity phenomenon which occurs when we set $q = 1+u$. To best describe this e-positivity, we first go back to the definition of multilabeled k^n-Dyck paths. Let $\mathrm{MD}^r_{k^n}$ be the set of multilabeled Dyck paths whose labels have r components. We can also define the set $\mathrm{LMD}^r_{k^n}$ to be the subset of $\mathrm{MD}^r_{k^n}$ in which the labels are lattice. Given an element $P \in \mathrm{MD}^r_{k^n}$ we construct a composition $\eta(P)$ as follows:

Suppose the top path of P is given by $N^{a_1}E^{b_1}\cdots N^{a_k}E^{a_k}$, where all a_i and b_i are nonzero; and let $w = w^1 \cdots w^n$ be the sequence of labels when read from bottom to top. Let c_i be the number of nonascents between w^i and w^{i+1}:

$$c_i = \#\{j : w^i_j \geq w^{i+1}_j\}.$$

Construct a new path $N^{a_1}E^{b_1-c_1}N^{a_2}E^{b_2-c_2}\cdots N^{a_k}E^{a_k} = N^{a'_1}E^{b'_1}\cdots N^{a'_r}E^{b'_r}$, where all a'_i and b'_i are nonzero. Then define $\eta(P) = (a'_1, \ldots, a'_r)$.

We now let $\mathrm{Area}(P)$ denote the set of area cells of P. Then we can rewrite Proposition 3 as follows:

$$\nabla_{\boldsymbol{v}_1}\cdots\nabla_{\boldsymbol{v}_k} e_n(\boldsymbol{z})\Big|_{t=1} = \sum_{P\in \mathrm{LMD}^k_{k^n}} q^{|\,\mathrm{Area}(P)|} s_{\lambda^1(P)}(\boldsymbol{v}_1)\cdots s_{\lambda^k(P)}(\boldsymbol{v}_k) e_{\eta(P)}(\boldsymbol{z}) \tag{8.4}$$

The e-positivity phenomenon would give the following conjecture.

Conjecture 7. *There exists a statistic d such that for any n and k, we have*

$$\nabla_{\boldsymbol{v}_1}\cdots\nabla_{\boldsymbol{v}_k} e_n(\boldsymbol{z})\Big|_{q=1+u} = \sum_{P\in \mathrm{LMD}^k_{k^n}}\ \sum_{S\subseteq \mathrm{Area}(P)} u^{|S|} t^{d(P,S)} s_{\lambda^1(P)}(\boldsymbol{v}_1)\cdots s_{\lambda^k(P)}(\boldsymbol{v}_k) e_{\eta(P)}(\boldsymbol{z})$$

A similar e-positivity can be found for applications to power sums.

Conjecture 8. *There exists a statistic d_R such that for any n and k, we have*

$$\begin{aligned}\nabla_{\boldsymbol{v}_1}\cdots\nabla_{\boldsymbol{v}_k}\nabla(-1)^{n-1}p_n(\boldsymbol{z})\Big|_{q=1+u} \\ = \sum_{P\in \mathrm{RLMD}^k_{1^n,k^n}}\ \sum_{S\subseteq \mathrm{Area}(P)} u^{|S|} t^{d(P,S)} s_{\lambda^1(P)}(\boldsymbol{v}_1)\cdots s_{\lambda^k(P)}(\boldsymbol{v}_k) e_{\eta(P)}(\boldsymbol{z})\end{aligned}$$

8.3. **The See-Conjecture.** There is an even stronger e-positivity which arises from setting $q = 1 + u$ and $t = 1 + v$.

Conjecture 9 (The See-Conjecture [BHIR23])**.** *The coefficients $C_{\lambda^0,\lambda^1,\ldots,\lambda^k}(u, v)$ appearing in the expansion*

$$\nabla_{\boldsymbol{v}_1}\cdots\nabla_{\boldsymbol{v}_k} e_n(\boldsymbol{z})\Big|_{\substack{q=1+u\\ t=1+v}} = \sum_{\lambda^0,\lambda^1,\ldots,\lambda^k\vdash n} C_{\lambda^0,\lambda^1,\ldots,\lambda^k}(u,v) e_{\lambda^0}(\boldsymbol{z}) e_{\lambda^1}(\boldsymbol{v}_1)\cdots e_{\lambda^k}(\boldsymbol{v}_k)$$

are Schur positive symmetric functions in u and v.

The name "See-Conjecture" describes the fact that we have a simultaneous Schur positivity in u and v, and e-positivity in the variables $\boldsymbol{z}, \boldsymbol{v}_1, \ldots, \boldsymbol{v}_k$. A seemingly stronger conjecture is presented in Section 9.

8.4. **Other special cases.** Recall that (from Sections 1 and 2) we have

$$\langle \nabla_{\boldsymbol{v}} e_n(\boldsymbol{z}), s_{n-k,1^k}(\boldsymbol{v}) \rangle = \Delta'_{e_k} e_n(\boldsymbol{z}). \tag{8.5}$$

This means that the symmetric function side of the Delta Conjecture appears as a special case in the expansion of $\nabla_{\boldsymbol{v}} e_n(\boldsymbol{z})$. By Proposition 3, a basic construction (see [HHL+05] or [Hag08, Chapter 6, p. 99-101]) well-known to researchers in this area shows that the left-hand-side of (8.5) equals the right-hand-side of the $k=1$ case of (8.2), restricted to those $P \in \mathrm{MD}_{1^n}$ whose reading word in the $\boldsymbol{v}$ variables is a proper shuffle of the increasing sequence of "big cars" $k+1, k+2, \ldots, n$ and the decreasing sequence of "small cars" $k, k-1, \ldots, 1$. Given such a P, assume row_i contains a big car b (in the $\boldsymbol{v}$ variables). The shuffle condition then implies the v variable car in row_{i+1} is less than b. The defining condition (8.1) now implies that if c is the car in row_i in the $\boldsymbol{z}$ variables, the car in row_{i+1} is greater than c. Hence each of the rows containing big cars in the $\boldsymbol{v}$ variables corresponds to a moveable valley, as in Figure 5.

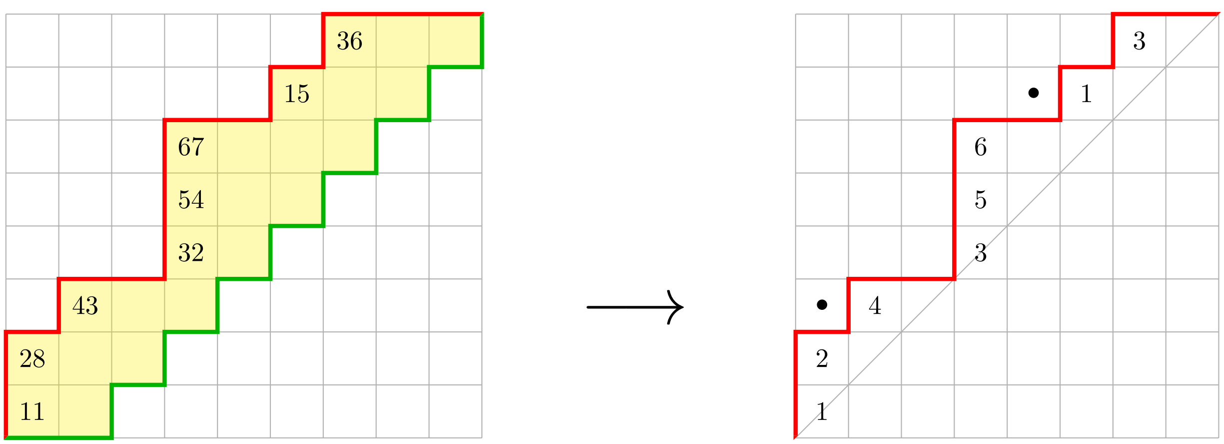

Figure 5. A multilabeled 1^8-Dyck path with Y-reading word $12834567 \in 123456 \sqcup 87$ and the corresponding valley-decorated labeled Dyck path. The rows above the big cars 8 and 7 contain a dot, indicating a moveable valley.

This all shows that finding and proving a t-statistic to combine with area to generate the monomial expansion of $\nabla_{\boldsymbol{v}} e_n(\boldsymbol{z})$ could be viewed as containing the problem of proving the Valley version of the Delta Conjecture.

9. Theta operators

We recall that for any symmetric function $f(\boldsymbol{z})$, we let $f^*(\boldsymbol{z}) = f[\boldsymbol{z}/M]$ with $M = (1-q)(1-t)$. The $*$-scalar product may be defined by setting

$$\langle f(\boldsymbol{z}), g(\boldsymbol{z})\rangle_* = \langle f(\boldsymbol{z}), \omega(g)[M\boldsymbol{z}]\rangle,$$

where on the right-hand side we have a Hall-inner product. Modified Macdonald polynomials are orthogonal under the $*$-scalar product and give

$$\langle \widetilde{H}_\mu(q,t;\boldsymbol{z}), \widetilde{H}_\mu(q,t;\boldsymbol{z})\rangle = w_\mu,$$

where

$$w_\mu = w_\mu(q,t) = \prod_{c\in\mu}(q^{\mathrm{arm}(c)} - t^{\mathrm{leg}(c)+1})(t^{\mathrm{leg}(c)} - q^{\mathrm{arm}(c)+1}).$$

One of the important specializations of the modified Macdonald basis is

$$\widetilde{H}_\mu[q,t;1-u] = \prod_{(i,j)\in\mu} 1 - uq^i t^j.$$

Using super nabla operators, we see that on symmetric functions with no constant term,

$$\Pi = (1-u)^{-1}\nabla_{1-u}\Big|_{u=1},$$

and Theta operators can be written as a specialization of the more general operator

$$\widetilde{\Theta}_g(u) = \nabla_{1-u} g^* \nabla_{1-u}^{-1} \qquad \text{from which} \qquad \Theta_g = \widetilde{\Theta}_g(1).$$

We also see that by Macdonald-Koornwinder reciprocity,

$$\nabla_M = M\Delta_{e_1}\Pi.$$

This operator appears quite naturally in the theory of modified Macdonald polynomials. For instance, one has $\nabla_M e_n^* = e_n$. For a given symmetric function F, we denote by Ξ the operator (studied in [IR22]) which gives

$$\Xi F(\boldsymbol{z}) = \nabla_M F^*(\boldsymbol{z}).$$

Since

$$e_\lambda^* = \sum_\mu \frac{\widetilde{H}_\mu}{w_\mu}\langle \widetilde{H}_\mu, e_\lambda^*\rangle_* = \sum_\mu \frac{\widetilde{H}_\mu}{w_\mu}\langle \widetilde{H}_\mu, h_\lambda\rangle,$$

we find that

$$\Xi e_\lambda(\boldsymbol{z}) = \langle \nabla_{\boldsymbol{v}} e_n, h_\lambda(\boldsymbol{v})\rangle,$$

or equivalently

$$\nabla_{\boldsymbol{v}} e_n(\boldsymbol{z}) = \sum_{\lambda\vdash n} m_\lambda(\boldsymbol{v})\Xi e_\lambda(\boldsymbol{z}).$$

At $t=1$, we can use multilabeled Dyck paths to give both an elementary and monomial expansion as follows: Let $\mathrm{MD}^1_{1^n}(\lambda)$ be the subset of $\mathrm{MD}^1_{1^n}$ where the labels form a rearrangement of $1^{\lambda_1}, 2^{\lambda_2}, \ldots, n^{\lambda_n}$. Then, as was found in [IR22],

$$\Xi e_\lambda(\boldsymbol{z})\Big|_{t=1} = \sum_{P\in\mathrm{MD}^1_{1^n}(\lambda)} q^{\mathrm{area}(P)} e_{\eta(P)}(\boldsymbol{z}).$$

On the other hand, DAdderio, Iraci, LeBorgne, Romero, and Vanden Wyngaerd give a conjectural expansion of this symmetric function in terms of tiered trees [DIL+22]. We now describe this conjecture.

9.1. **Rooted tiered trees and Theta operators.** A *graph* G is be a pair (V, E), with V a finite set of *vertices* and $E \subseteq \binom{V}{2}$ a set of *edges* (hence no loops nor multiple edges). A *rooted graph* is a graph (V, E) with a distinguished vertex $r \in V$ which we call its *root*. A *tree* is a connected graph with no circuits. Tiered trees were first defined in [DGGS19], and then in [DIL+22], where the definition was extended to trees with repeated labels, requiring an extra condition.

A *rooted tiered tree* is a tree $T = (V, E)$ consisting of vertices V and edges E with a level function $\mathrm{lv} : V \to \mathbb{N}$ and a labeling $w : V \to \mathbb{N}_+$ such that the following hold:

(a) If $\{i, j\} \in E$, then $\mathrm{lv}(i) \neq \mathrm{lv}(j)$.
(b) If $\{i, j\} \in E$ and $\mathrm{lv}(i) < \mathrm{lv}(j)$, then $w(i) < w(j)$.
(c) If $p(i) = p(j)$ and $\mathrm{lv}(i) = \mathrm{lv}(j)$, then $w(i) \neq w(j)$. Here $p(i)$ denotes the parent of i, which is the unique neighbor of i closest to the root.
(d) The root r is the only vertex at level 0 ($\mathrm{lv}^{-1}(0) = \{r\}$).

A rooted α-tree is one in which the number of vertices at level i is given by α_i. The collection of all rooted, tiered α-trees will be denoted by $\mathrm{RTT}(\alpha)$; and the collection of all rooted, tiered α-trees where the root is the unique vertex labeled with 0 will be denoted by $\mathrm{RTT}_0(\alpha)$.

Recall that j is a descendant of i if there is a $k > 0$ for which $p^k(j) = i$. We will say that two vertices i and j are compatible if $\mathrm{lv}(i) < \mathrm{lv}(j)$ and $w(i) < w(j)$, or if the reverse inequalities hold. This means that between i and j, both levels and labels increase, or they both decrease.

Definition 2. *An inversion of T is a pair (i, j) of non-root vertices for which the following hold:*

(1) j is a descendant of i,
(2) j is compatible with $p(i)$, and
(3) either $w(j) < w(i)$ or $w(j) = w(i) \wedge \mathrm{lv}(j) > \mathrm{lv}(i)$.

The monomial weight of a labeled tiered tree T with vertices $1, 2, \ldots, n$ is given by $\boldsymbol{z}^T = z_{w(1)} \cdots z_{w(n)}$ (with $z_0 = 1$).

Conjecture 10 (The Theta Trees Conjecture [DIL+22])**.** *For any composition α,*

$$\Theta_{e_\alpha} e_1(\boldsymbol{z})\Big|_{t=1} = \sum_{T \in \mathrm{RTT}(\alpha)} q^{\mathrm{inv}(T)} \boldsymbol{z}^T.$$

Also, one has

$$\Xi e_\alpha(\boldsymbol{z})\Big|_{t=1} = \sum_{\mathrm{RTT}_0(\alpha)} q^{\mathrm{inv}(T)} \boldsymbol{z}^T.$$

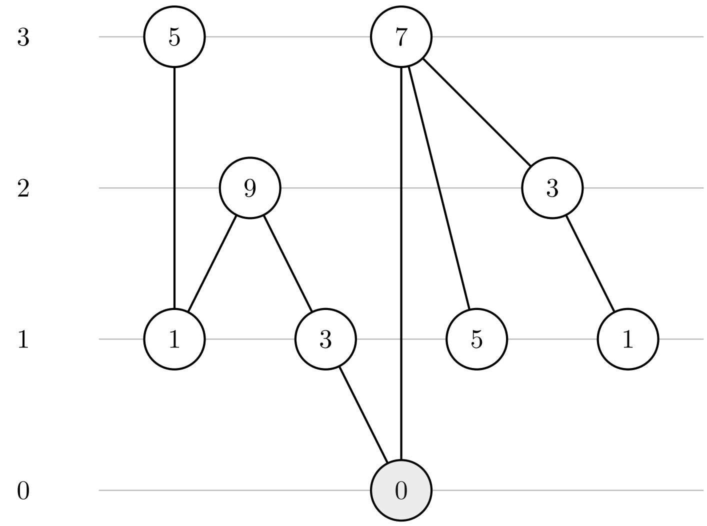

FIGURE 6. An element of $\mathrm{RTT}_0(4,2,2)$ with 6 inversions.

One special case suggests a deeper connection between these symmetric functions and certain representations of quivers. It was shown that the identity in the conjecture holds for $\Theta_{e_1^n} e_1(\boldsymbol{z})\Big|_{t=1}$ [DIL+22]. In order to prove this, however, it was essential to view this symmetric function as giving Kac polynomials for dandelion quivers: the coefficient of m_μ is the number of absolutely indecomoposable representations over $\mathbb{F}_q$ with dimension vector depending on μ. There is then the question of whether general Theta operator expressions can be interpreted in this way to give a relation between Theta operators and representations of quivers over finite fields.

The following remains an open question:

Conjecture 11. *For any α, we have*

$$\sum_{P \in \mathrm{MD}^1_{1^n}(\alpha)} q^{\mathrm{area}(P)} e_{\eta(P)}(\boldsymbol{z}) = \sum_{\mathrm{RTT}_0(\alpha)} q^{\mathrm{inv}(T)} \boldsymbol{z}^T.$$

Alternatively, there is a statistic-preserving bijection

$$\phi : \mathrm{RTT}_0(\alpha) \to \{\pi \in \mathrm{MD}_{1^n} \mid \textit{the } \boldsymbol{v} \textit{ labels give the monomial } \boldsymbol{v}^\alpha\ \}$$

That is, ϕ is a function from 0-rooted tiered trees to two-labeled Dyck paths that sends the inv *of the tree to the* area *of the path, the labels of the tree to the $\boldsymbol{z}$-labels of the path, and the levels of the tree to the $\boldsymbol{v}$-labels of the path.*

Finding such a bijection would automatically prove Conjecture 10. One of the main difficulties here lies in the fact that, even for $n = 2$, there is no map between these two sets that sends $(w(i), \mathrm{lv}(i))$ to (x_i, y_i) for each $i \in V$; that is, we cannot simultaneously send the label and the level of every vertex to the two labels assigned to the same step.

One of the motivations for studying the expansion of Πe^*_λ is that $\Theta_{e_\lambda} \Pi e^*_\mu = \Pi e^*_\lambda e^*_\mu$. This means that one can understand applications of Theta operators to any symmetric function by understanding the basis $\{\Pi e^*_\lambda\}_\lambda$. The symmetric function $\nabla_M e^*_\lambda = \Xi e_\lambda$ gives a slight modification which produces positivities. In particular, analogous to the positivity observed in subsection 8.3,

Conjecture 12 (General See-Conjecture [BHIR23])**.** *For any partition $\mu \vdash n$, there exists stable coefficients $C^\mu_{\lambda^0,\lambda^1,\ldots,\lambda^k}(u,v)$ which are Schur positive polynomials in u and v such that*

$$\nabla_{\boldsymbol{v}_1} \cdots \nabla_{\boldsymbol{v}_k} \Xi e_\mu(\boldsymbol{z})\Big|_{\substack{q=1+u\\ t=1+v}} = \sum_{\lambda^0,\lambda^1,\ldots,\lambda^k \vdash n} C^\mu_{\lambda^0,\lambda^1,\ldots,\lambda^k}(u,v) e_{\lambda^0}(\boldsymbol{z}) e_{\lambda^1}(\boldsymbol{v}_1) \cdots e_{\lambda^k}(\boldsymbol{v}_k).$$

9.2. **On the modified Schur function s^*_λ.** Since for any partition λ,

$$s^*_{\lambda'}(\boldsymbol{z}) = \sum_\mu \frac{\widetilde{H}_\mu(\boldsymbol{z})}{w_\mu} \langle \widetilde{H}_\mu, s^*_{\lambda'} \rangle_* = \sum_\mu \frac{\widetilde{H}_\mu(\boldsymbol{z})}{w_\mu} \langle \widetilde{H}_\mu, s_\lambda \rangle,$$

we have

$$\Xi s_{\lambda'}(\boldsymbol{z}) = \langle \nabla_{\boldsymbol{v}} e_n(\boldsymbol{z}), s_\lambda(\boldsymbol{v}) \rangle.$$

From Equation 8.4, we see that

$$\Xi s_{\lambda'}(\boldsymbol{z})\Big|_{t=1} = \sum_{\substack{P \in \mathrm{LMD}_{1^n}\\ \lambda^1(P)=\lambda}} q^{\mathrm{area}(P)} e_{\eta(P)}(\boldsymbol{z}).$$

As an example, Figure 7 has a Dyck path with labels given by a lattice word. The lattice word gives the partition $\lambda = (3,2,1,1)$, and thus appears as one of the objects enumerated by $\Xi s_{(4,2,1)}(\boldsymbol{z})\Big|_{t=1}$.

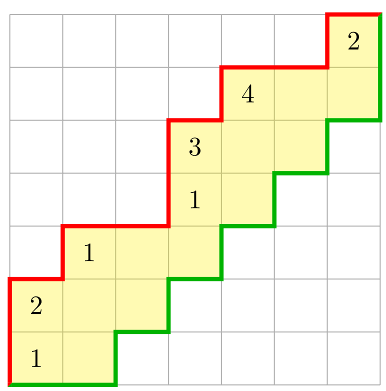

FIGURE 7. An element of $P \in \mathrm{LMD}_{1^6}$ with associated lattice word $w = 1211342$ and partition $\lambda^1(P) = (3,2,1,1)$. The composition associated to this path is $\eta(P) = (3,2,1,1)$.

Here, we have another example of a symmetric function which exhibits the e-positivity phenomenon under the substitution $q = 1+u$.

Conjecture 13 ([IR22]). *The symmetric functions $\Xi s_{\lambda'}(\boldsymbol{z})$ (and therefore $\Xi e_\lambda(\boldsymbol{z})$) exhibit the e-positivity phenomenon. This would imply the existence of a statistic d for which we have*

$$\Xi s_{\lambda'}(\boldsymbol{z})\Big|_{q=1+u} = \sum_{\substack{P\in \mathrm{LMD}^1_{1^n} \\ \lambda^1(P)=\lambda}} \sum_{S\subseteq \mathrm{Area}(P)} u^{|S|} t^{d(P,S)} e_{\eta(P)}(\boldsymbol{z}).$$

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