

Constant Term Methods in the Theory of Tesler matrices and Macdonald Polynomial Operators

by

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ABSTRACT

The Tesler matrices with hook sums (a_1, a_2, \dots, a_n) are non-negative integral upper triangular matrices, whose i^{th} diagonal element plus the sum of the entries in the arm of its (french) hook minus the sum of the entries in its leg is equal to a_i for all i . In a recent paper [6], the second author expressed the Hilbert series of the Diagonal Harmonic modules as a weighted sum of the family of Tesler matrices with hook weights $(1, 1, \dots, 1)$. In this paper we use the constant term algorithm developed by the third author to obtain a Macdonald polynomial interpretation of these weighted sum of Tesler matrices for arbitrary hook weights. In particular we also obtain new and illuminating proofs of the results in [6].

I. Introduction

Denoting by \mathcal{UP} the collection of upper triangular matrices with non-negative integer entries, let us set for a given integral vector (a_1, a_2, \dots, a_n)

$$\mathcal{T}(a_1, a_2, \dots, a_n) = \left\{ A = \|a_{i,j}\|_{i,j=1}^n \in \mathcal{UP} : a_{s,s} + \sum_{j=s+1}^n a_{s,j} - \sum_{i=1}^{s-1} a_{i,s} = a_s \quad \forall \quad 1 \leq s \leq n \right\}. \quad \text{I.1}$$

We will here and after refer to this as the collection of Tesler matrices with hook sums (a_1, a_2, \dots, a_n) .

To present our results we need to use plethystic notation. Readers unfamiliar with this notation are urged to read at least the beginning of the first section where we include an introductory Macdonald polynomial “tool kit”.

The modified Macdonald polynomials $\{\tilde{H}_\mu[X; q, t]\}_\mu$ we work with here are the unique symmetric function basis which satisfies the two triangularity conditions

$$a) \quad \tilde{H}_\mu = \sum_{\lambda \leq \mu} s_\lambda \left[\frac{X}{t-1} \right] a_{\lambda\mu}(q, t), \quad b) \quad \tilde{H}_\mu = \sum_{\lambda \geq \mu} s_\lambda \left[\frac{X}{1-q} \right] b_{\lambda\mu}(q, t), \quad \text{I.2}$$

together with the normalizing condition

$$\tilde{H}_\mu \Big|_{s_n} = 1 \quad \text{for all } \mu. \quad \text{I.3}$$

Setting for a given partition $\mu = (\mu_1, \mu_2, \dots, \mu_k)$

$$n(\mu) = \sum_{i=1}^k (i-1)\mu_i,$$

the operator ∇ was defined in [1] by setting

$$\nabla \tilde{H}_\mu = T_\mu \tilde{H}_\mu \quad (\text{with } T_\mu = t^{n(\mu)} q^{n(\mu')}). \quad \text{I.4}$$

Now it was conjectured in [3] and proved in [7] that the Hilbert series of the Diagonal Harmonics modules is given by the polynomial

$$\partial_{p_1}^n \nabla e_n.$$

With this notation the result in [6] may be expressed in the form

$$\partial_{p_1}^n \nabla e_n = \left(-\frac{1}{M}\right)^n \sum_{A \in \mathcal{T}(1,1,\dots,1)} wt(A), \quad \text{I.5}$$

where for convenience we set for $A = \|a_{i,j}\|_{i,j=1}^n$

$$wt(A) = \prod_{i,j} wt(a_{i,j}), \quad wt(a) = \begin{cases} -M \frac{q^a - t^a}{q-t} & \text{if } a > 0 \\ 1 & \text{if } a = 0 \end{cases}, \quad M = (1-t)(1-q). \quad \text{I.6}$$

In the same paper [6] it is also shown that

$$\partial_{p_1}^n \nabla^m e_n = \left(-\frac{1}{M}\right)^n \sum_{A \in \mathcal{T}(1,m,\dots,m)} wt(A) \quad (\text{for all } m \geq 1). \quad \text{I.7}$$

Note further that, as long as the hook sums (a_1, a_2, \dots, a_n) are strictly positive, each row of a matrix $A \in \mathcal{T}(a_1, a_2, \dots, a_n)$ has to have at least one positive element. This is quite evident from the definition in I.1. Thus, in view of I.6, in spite of the denominator factor $(-M)^n$, the expression

$$P_{a_1, a_2, \dots, a_n}(q, t) = \left(-\frac{1}{M}\right)^n \sum_{A \in \mathcal{T}(a_1, a_2, \dots, a_n)} wt(A) \quad \text{I.8}$$

will necessarily evaluate to a polynomial. Further experimentations revealed that when $a_1 \leq a_2 \leq \dots \leq a_n$ the polynomial $P_{a_1, a_2, \dots, a_n}(q, t)$ turns out to have non-negative integer coefficients. One of our identities may lead to a Hilbert Scheme proof of this non-negativity. More precisely, we obtain the following equality

$$P_{a_1, a_2, \dots, a_n}(q, t) = (-1)^n M \partial_{p_1} \nabla^{-a_n} \partial_{p_1} \nabla^{a_n - a_{n-1}} \dots \partial_{p_1} \nabla^{a_3 - a_2} \partial_{p_1} \nabla^{a_2 - a_1} \partial_{p_1} \nabla^{a_1} p_{n+1} \left[\frac{X}{M} \right], \quad \text{I.9}$$

which (as we shall see) is easily shown to contain both I.5 and I.7.

The proof of I.9 is based on two identities which are of interest in their own right. The first of these identities involves the plethystic operators D_a introduced in [4], which may simply be defined by setting for any symmetric function $F[X]$

$$D_a F[X] = F\left[X + \frac{M}{z}\right] \Omega[-zX] \Big|_{z^a}. \quad \text{I.10}$$

This given, we will show in section 2 that

$$P_{a_1, a_2, \dots, a_n}(q, t) = \left(-\frac{1}{M}\right)^n D_{a_1-1} D_{a_2-1} \dots D_{a_n-1} e_{n+1} \left[\frac{X}{M} \right] \Big|_{X \rightarrow M}. \quad \text{I.11}$$

In addition, also in section 2, we will give a new proof of I.5 as well as a proof of the surprising identity

$$\left(-\frac{1}{M}\right)^{n-1} \sum_{A \in \mathcal{T}(n-1, -1, -1, \dots, -1)} wt(A) = \left(-\frac{1}{M}\right)^{n-1} D_{-1}^n p_n \left[\frac{X}{M} \right] = \partial_{p_1}^{n-1} \nabla e_{n-1}. \quad \text{I.12}$$

We should note that both these two results will be derived by manipulations involving only identities from our ‘‘tool kit’’.

Section 3 will be dedicated to the proof of a truly remarkable recursive algorithm for the construction of the polynomials $P_{a_1, a_2, \dots, a_n}(q, t)$. To state this result we need notation.

To begin, for any indeterminate z let us set

$$f(z) = \frac{(1-z)(1-zqt)}{(1-zt)(1-zq)}. \quad \text{I.13}$$

Next let \mathcal{T}_n be the set of all Standard Young tableaux with labels $0, 1, 2, \dots, n$. For a given $T \in \mathcal{T}_n$, we let $w_T(k) = q^{j-1}t^{i-1}$ if the label k of T is in the i -th row j -th column. We also denote by S_T the substitution set $\{z_k^{-1} = w_T(k) : 1 \leq k \leq n\}$. For instance, for $T = \begin{pmatrix} 1 & 4 \\ 0 & 2 & 3 \end{pmatrix}$ we have

$$S_T = \{z_1^{-1} = t, z_2^{-1} = q, z_3^{-1} = q^2, z_4^{-1} = qt\}.$$

This given, it will be shown in section 3 that

$$P_{a_1, a_2, \dots, a_n}(q, t) = \sum_{T \in \mathcal{T}_n} \mathcal{H}_n \prod_{1 \leq k \leq n} (1 - z_k w_T(k)) z_k^{-a_k} \Big|_{S_T}, \quad \text{I.14}$$

where

$$\mathcal{H}_n = \left(-\frac{1}{M}\right)^n \prod_{1 \leq i \leq n} f(z_i) \prod_{1 \leq i < j \leq n} f(z_j/z_i).$$

We must emphasize that I.14 is not a formula but rather an algorithm, since it should be understood that each summand must be separately constructed by progressive applications of the substitutions in the set S_T , and in the successive order $k = 1, 2, \dots, n$. This provision notwithstanding we will show that the combination of I.11 and I.14 proves I.9.

1. A Macdonald Polynomial tool kit

The space of symmetric polynomials will be denoted Λ . The subspace of homogeneous symmetric polynomials of degree m will be denoted by Λ^m . We will seldom work with symmetric polynomials expressed in terms of variables but rather express them in terms of one of the six classical symmetric function bases

- (1) “*power*” $\{p_\mu\}_\mu$, (2) “*monomial*” $\{m_\mu\}_\mu$, (3) “*homogeneous*” $\{h_\mu\}_\mu$,
 (4) “*elementary*” $\{e_\mu\}_\mu$, (5) “*forgotten*” $\{f_\mu\}_\mu$ and (6) “*Schur*” $\{s_\mu\}_\mu$.

We recall that the fundamental involution ω may be defined by setting for the power basis indexed by $\mu = (\mu_1, \mu_2, \dots, \mu_k) \vdash n$

$$\omega p_\mu = (-1)^{n-k} p_\mu = (-1)^{|\mu| - l(\mu)} p_\mu \quad \text{I.1}$$

where for any vector $v = (v_1, v_2, \dots, v_k)$ we set $|v| = \sum_{i=1}^k v_i$ and $l(v) = k$.

In dealing with symmetric function identities, specially with those arising in the Theory of Macdonald Polynomials, we find it convenient and often indispensable to use plethystic notation. This device has a straightforward definition which can be verbatim implemented in MAPLE or MATHEMATICA for computer experimentation. We simply set for any expression $E = E(t_1, t_2, \dots)$ and any power symmetric function p_k

$$p_k[E] = E(t_1^k, t_2^k, \dots). \quad \text{I.2}$$

This given, for any symmetric function F we set

$$F[E] = Q_F(p_1, p_2, \dots) \Big|_{p_k \rightarrow E(t_1^k, t_2^k, \dots)}, \quad 1.3$$

where Q_F is the polynomial yielding the expansion of F in terms of the power basis. Note that in writing $E(t_1, t_2, \dots)$ we are tacitly assuming that t_1, t_2, t_3, \dots are all the variables appearing in E and in writing $E(t_1^k, t_2^k, \dots)$ we intend that all the variables appearing in E have been raised to their k^{th} power.

A paradoxical but necessary property of plethystic substitutions is that 1.2 requires

$$p_k[-E] = -p_k[E]. \quad 1.4$$

This notwithstanding, we will still need to carry out ordinary changes of signs. To distinguish it from the “*plethystic*” minus sign, we will carry out the “*ordinary*” sign change by prepending our expressions with a superscripted minus sign, or as the case may be, by means of a new variables ϵ which outside of the plethystic bracket is simply replaced by -1 . For instance, these conventions give for $X_n = x_1 + x_2 + \dots + x_n$

$$p_k[-X_n] = (-1)^{k-1} \sum_{i=1}^n x_i^k, \quad 1.5$$

or, equivalently

$$p_k[-\epsilon X_n] = -\epsilon^k \sum_{i=1}^n x_i^k = (-1)^{k-1} \sum_{i=1}^n x_i^k.$$

In particular we get for $X = x_1 + x_2 + x_3 + \dots$

$$\omega p_k[X] = p_k[-X].$$

Thus for any symmetric function $F \in \Lambda$ and any expression E we have

$$\omega F[E] = F[-E] = F[-\epsilon E]. \quad 1.6$$

In particular, if $F \in \Lambda^{=k}$ we may also rewrite this as

$$F[-E] = \omega F[-E] = (-1)^k \omega F[E]. \quad 1.7$$

The formal power series

$$\Omega = \exp\left(\sum_{k \geq 1} \frac{p_k}{k}\right)$$

combined with plethystic substitutions will provide a powerful way of dealing with the many generating functions occurring in our manipulations.

Let us recall that in I.10 we have set

$$D_k F[X] = F\left[X + \frac{M}{z}\right] \Omega[-zX] \Big|_{z^k} \quad \text{for } -\infty < k < +\infty, \quad 1.8$$

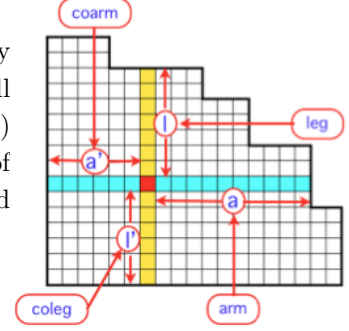
with

$$\Omega[-zX] = \sum_{m \geq 0} z^m h_m[-X] = \sum_{m \geq 0} (-1)^m z^m e_m[X]. \tag{1.9}$$

Here “ $|_{z^k}$ ” denotes the operation of taking the coefficient of z^k in the preceding expression, e_m and h_m denote the elementary and homogeneous symmetric functions indexed by m .

To present our Macdonald polynomial kit, it is convenient to identify partitions with their (french) Ferrers diagram. Given a partition μ and a cell $c \in \mu$, Macdonald introduces four parameters $l = l_\mu(c)$, $l' = l'_\mu(c)$, $a = a_\mu(c)$ and $a' = a'_\mu(c)$ called *leg*, *coleg*, *arm* and *coarm* which give the number of lattice cells of μ strictly NORTH, SOUTH, EAST and WEST of c , (see attached figure). Following Macdonald we will set

$$n(\mu) = \sum_{c \in \mu} l_\mu(c) = \sum_{c \in \mu} l'_\mu(c) = \sum_{i=1}^{l(\mu)} (i-1)\mu_i.$$



Denoting by μ' the conjugate of μ , the basic ingredients playing a role in the theory of Macdonald polynomials are

$$\begin{aligned} T_\mu &= t^{n(\mu)} q^{n(\mu')}, & B_\mu(q, t) &= \sum_{c \in \mu} t^{l'_\mu(c)} q^{a'_\mu(c)}, & \Pi_\mu(q, t) &= \prod_{c \in \mu; c \neq (0,0)} (1 - t^{l'_\mu(c)} q^{a'_\mu(c)}), \\ w_\mu(q, t) &= \prod_{c \in \mu} (q^{a_\mu(c)} - t^{l_\mu(c)+1})(t^{l_\mu(c)} - q^{a_\mu(c)+1}), \end{aligned} \tag{1.10}$$

together with a deformation of the Hall scalar product, which we call the “*star*” scalar product, defined by setting for the power basis

$$\langle p_\lambda, p_\mu \rangle_* = (-1)^{|\mu| - l(\mu)} \prod_i (1 - t^{\mu_i})(1 - q^{\mu_i}) z_\mu \chi(\lambda = \mu),$$

where z_μ gives the order of the stabilizer of a permutation with cycle structure μ .

The operators in 1.8 are connected to ∇ and the polynomials \tilde{H}_μ through the following basic identities:

$$\begin{aligned} (i) & \quad D_0 \tilde{H}_\mu = -D_\mu(q, t) \tilde{H}_\mu \\ (ii) & \quad D_k \underline{e}_1 - \underline{e}_1 D_k = M D_{k+1} \\ (iii) & \quad \nabla \underline{e}_1 \nabla^{-1} = -D_1 \\ (iv) & \quad \nabla^{-1} \partial_1 \nabla = \frac{1}{M} D_{-1} \\ (v) & \quad D_k \partial_1 - \partial_1 D_k = D_{k-1} \end{aligned} \tag{1.11}$$

where \underline{e}_1 is simply the operator “*multiplication by e_1* ”, and ∂_1 denotes its “*Hall*” scalar product adjoint.

Recall that for our version of the Macdonald polynomials the Macdonald Reciprocity formula states that

$$\frac{\tilde{H}_\alpha[1 + u D_\beta]}{\prod_{c \in \alpha} (1 - u t^{l'_\mu(c)} q^{a'_\mu(c)})} = \frac{\tilde{H}_\beta[1 + u D_\alpha]}{\prod_{c \in \beta} (1 - u t^{l'_\mu(c)} q^{a'_\mu(c)})} \quad (\text{for all pairs } \alpha, \beta). \tag{1.12}$$

We will use here several special evaluations of 1.12. To begin, canceling the common factor $(1 - u)$ out of the denominators on both sides of 1.12 and then setting $u = 1$ gives

$$\frac{\tilde{H}_\alpha[MB_\beta]}{\Pi_\alpha} = \frac{\tilde{H}_\beta[MB_\alpha]}{\Pi_\beta} \quad (\text{for all pairs } \alpha, \beta). \tag{1.13}$$

On the other hand replacing u by $1/u$ and letting $u = 0$ in 1.12 gives

$$(-1)^{|\alpha|} \frac{\tilde{H}_\alpha[D_\beta]}{T_\alpha} = (-1)^{|\beta|} \frac{\tilde{H}_\beta[D_\alpha]}{T_\beta} \quad (\text{for all pairs } \alpha, \beta). \tag{1.14}$$

Since for β the empty partition we can take $\tilde{H}_\beta = 1$ and $D_\beta = -1$, 1.12 in this case reduces to

$$\tilde{H}_\alpha[1-u] = \prod_{c \in \alpha} (1 - ut^{l'} q^{a'}) = (1-u) \sum_{r=0}^{n-1} (-u)^r e_r[B_\mu - 1]. \tag{1.15}$$

This identity yields the coefficients of hook Schur functions in the expansion.

$$\tilde{H}_\mu[X; q, t] = \sum_{\lambda \vdash |\mu|} s_\mu[X] \tilde{K}_{\lambda\mu}(q, t). \tag{1.16}$$

Recall that the addition formula for Schur functions gives

$$s_\mu[1-u] = \begin{cases} (-u)^r (1-u) & \text{if } \mu = (n-r, 1^r) \\ 0 & \text{otherwise} \end{cases} \tag{1.17}$$

Thus 1.16, with $X = 1-u$, combined with 1.15 gives for $\mu \vdash n$

$$\langle \tilde{H}_\mu, s_{(n-r, 1^r)} \rangle = e_r[B_\mu - 1]$$

and the identity $e_r h_{n-r} = s_{(n-r, 1^r)} + s_{(n-r-1, 1^{r-1})}$ gives

$$\langle \tilde{H}_\mu, e_r h_{n-r} \rangle = e_r[B_\mu]. \tag{1.19}$$

Since for $\beta = (1)$ we have $\tilde{H}_\beta = 1$ and $\Pi_\beta = 1$, formula 1.13 reduces to the surprisingly simple identity

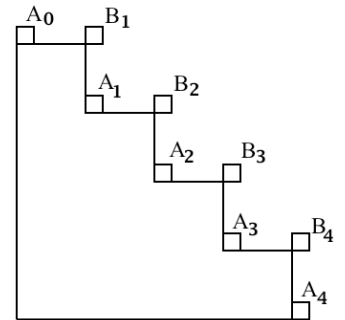
$$\tilde{H}_\alpha[M] = MB_\alpha \Pi_\alpha. \tag{1.20}$$

Last but not least we must also recall that we have the Pieri formulas

$$a) \quad e_1 \tilde{H}_\nu = \sum_{\mu \leftarrow \nu} d_{\mu\nu} \tilde{H}_\mu, \quad b) \quad e_1^\perp \tilde{H}_\mu = \sum_{\nu \rightarrow \mu} c_{\mu\nu} \tilde{H}_\nu, \tag{1.21}$$

Here $\nu \rightarrow \mu$ simply means that the sum is over ν 's obtained from μ by removing a corner cell and $\mu \leftarrow \nu$ means that the sum is over μ 's obtained from ν by adding a corner cell.

The final ingredient we need, to carry out our proofs are expressions, proved in [11], for the coefficients $d_{\mu,\nu}$ in terms of the corner weights of the partition μ . More precisely, let μ have m outer corners A_0, A_1, \dots, A_m (see adjacent figure) and inner corners B_1, B_2, \dots, B_m . Then the weight of A_i is $x_i = t^{\alpha_i+1} q^{\beta_i}$ with α_i and β_i the coleg and coarm of B_i in μ . The weight of B_i will then be $u_i = t^{\alpha_i} q^{\beta_i}$. It is also convenient to set $\beta_0 = 0$ and $\alpha_{m+1} = 0$.



This given, in the proof of I.9 we will make use of the following two identities

$$x_0 + x_1 + \cdots + x_m - u_1 - \cdots - u_m = -MB_\mu(q, t) + 1 \quad 1.22$$

and

$$d_{\mu^i \nu} = \frac{1}{x_i} \frac{\prod_{j=1}^m (1 - u_j/x_i)}{\prod_{j=0, j \neq i}^m (1 - x_j/x_i)} \quad 1.23$$

where μ^i is the partition obtained by adding to ν corner A_i . The proof of both identities is given in [11]. We must note that 1.22 follows easily from the geometry of the above diagram, while 1.23 results from massive cancellations occurring in the original Macdonald Pieri formulas.

It will also be useful to know that these two Pieri coefficients are related by the identity

$$d_{\mu\nu} = Mc_{\mu\nu} \frac{w_\nu}{w_\mu}. \quad 1.24$$

Recall that our Macdonald Polynomials satisfy the orthogonality condition

$$\langle \tilde{H}_\lambda, \tilde{H}_\mu \rangle_* = \chi(\lambda = \mu) w_\mu(q, t). \quad 1.25$$

The $*$ -scalar product, is simply related to the ordinary Hall scalar product by setting for all pairs of symmetric functions f, g

$$\langle f, g \rangle_* = \langle f, \omega \phi g \rangle, \quad 1.26$$

where it has been customary to let ϕ be the operator defined by setting for any symmetric function f

$$\phi f[X] = f[MX]. \quad 1.27$$

Note that the inverse of ϕ is usually written in the form

$$f^*[X] = f[X/M]. \quad 1.28$$

In particular we also have for all symmetric functions f, g

$$\langle f, g \rangle = \langle f, \omega g^* \rangle_* \quad 1.29$$

The orthogonality relations in 1.25 yield the ‘‘Cauchy’’ identity for our Macdonald polynomials in the form

$$\Omega \left[-\epsilon \frac{XY}{M} \right] = \sum_{\mu} \frac{\tilde{H}_\mu[X] \tilde{H}_\mu[Y]}{w_\mu}, \quad 1.30$$

which restricted to its homogeneous component of degree n in X and Y reduces to

$$e_n \left[\frac{XY}{M} \right] = \sum_{\mu \vdash n} \frac{\tilde{H}_\mu[X] \tilde{H}_\mu[Y]}{w_\mu}. \quad 1.31$$

Note that the orthogonality relations in 1.25 yield us the following Macdonald polynomial expansions

Proposition 1.1

For all $n \geq 1$ we have

$$\begin{aligned}
a) \quad e_n \left[\frac{X}{M} \right] &= \sum_{\mu \vdash n} \frac{\tilde{H}_\mu[X]}{w_\mu}, & b) \quad h_k \left[\frac{X}{M} \right] e_{n-k} \left[\frac{X}{M} \right] &= \sum_{\mu \vdash n} \frac{e_k[B_\mu] \tilde{H}_\mu[X]}{w_\mu}, & c) \quad h_n \left[\frac{X}{M} \right] &= \sum_{\mu \vdash n} \frac{T_\mu \tilde{H}_\mu[X]}{w_\mu} \\
d) \quad (-1)^{n-1} p_n &= (1-t^n)(1-q^n) \sum_{\mu \vdash n} \frac{\Pi_\mu \tilde{H}_\mu[X]}{w_\mu} \\
e) \quad e_1[X/M]^n &= \sum_{\mu \vdash n} \frac{\tilde{H}_\mu[X]}{w_\mu} \langle \tilde{H}_\mu, e_1^n \rangle \\
f) \quad e_n &= \sum_{\mu \vdash n} \frac{\tilde{H}_\mu[X] M B_\mu \Pi_\mu}{w_\mu}
\end{aligned} \tag{1.32}$$

Finally it is good to keep in mind, for future use, that we have for all partitions μ

$$T_\mu \omega \tilde{H}_\mu[X; 1/q, 1/t] = \tilde{H}_\mu[X; q, t]. \tag{1.33}$$

Remark 1.1

It was conjectured in [3] and proved in [7] that the bigraded Frobenius characteristic of the diagonal Harmonics of S_n is given by the symmetric function

$$DH_n[X; q, t] = \sum_{\mu \vdash n} \frac{T_\mu \tilde{H}_\mu(X; q, t) M B_\mu(q, t) \Pi_\mu(q, t)}{w_\mu(q, t)}. \tag{1.34}$$

Surprisingly the intricate rational function on the right hand side is none other than ∇e_n . To see this we simply combine the relation in 1.20 with the degree n restricted Macdonald-Cauchy formula 1.31 obtaining

$$e_n[X] = e_n \left[\frac{XM}{M} \right] = \sum_{\mu \vdash n} \frac{\tilde{H}_\mu[X] M B_\mu \Pi_\mu}{w_\mu}. \tag{1.35}$$

This is perhaps the simplest way to prove 1.32 f). This discovery is precisely what led to the introduction of ∇ in the first place.

2. Iterated plethystic operators identities

Our first goal here is to obtain our new proof of I.5 based on the connection between Tesler matrices and plethystic operators. The basic ingredient in this approach is provided by the following

Proposition 2.1

For any symmetric function $F[X]$ and any sequence of integers a_1, a_2, \dots, a_n we have

$$\begin{aligned}
D_{a_n} \cdots D_{a_2} D_{a_1} F[X] \Big|_{X=M} &= F \left[M + \sum_{i=1}^n \frac{M}{z_i} \right] \prod_{i=1}^n \Omega[-z_i M] \prod_{1 \leq i < j \leq n} \Omega \left[-\frac{z_i}{z_j} M \right] \Big|_{z_1^{a_1} z_2^{a_2} \cdots z_n^{a_n}} \\
&= F \left[M + \sum_{i=1}^n \frac{M}{z_i} \right] \prod_{i=1}^n \frac{(1-z_i)(1-tqz_i)}{(1-tz_i)(1-qz_i)} \prod_{1 \leq i < j \leq n} \frac{(1-\frac{z_i}{z_j})(1-tq\frac{z_i}{z_j})}{(1-t\frac{z_i}{z_j})(1-q\frac{z_i}{z_j})} \Big|_{z_1^{a_1} z_2^{a_2} \cdots z_n^{a_n}}
\end{aligned} \tag{2.1}$$

Proof

The definition in 1.8 gives

$$D_{a_1}F[X] = F[X + \frac{M}{z_1}]\Omega[-z_1X] \Big|_{z_1^{a_1}}$$

and using 1.8 again we get

$$\begin{aligned} D_{a_2}D_{a_1}F[X] &= F[X + \frac{M}{z_1} + \frac{M}{z_2}]\Omega[-z_1(X + \frac{M}{z_2})]\Omega[-z_2X] \Big|_{z_1^{a_1}z_2^{a_2}} \\ &= F[X + \frac{M}{z_1} + \frac{M}{z_2}]\Omega[-\frac{z_1}{z_2}M]\Omega[-z_1X]\Omega[-z_2X] \Big|_{z_1^{a_1}z_2^{a_2}} \end{aligned}$$

This given, it is easy to see that the successive actions of $D_{a_3} \cdots D_{a_k}$ will eventually yield the identity

$$D_{a_n} \cdots D_{a_2}D_{a_1}F[X] = F[X + \sum_{i=1}^n \frac{M}{z_i}] \prod_{1 \leq i < j \leq n} \Omega[-\frac{z_i}{z_j}M] \prod_{i=1}^n \Omega[-z_iX] \Big|_{z_1^{a_1}z_2^{a_2} \cdots z_n^{a_n}}.$$

Setting $X = M$ gives the first equality in 1.6, but then the second equality holds as well since for any indeterminate v we have

$$\Omega[-vM] = \Omega[vt + vq - v - qtv] = \frac{(1-v)(1-qtv)}{(1-tv)(1-qv)}. \quad 2.2$$

Two immediate corollaries of Proposition 2.1 are identities I.11 and I.12:

Theorem 2.1

$$P_{a_1, a_2, \dots, a_n}(q, t) = (-\frac{1}{M})^n D_{a_1-1}D_{a_2-1} \cdots D_{a_n-1}e_{n+1}[\frac{X}{M}] \Big|_{X \rightarrow M} \quad 2.3$$

$$(-\frac{1}{M})^{n-1} \sum_{A \in \mathcal{T}(n-1, -1, -1, \dots, -1)} wt[A] = (-\frac{1}{M})^{n-1} D_{-1}^n p_n[\frac{X}{M}] = \partial_{p_1}^{n-1} \nabla e_{n-1} \quad 2.4$$

Proof

Setting $F[X] = e_{n+1}[\frac{X}{M}]$ the second equality in 2.1 gives

$$\begin{aligned} D_{a_n} \cdots D_{a_2}D_{a_1}F[X] \Big|_{X=M} &= e_{n+1} \left[1 + \sum_{i=1}^n \frac{1}{z_i} \right] \prod_{i=1}^n \Omega[-z_iM] \prod_{1 \leq i < j \leq n} \Omega[-\frac{z_i}{z_j}M] \Big|_{z_1^{a_1}z_2^{a_2} \cdots z_n^{a_n}} \\ &= \prod_{i=1}^n \Omega[-z_iM] \prod_{1 \leq i < j \leq n} \Omega[-\frac{z_i}{z_j}M] \Big|_{z_1^{a_1+1}z_2^{a_2+1} \cdots z_n^{a_n+1}}, \end{aligned} \quad 2.5$$

where the second equality is due to the identity

$$e_{n+1} \left[1 + \sum_{i=1}^n \frac{1}{z_i} \right] = \frac{1}{z_1 z_2 \cdots z_n}.$$

Now note that the generic summand produced by the expression

$$H(b_1, b_2, \dots, b_n) = \prod_{i=1}^n \Omega[-z_iM] \prod_{1 \leq i < j \leq n} \Omega[-\frac{z_i}{z_j}M] \Big|_{z_1^{b_1}z_2^{b_2} \cdots z_n^{b_n}} \quad 2.6$$

after we expand all its factors according to 1.9 is

$$\prod_{i=1}^n z_i^{p_i} h_{p_i}[-M] \prod_{1 \leq i < j \leq n} (z_i/z_j)^{p_{ij}} h_{p_{ij}}[-M] \Big|_{z_1^{b_1} z_2^{b_2} \dots z_n^{b_n}}.$$

This forces the equalities

$$p_s + \sum_{j=s+1}^n p_{sj} - \sum_{i=1}^{s-1} p_{is} = b_s \quad (\text{for all } 1 \leq s \leq n). \quad 2.7$$

We may thus associate to each summand an upper triangular matrix $A = \|a_{i,j}\|_{i,j=1}^n$ by setting

$$a_{i,j} = \begin{cases} p_i & \text{if } i = j, \\ p_{ij} & \text{if } i < j, \\ 0 & \text{otherwise.} \end{cases}$$

Since it can be shown that

$$h_a[-M] = \begin{cases} -M \frac{q^a - t^a}{q-t} & \text{if } a \geq 1 \\ 1 & \text{if } a = 0 \end{cases}$$

we immediately derive from this observation, and the definition in I.8 that

$$\prod_{i=1}^n \Omega[-z_i M] \prod_{1 \leq i < j \leq n} \Omega[-\frac{z_i}{z_j} M] \Big|_{z_1^{b_1} z_2^{b_2} \dots z_n^{b_n}} = \sum_{A \in T(b_1, b_2, \dots, b_n)} wt(A). \quad 2.8$$

Thus 2.3 follows from 2.5 with the replacements $a_i \rightarrow a_i - 1$.

To prove 2.4 we start with the definition of D_{-1} and get

$$D_{-1} p_n \left[\frac{X}{M} \right] = \left(p_n \left[\frac{X}{M} \right] + \frac{1}{z^n} \right) \Omega[-zX] \Big|_{z^{-1}} = (-1)^{n-1} e_{n-1}.$$

Using this and 1.11 (iv) we derive that

$$\left(-\frac{1}{M} \right)^{n-1} D_{-1}^n p_n \left[\frac{X}{M} \right] = \left(\frac{1}{M} \right)^{n-1} D_{-1}^{n-1} e_{n-1} = \nabla^{-1} \partial_{p_1}^{n-1} \nabla e_{n-1} = \partial_{p_1}^{n-1} \nabla e_{n-1}. \quad 2.9$$

On the other hand Proposition 2.1 with $a_1 = a_2 = \dots = a_n = -1$ and $F[X] = p_n \left[\frac{X}{M} \right]$ gives

$$D_{-1}^n p_n \left[\frac{X}{M} \right] \Big|_{X=M} = \left(1 + \sum_{i=1}^n \frac{1}{z_i^n} \right) \prod_{i=1}^n \Omega[-z_i M] \prod_{1 \leq i < j \leq n} \Omega[-\frac{z_i}{z_j} M] \Big|_{z_1^{-1} z_2^{-1} \dots z_n^{-1}} \quad 2.10$$

and 2.9 gives

$$\partial_{p_1}^{n-1} \nabla e_{n-1} = \left(-\frac{1}{M} \right)^{n-1} \left(1 + \sum_{i=1}^n \frac{1}{z_i^n} \right) \prod_{i=1}^n \Omega[-z_i M] \prod_{1 \leq i < j \leq n} \Omega[-\frac{z_i}{z_j} M] \Big|_{z_1^{-1} z_2^{-1} \dots z_n^{-1}}. \quad 2.11$$

Note next that if we set $F = 1$ and $a_1 = a_2 = \dots = a_n = -1$ in 2.1 we obtain

$$0 = D_{-1}^n 1 \Big|_{X=M} = \prod_{i=1}^n \Omega[-z_i M] \prod_{1 \leq i < j \leq n} \Omega[-\frac{z_i}{z_j} M] \Big|_{z_1^{-1} z_2^{-1} \dots z_n^{-1}} \quad 2.12$$

that eliminates one of the terms in 2.10. We claim that the only term that survives in 2.10 is the one yielded by $\frac{1}{z_1^n}$. That is we have

$$\frac{1}{z_k^n} \prod_{i=1}^n \Omega[-Mz_i] \prod_{1 \leq i < j \leq n} \Omega[-Mz_i/z_j] \Big|_{z_1^{-1} z_2^{-1} \dots z_n^{-1}} = 0 \quad (\text{for all } 2 \leq k \leq n). \quad 2.13$$

To see this let us rewrite the LHS in the expanded form, that is

$$\frac{1}{z_k^n} \prod_{i=1}^n \sum_{p_i \geq 0} z_i^{p_i} h_{p_i}[-M] \prod_{1 \leq i < j \leq n} \sum_{r_{i,j} \geq 0} \frac{z_i^{r_{i,j}}}{z_j^{r_{i,j}}} h_{r_{i,j}}[-M] \Big|_{z_1^{-1} z_2^{-1} \dots z_n^{-1}}.$$

The exponent of z_1 in the generic term of the product of these geometric series must satisfy the equation

$$p_1 + \sum_{j=2}^n r_{1,j} = -1.$$

This is, of course impossible, causing 2.13 to be true precisely as asserted.

Now using 2.13 and 2.12 in 2.11 gives

$$\partial_{p_1}^{n-1} \nabla e_{n-1} = \left(-\frac{1}{M}\right)^{n-1} \prod_{i=1}^n \Omega[-z_i M] \prod_{1 \leq i < j \leq n} \Omega\left[-\frac{z_i}{z_j} M\right] \Big|_{z_1^{n-1} z_2^{-1} \dots z_n^{-1}} \quad 2.14$$

and this, combined with 2.8 and 2.9 proves that

$$\partial_{p_1}^{n-1} \nabla e_{n-1} = \left(-\frac{1}{M}\right)^{n-1} D_{-1}^n p_n \left[\frac{X}{M}\right] = \left(-\frac{1}{M}\right)^{n-1} \sum_{A \in \mathcal{T}(n-1, -1, -1, \dots, -1)} w_H[A]$$

and our proof is complete.

Our proof of I.5 is more elaborate and requires the following auxiliary identity.

Proposition 2.2

For any symmetric function $F[X]$ we have

$$\sum_{\mu \vdash n+1} \frac{MB_\mu \Pi_\mu}{w_\mu} F[MB_\mu] = \Delta_{e_n} F[X] \Big|_{X=M}. \quad 2.15$$

Proof

We need only prove this for $F[X] = \tilde{H}_\gamma[X]$ for arbitrary γ . In this case 2.15 becomes

$$\sum_{\mu \vdash n+1} \frac{MB_\mu \Pi_\mu}{w_\mu} \tilde{H}_\gamma[MB_\mu] = \tilde{H}_\gamma[M] e_n[B_\gamma] = MB_\gamma \Pi_\gamma e_n[B_\gamma]. \quad 2.16$$

Since by the reciprocity in 1.13 we have

$$\frac{\tilde{H}_\gamma[MB_\mu]}{\Pi_\gamma} = \frac{\tilde{H}_\mu[MB_\gamma]}{\Pi_\mu},$$

2.16 becomes

$$\Pi_\gamma \sum_{\mu \vdash n+1} \frac{MB_\mu}{w_\mu} \tilde{H}_\mu[MB_\gamma] = MB_\gamma \Pi_\gamma e_n[B_\gamma].$$

Or better

$$\sum_{\mu \vdash n+1} \frac{MB_\mu}{w_\mu} \tilde{H}_\mu[MB_\gamma] = MB_\gamma e_n[B_\gamma]. \quad 2.17$$

But recall that we have

$$B_\mu = \sum_{\nu \rightarrow \mu} c_{\mu\nu}$$

and 2.17 becomes

$$\sum_{\mu \vdash n+1} \frac{M}{w_\mu} \tilde{H}_\mu[MB_\gamma] \sum_{\nu \rightarrow \mu} c_{\mu\nu} = MB_\gamma e_n[B_\gamma]. \quad 2.18$$

Now for the left hand side we have

$$\begin{aligned} LHS &= \sum_{\nu \vdash n} \frac{1}{w_\nu} \sum_{\mu \leftarrow \nu} \frac{Mw_\nu}{w_\mu} c_{\mu\nu} \tilde{H}_\mu[MB_\gamma] \\ &\text{(by 1.24)} = \sum_{\nu \vdash n} \frac{1}{w_\nu} \sum_{\mu \leftarrow \nu} d_{\mu\nu} \tilde{H}_\mu[MB_\gamma] \\ &\text{(by 1.21 a)} = \sum_{\nu \vdash n} \frac{1}{w_\nu} e_1[MB_\gamma] \tilde{H}_\nu[MB_\gamma] \\ &\text{(by 1.32 a)} = e_1[MB_\gamma] e_n \left[\frac{MB_\gamma}{M} \right] = MB_\gamma e_n[B_\gamma] = RHS \quad (!!!!) \end{aligned}$$

and our proof is complete.

As a corollary we obtain

Proposition 2.3

For $F \in \Lambda^{=k}$ with $k \leq n$ we have

$$\sum_{\mu \vdash n+1} \frac{MB_\mu \Pi_\mu}{w_\mu} F[MB_\mu] = \begin{cases} \nabla F[X] \Big|_{X=M} & \text{if } k = n \\ 0 & \text{if } k < n \end{cases} \quad 2.19$$

Proof

From 2.15 we get that the left hand side of 2.19 is

$$\Delta_{e_n} F[X] \Big|_{X=M}$$

but for a symmetric function $F[X]$ of degree n we have $\Delta_{e_n} F[X] = \nabla F[X]$. Thus the first alternative in 2.19 is immediate. On the other hand for $k < n$ the expansion of F in the Macdonald basis will involve $\tilde{H}'_\gamma s$ with $\gamma \vdash k$ and even before we make the evaluation at $X = M$ the identity

$$\Delta_{e_n} \tilde{H}_\gamma[X] = e_n[B_\gamma] \tilde{H}_\gamma[S] = 0 \quad (\text{for all } \gamma \vdash k < n)$$

forces $\Delta_{e_n} F = 0$, yielding the second alternative in 2.19.

We are now in a position to give our new and direct proof of I.5.

Theorem 2.2

$$\partial_{p_1}^n \nabla e_n = \left(-\frac{1}{M}\right)^n D_0^n e_{n+1} \left[\frac{X}{M}\right] \Big|_{X=M}. \quad 2.20$$

Proof

From 1.32 a) and 1.11 (i) we get

$$\begin{aligned} D_0^n e_{n+1} \left[\frac{X}{M}\right] \Big|_{X=M} &= \left(D_0^n \sum_{\mu \vdash n+1} \frac{\tilde{H}_\mu[X; q, t]}{w_\mu}\right) \Big|_{X=M} = \sum_{\mu \vdash n+1} \frac{\tilde{H}_\mu[X; q, t] (1 - MB_\mu)^n}{w_\mu} \Big|_{X=M} \\ &= \sum_{k=0}^n \binom{n}{k} (-M)^k \sum_{\mu \vdash n+1} \frac{MB_\mu \Pi_\mu}{w_\mu} B_\mu^k \end{aligned}$$

and Proposition 2.3 with $F = e_1 \left[\frac{X}{M}\right]^k$ and $0 \leq k \leq n$ gives

$$\begin{aligned} \left(-\frac{1}{M}\right)^n D_0^n e_{n+1} \left[\frac{X}{M}\right] \Big|_{X=M} &= \nabla e_1^n \left[\frac{X}{M}\right] \Big|_{X=M} \\ \text{(by 1.32 e) and the definition of } \nabla &= \sum_{\mu \vdash n} \frac{T_\mu \tilde{H}_\mu[X]}{w_\mu} \langle \tilde{H}_\mu, e_1^n \rangle \Big|_{X=M} \\ \text{(by 1.20)} &= \sum_{\mu \vdash n} \frac{T_\mu MB_\mu \Pi_\mu}{w_\mu} \langle \tilde{H}_\mu, e_1^n \rangle \\ \text{(by 1.32 f) and } \partial_{p_1} = e_1^\perp &= \partial_{p_1}^n \nabla e_n \end{aligned}$$

This proves 2.20 and completes our argument.

3. Constant terms and positivity of Tesler matrix polynomials

Our first goal here is to establish I.9. It will be convenient to write it here in the form

$$P_{a_1, a_2, \dots, a_n}(q, t) = M \partial_{p_1} \left(\prod_{i=1}^n \nabla^{-a_i} \partial_{p_1} \nabla^{a_i} \right) (-1)^n p_{n+1} \left[\frac{X}{M}\right] \quad 3.1$$

provided it is understood that the operator factors are successively applied from right to left, starting with $\nabla^{-a_n} \partial_{p_1} \nabla^{a_n}$ and ending with $\nabla^{-a_1} \partial_{p_1} \nabla^{a_1}$.

Our starting point is a semi-combinatorial interpretation of the right hand side. To state it we need some notation. As in the introduction, we let \mathcal{T}_n be the set of all Standard Young tableaux (SYTs for short) with labels $0, 1, 2, \dots, n$. For a given $T \in \mathcal{T}_n$, we let $w_T(k) = q^{j-1} t^{i-1}$ if the label k of T is in the i -th row j -th column and let $sh(T)$ denote the partition giving the shape of T . Recall that we have set

$$S_T = \{z_k^{-1} = w_T(k) : 1 \leq k \leq n\}. \quad 3.2$$

For a tableau $T \in \mathcal{T}_n$ we let $T^{(i)}$ denote the tableau in \mathcal{T}_i obtained by removing $i+1, i+2, \dots, n$ from T . It will be convenient to also let $T^{(n)} = T$. This given, for any $0 \leq i \leq n$ we set

$$\begin{aligned} c_{T^{(i)}/T^{(i-1)}} &= c_{sh(T^{(i)}), sh(T^{(i-1)})}, & d_{T^{(i)}/T^{(i-1)}} &= d_{sh(T^{(i)}), sh(T^{(i-1)})} \\ w_{T^{(i)}} &= w_{sh(T^{(i)})}, & B_{T^{(i)}} &= B_{sh(T^{(i)})}, & \Pi_{T^{(i)}} &= \Pi_{sh(T^{(i)})}. \end{aligned} \quad 3.3$$

Note that if $\mu \vdash n+1$ then I.4 and 1.21 b) give

$$\nabla^{-a} \partial_{p_1} \nabla^a \tilde{H}_\mu = \sum_{\nu \rightarrow \mu} c_{\mu, \nu} \left(\frac{T_\mu}{T_\nu} \right)^a \tilde{H}_\nu. \quad 3.4$$

Since the monomial $\frac{T_\mu}{T_\nu}$ is none other than the weight of the corner cell that we must remove from μ to get ν . To avoid notational conflicts let us set, for a moment $\frac{T_\mu}{T_\nu} = wt(\mu/\nu)$. Now it is easily seen that $wt(sh(T^{(k)})/sh(T^{(k-1)}))$ is precisely the weight of the cell that contains k in $T^{(k)}$, which was earlier denoted $w_T(k)$. Thus here in the following we can set

$$wt(sh(T^{(k)})/sh(T^{(k-1)})) = w_T(k). \quad 3.5$$

This given, we can now give the right hand side of 3.1 the following standard tableau expansion.

Proposition 3.1

$$M \partial_{p_1} \left(\prod_{i=1}^n \nabla^{-a_i} \partial_{p_1} \nabla^{a_i} \right) (-1)^n p_{n+1} \left[\frac{X}{M} \right] = \frac{1}{M^n} \sum_{T \in \mathcal{T}_n} \prod_{k=1}^n w_T(k)^{a_k} (1 - w_T(k)) d_{T^{(k)}/T^{(k-1)}} \quad 3.6$$

Proof

Using 1.32 d) the left hand side of 3.6 becomes

$$LHS = \sum_{\mu \vdash n+1} \frac{M \Pi_\mu}{w_\mu} \left(\partial_{p_1} \prod_{i=1}^n \nabla^{-a_i} \partial_{p_1} \nabla^{a_i} \right) \tilde{H}_\mu[X]. \quad 3.7$$

For $n=1$ we have for $\mu = (1, 1)$ and $\mu = (2)$

$$\partial_{p_1} \nabla^{-a_1} \partial_{p_1} \nabla^{a_1} \tilde{H}_{(1,1)}[X] = t^{a_1} c_{(1,1),(1)}, \quad \partial_{p_1} \nabla^{-a_1} \partial_{p_1} \nabla^{a_1} \tilde{H}_{(2)}[X] = q^{a_1} c_{(2),(1)}.$$

Thus we can inductively assume that for $\nu \vdash n$

$$\partial_{p_1} \left(\prod_{i=1}^{n-1} \nabla^{-a_i} \partial_{p_1} \nabla^{a_i} \right) \tilde{H}_\nu[X] = \sum_{T \in \mathcal{T}_{n-1}; sh(T)=\nu} \prod_{k=1}^{n-1} w_T(k)^{a_k} c_{T^{(k)}/T^{(k-1)}}. \quad 3.8$$

But for $\mu \vdash n+1$ we have

$$\nabla^{-a_n} \partial_{p_n} \nabla^{a_1} \tilde{H}_\mu[X] = \sum_{\nu \rightarrow \mu} wt(\mu/\nu)^{a_n} c_{\mu, \nu} \tilde{H}_\nu[X]$$

and 3.8 gives

$$\partial_{p_1} \left(\prod_{i=1}^n \nabla^{-a_i} \partial_{p_1} \nabla^{a_i} \right) \tilde{H}_\mu[X] = \sum_{\nu \rightarrow \mu} wt(\mu/\nu)^{a_n} c_{\mu, \nu} \sum_{T \in \mathcal{T}_{n-1}; sh(T)=\nu} \prod_{k=1}^{n-1} w_T(k)^{a_k} c_{T^{(k)}/T^{(k-1)}}.$$

Now this can clearly be rewritten as

$$\partial_{p_1} \left(\prod_{i=1}^n \nabla^{-a_i} \partial_{p_1} \nabla^{a_i} \right) \tilde{H}_\mu[X] = \sum_{T \in \mathcal{T}_n; sh(T)=\mu} \prod_{k=1}^n w_T(k)^{a_k} c_{T^{(k)}/T^{(k-1)}}, \quad 3.9$$

completing the induction. Using 3.9 in 3.7 yields that

$$\begin{aligned} LHS &= \sum_{\mu \vdash n+1} \frac{M \Pi_\mu}{w_\mu} \sum_{T \in \mathcal{T}_n; sh(T)=\mu} \prod_{k=1}^n w_T(k)^{a_k} c_{T^{(k)}/T^{(k-1)}} \\ &= \sum_{T \in \mathcal{T}_n} \Pi_T \frac{M}{w_T} \prod_{k=1}^n w_T(k)^{a_k} c_{T^{(k)}/T^{(k-1)}}. \end{aligned} \quad 3.10$$

Since the definition in 1.10 gives $W_{(1)} = M$ we can write

$$\Pi_T \frac{M}{w_T} = \prod_{k=1}^n (1 - cw(T^{(k)})) \frac{w_{T^{(k-1)}}}{w_{T^{(k)}}}$$

and 3.10 becomes

$$\begin{aligned} LHS &= \sum_{T \in \mathcal{T}_n} \prod_{k=1}^n w_T(k)^{a_k} (1 - cw(T^{(k)})) \frac{w_{T^{(k-1)}}}{w_{T^{(k)}}} c_{T^{(k)}/T^{(k-1)}} \\ (\text{Using 1.24}) &= \frac{1}{M^n} \sum_{T \in \mathcal{T}_n} \prod_{k=1}^n w_T(k)^{a_k} (1 - cw(T^{(k)})) d_{T^{(k)}/T^{(k-1)}}, \end{aligned}$$

completing our proof of 3.6.

Now recall that we have shown in 2.8 that

$$\sum_{A \in T(a_n, a_{n-1}, \dots, a_1)} wt(A) = \prod_{i=1}^n \frac{1}{z_i^{a_i}} \Omega[-z_i M] \prod_{1 \leq i < j \leq n} \Omega[-\frac{z_j}{z_i} M] \Big|_{z_1^0 z_2^0 \dots z_n^0} \quad 3.11$$

which 2.2 allows us to rewrite in the form

$$\sum_{A \in T(a_n, a_{n-1}, \dots, a_1)} wt(A) = \prod_{i=1}^n \frac{1}{z_i^{a_i}} \frac{(1 - z_i)(1 - qtz_i)}{(1 - tz_i)(1 - qz_i)} \prod_{1 \leq i < j \leq n} \frac{(1 - z_j/z_i)(1 - qtz_j/z_i)}{(1 - tz_j/z_i)(1 - qz_j/z_i)} \Big|_{z_1^0 z_2^0 \dots z_n^0}. \quad 3.12$$

Thus our proof of 3.1 will be complete by showing that

$$\begin{aligned} \prod_{i=1}^n \frac{1}{z_i^{a_i}} \frac{(1 - z_i)(1 - qtz_i)}{(1 - tz_i)(1 - qz_i)} \prod_{1 \leq i < j \leq n} \frac{(1 - z_j/z_i)(1 - qtz_j/z_i)}{(1 - tz_j/z_i)(1 - qz_j/z_i)} \Big|_{z_1^0 z_2^0 \dots z_n^0} &= \\ &= \sum_{T \in \mathcal{T}_n} \prod_{k=1}^n w_T(k)^{a_k} (1 - w_T(k)) d_{T^{(k)}/T^{(k-1)}}. \end{aligned} \quad 3.13$$

Remark 3.1

We will find it convenient to note that the factor $d_{T^{(k)}/T^{(k-1)}}$ and 3.13 itself can be given a revealing expression by manipulations with our “ Ω ” symbol. To see this we start by rewriting 1.23 in terms of Ω as follows

$$\begin{aligned}
d_{\mu^{(i)}, \nu} &= \frac{1}{x_i} \frac{\prod_{j=1}^m (1 - u_j/x_i)}{\prod_{j=0, j \neq i}^m (1 - x_j/x_i)} = z \frac{\prod_{j=1}^m (1 - zu_j)}{\prod_{j=0, j}^m (1 - zx_j)} (1 - zx_i) \Big|_{z=1/x_i} \\
&= z \Omega [z(x_0 + \cdots + x_m - u_1 - \cdots - u_m)] (1 - zx_i) \Big|_{z=1/x_i}.
\end{aligned}$$

Taking account that x_i is the weight of the cell that contains k in $T^{(k)}$, using 1.22 we obtain

$$\begin{aligned}
d_{T^{(k)}/T^{(k-1)}} &= z \Omega [-z(MB_{T^{(k-1)}} - 1)] (1 - zw_T(k)) \Big|_{1/z=w_T(k)} \\
&= z \Omega [-zMB_{T^{(k-1)}} + z - zw_T(k)] \Big|_{1/z=w_T(k)} \\
&= \frac{1}{w_T(k)} \Omega \left[-\frac{1}{w_T(k)} MB_{T^{(k-1)}} + \frac{1}{w_T(k)} - 1 \right] \\
&= \frac{1}{w_T(k)(1-\frac{1}{w_T(k)})} \Omega \left[-\frac{1}{w_T(k)} MB_{T^{(k-1)}} - 1 \right].
\end{aligned}$$

Using 1.22 again this may be rewritten as

$$(1 - w_T(k)) d_{T^{(k)}/T^{(k-1)}} = -\Omega \left[-\frac{1}{w_T(k)} M(1 + w_T(1) + \cdots + w_T(k-1)) - 1 \right]. \quad 3.14$$

Thus 3.13 becomes none other than

$$\begin{aligned}
&\prod_{i=1}^n \frac{1}{z_i^{a_i}} \frac{(1 - z_i)(1 - qtz_i)}{(1 - tz_i)(1 - qz_i)} \prod_{1 \leq i < j \leq n} \frac{(1 - z_j/z_i)(1 - qtz_j/z_i)}{(1 - tz_j/z_i)(1 - qz_j/z_i)} \Big|_{z_1^0 z_2^0 \cdots z_n^0} = \\
&= \sum_{T \in \mathcal{T}_n} (-1)^n \prod_{k=1}^n w_T(k)^{a_k} \Omega \left[-\frac{1}{w_T(k)} M(1 + w_T(1) + \cdots + w_T(k-1)) - 1 \right] \\
&= \sum_{T \in \mathcal{T}_n} (-1)^n \prod_{k=1}^n w_T(k)^{a_k} \Omega \left[-M/w_T(k) \right] \prod_{k=1}^n \Omega \left[-M \left(\sum_{h=1}^{k-1} w_T(h)/w_T(k) \right) - 1 \right].
\end{aligned}$$

In summary, the proof of 3.1 is thus reduced to verify the constant term identity

$$\begin{aligned}
&\frac{1}{M^n} \prod_{i=1}^n \frac{1}{z_i^{a_i}} \frac{(1 - z_i)(1 - qtz_i)}{(1 - tz_i)(1 - qz_i)} \prod_{1 \leq i < j \leq n} \frac{(1 - z_j/z_i)(1 - qtz_j/z_i)}{(1 - tz_j/z_i)(1 - qz_j/z_i)} \Big|_{z_1^0 z_2^0 \cdots z_n^0} = \\
&= \sum_{T \in \mathcal{T}_n} \left(-\frac{1}{M}\right)^n \prod_{k=1}^n w_T(k)^{a_k} \Omega \left[-M \sum_{1 \leq k \leq n} w_T(k)^{-1} - M \sum_{1 \leq h < k \leq n} w_T(h)/w_T(k) - n \right]. \quad 3.15
\end{aligned}$$

This will be done by means of the partial fraction algorithm developed in [9]. To benefit the reader who is unfamiliar with this algorithm we will briefly review here its basic steps.

Firstly, to avoid ordinary convergence problems we need to work in the field of iterated formal Laurent series. The definition of this field is recursive and is determined by a chosen total order of all the variables appearing in our given “kernel” \mathcal{H} . In the applications we are to compute the constant term of \mathcal{H} , usually denoted $\mathcal{H}_{=0}$. To be precise this operation will involve only a specific subset of the variables. For simplicity let us assume this subset to be z_1, z_2, \dots, z_k , and here we use the notation $\mathcal{H} \Big|_{z_1^0 z_2^0 \cdots z_k^0}$.

The first operation consists in expanding \mathcal{H} as a formal iterated Laurent series and selecting the terms that do not contain any of the variables z_1, z_2, \dots, z_k . This is done by a succession of a single variable constant term extractions.

Supposing that our variables, in the chosen total order, are z_1, z_2, \dots, z_n . Then, for a given field of scalars K the initial field is $K((z_1))$ consisting of formal Laurent series in z_1 with coefficients in K , that is the series in which z_1 appears with a negative exponent only in a finite number of terms. In symbols

$$K((z_1)) = \left\{ \sum_{m \geq M_0} a_m z_1^m : a_m \in K \right\}.$$

This given, recursively we define the field of iterated Laurent series $K((z_1))((z_2)) \cdots ((z_n))$ to be the field of formal Laurent series in z_n with coefficients in $K((z_1))((z_2)) \cdots ((z_{n-1}))$. The fundamental fact is that the total order allows us to imbed the field of rational functions $K(z_1, z_2, \dots, z_n)$ as a subfield of $K((z_1))((z_2)) \cdots ((z_n))$. We shall only describe here how this imbedding is carried out but leave all the matters of consistency to the original works [9], [10]. The important fact is that under this imbedding all the identities in $K(z_1, z_2, \dots, z_n)$ become identities in $K((z_1))((z_2)) \cdots ((z_n))$.

We will begin with the recipe for converting each rational function in the given variables into a formal Laurent series. The rational functions we will work with here may all be written in the form

$$F = \frac{P}{(1 - m_1)(1 - m_2) \cdots (1 - m_\ell)}$$

with P a Laurent polynomial and m_1, m_2, \dots, m_ℓ monomials in the given variables. Our first need is to be able to decide whether a given factor $\frac{1}{1 - m_i}$ should be converted to

$$a) \sum_{s \geq 0} m_i^s \quad \text{or} \quad b) - \sum_{s \geq 1} \frac{1}{m_i^s} \quad \left(= \frac{-\frac{1}{m_i}}{1 - \frac{1}{m_i}} \right)$$

The decision is based on the idea that the total order forces one of the two “*formal*” inequalities $m_i < 1$ or $m_i > 1$ to be true. In the first case, we choose a) (the “*ordinary form*”) and in the second case, we choose b) (the “*dual form*”). The criterion is as follows: we scan through the variables occurring in the monomial m_i . Suppose m_i has the variable z_j but not z_{j+1}, \dots, z_n . Then $m_i < 1$ if z_j has positive exponent and $m_i > 1$ if z_j has negative exponent.

For simplicity of notation we will avoid using summations and simply rewrite the given rational function in the form

$$F = P \times \left(\prod_{m_i < 1} \frac{1}{1 - m_i} \right) \times \left(\prod_{m_j > 1} \frac{-\frac{1}{m_j}}{1 - \frac{1}{m_j}} \right). \quad 3.16$$

We shall refer to this symbolic expression as the “*proper form*” of F .

To compute $F|_{z_1^0 z_2^0 \cdots z_k^0}$ by the partial fraction algorithm, at each step we use a partial fraction expansion to eliminate one of the variables z_1, z_2, \dots, z_k .

To see how this is done, assume that to begin we have chosen to eliminate the variable z . This given, by suitable manipulations we rewrite our rational function in the form

$$F = Q(z) + \frac{R(z)}{(1 - zU_1) \cdots (1 - zU_h)(z - V_1) \cdots (z - V_k)}$$

with $Q(z)$ a Laurent polynomial, $R(z)$ a polynomial of degree less than $h + k$ and U_1, U_2, \dots, U_h as well as V_1, V_2, \dots, V_k are monomials not containing z . The nature of the denominator will be determined by the requirement that

$$zU_i < 1 \quad \text{for } 1 \leq i \leq h \quad \text{and} \quad V_j/z < 1 \quad \text{for } 1 \leq j \leq k.$$

The next step is to derive the partial fraction expansion:

$$F = Q(z) + \sum_{i=1}^h \frac{A_i}{(1 - zU_i)} + \sum_{j=1}^k \frac{B_j}{(z - V_j)} \quad 3.17$$

which, as customary, is obtained by setting

$$A_i = (1 - zU_i)F(z) \Big|_{z=1/U_i} \quad \text{and} \quad B_j = (z - V_j)F(z) \Big|_{z=V_j}.$$

This immediately yields the equalities

$$F \Big|_{z^0} = Q(z) \Big|_{z^0} + \sum_{i=1}^h A_i. \quad 3.18$$

The reason for this is that $V_j/z < 1$ by assumption, so that the proper form of the last summation in 3.17 will be

$$\sum_{j=1}^k \frac{B_j/x}{(1 - V_j/z)}$$

and we see that the corresponding series contains only negative powers of z and thus yields no contribution to $F \Big|_{z^0}$. For this reason we say that the denominator factors $(1 - zU_i)$ are “*contributing*” and the factors $(z - V_j)$ are “*not contributing*”.

Keeping all this in mind, we will proceed to establish, by these methods, an auxiliary constant term result which at the same time will yield us both I.14 and 3.15.

To begin, for convenience, since we plan to eliminate the variables z_1, z_2, \dots, z_n in their natural order, we will write our kernel in the form

$$\mathcal{H}_n(z; q, t) = \frac{1}{(-M)^n} \prod_{1 \leq i \leq n} f(z_i) \prod_{1 \leq i < j \leq n} f(z_j/z_i). \quad 3.19$$

with

$$f(z) = \frac{(1 - z)(1 - qtz)}{(1 - tz)(1 - qz)}. \quad 3.20$$

This given the auxiliary constant term result may be stated as follows.

Theorem

Suppose $F_n(z) = \sum_{a_i \geq 1} c_{a_1, \dots, a_n} z_1^{-a_1} \cdots z_n^{-a_n}$ contains only negative powers in the z 's and c_{a_1, \dots, a_n} do not depend on the z 's. Then

$$\mathcal{H}_n(z; q, t) F_n(z) \Big|_{z^0} = \sum_{T \in \mathcal{T}_n} \mathcal{H}_n(z; q, t) F_n(z) \prod_{1 \leq k \leq n} (1 - z_k w_T(k)) \Big|_{S_T}. \quad 3.21$$

Here it must be understood that the substitution S_T has to be carried out iteratively. That means we first multiply by $(1 - z_1 w_T(1))$ and make the substitution for z_1 , then we multiply by $(1 - z_2 w_T(2))$ and make the substitution for z_2 , and so on, always keeping track of the cancellations that occur.

Proof

We will proceed by induction on n . For $n = 1$ we need to compute the constant term

$$\mathcal{H}_1(z; q, t) = \frac{1}{-M} \frac{(1 - z_1)(1 - qtz_1)}{(1 - qz_1)(1 - tz_1)} F_1(z) \Big|_{z_1^0}.$$

This is a proper rational function in z_1 and both denominator factors $(1 - qz_1)$ and $(1 - tz_1)$ are contributing, so the constant term is equal to

$$\mathcal{H}_1(z; q, t) F_1(z) \cdot (1 - z_1 q) \Big|_{z_1 = q^{-1}} + \mathcal{H}_1(z; q, t) F_1(z) \cdot (1 - z_1 t) \Big|_{z_1 = t^{-1}}.$$

This agrees with the right hand side of 3.21 since

$$\mathcal{T}_2 = \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, [0 \ 1] \right\}, \text{ and } S_{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} = \{z_1^{-1} = t\}, S_{[0 \ 1]} = \{z_1^{-1} = q\}.$$

Now suppose the theorem holds for n and we need to show that it holds for $n + 1$. So we need to compute the constant term

$$\begin{aligned} \mathcal{H}_{n+1}(z; q, t) F_{n+1}(z) \Big|_{z_1^0 z_2^0 \cdots z_{n+1}^0} &= \prod_{1 \leq i \leq n+1} \frac{(1 - z_i)(1 - qtz_i)}{(1 - qz_i)(1 - tz_i)} \prod_{1 \leq i < j \leq n+1} \frac{(1 - z_j/z_i)(1 - qtz_j/z_i)}{(1 - qz_j/z_i)(1 - tz_j/z_i)} F_{n+1}(z) \Big|_{z_1^0 z_2^0 \cdots z_{n+1}^0} \\ &= \prod_{1 \leq i \leq n} \frac{(1 - z_i)(1 - qtz_i)}{(1 - qz_i)(1 - tz_i)} \prod_{1 \leq i < j \leq n} \frac{(1 - z_j/z_i)(1 - qtz_j/z_i)}{(1 - qz_j/z_i)(1 - tz_j/z_i)} F'(z) \Big|_{z_1^0 z_2^0 \cdots z_{n+1}^0} \\ &= \mathcal{H}_n(z; q, t) F'(z) \Big|_{z_1^0 z_2^0 \cdots z_{n+1}^0} \end{aligned}$$

where

$$F'(z) = F_{n+1}(z) \frac{(1 - z_{n+1})(1 - qtz_{n+1})}{(1 - qz_{n+1})(1 - tz_{n+1})} \prod_{1 \leq i \leq n} \frac{(1 - z_{n+1}/z_i)(1 - qtz_{n+1}/z_i)}{(1 - qz_{n+1}/z_i)(1 - tz_{n+1}/z_i)}$$

contains only negative powers in z_i for $i = 1, 2, \dots, n$. Thus the inductive hypothesis applies to the variables z_1, \dots, z_n , and we obtain

$$\mathcal{H}_{n+1}(z; q, t) F_{n+1}(z) \Big|_{z_1^0 z_2^0 \cdots z_{n+1}^0} = \sum_{T \in \mathcal{T}_n} \mathcal{H}_n(z; q, t) F'(z) \prod_{1 \leq k \leq n} (1 - z_k w_T(k)) \Big|_{S_T} \Big|_{z_{k+1}^0}.$$

Since every SYT on $\{0, \dots, n+1\}$ is uniquely obtained from an SYT on $\{0, \dots, n\}$ by adding $n+1$ to an outer corner, it is sufficient to show that for every $T \in \mathcal{T}_n$ we have

$$\mathcal{H}_n(z; q, t)F'(z) \prod_{1 \leq k \leq n} (1 - z_k w_T(k)) \Big|_{S_T} \Big|_{z_{n+1}^0} = \sum_{T'} \mathcal{H}_{n+1}(z; q, t)F(z) \prod_{1 \leq k \leq n+1} (1 - z_{n+1} w_{T'}(n+1)) \Big|_{S_{T'}},$$

where T' ranges over all SYTs obtained from T by adding $n+1$ to an outer corner.

When taking constant term with respect to z_{n+1} , we only need to consider factors containing z_{n+1} , so we only need to work with the product

$$\begin{aligned} F_{n+1}(z) \frac{(1 - z_{n+1})(1 - qtz_{n+1})}{(1 - qz_{n+1})(1 - tz_{n+1})} \prod_{1 \leq k \leq n} \frac{(1 - z_{n+1}/z_k)(1 - qtz_{n+1}/z_k)}{(1 - qz_{n+1}/z_k)(1 - tz_{n+1}/z_k)} \Big|_{S_T} &= \\ = F_{n+1}(z) \Big|_{S_T} \frac{(1 - z_{n+1})(1 - qtz_{n+1})}{(1 - qz_{n+1})(1 - tz_{n+1})} \prod_{1 \leq k \leq n} \frac{(1 - z_{n+1}w_T(k))(1 - qtz_{n+1}w_T(k))}{(1 - qz_{n+1}w_T(k))(1 - tz_{n+1}w_T(k))}. \end{aligned}$$

This is a proper rational function in z_{n+1} , since $F_{n+1}(z)$ contains only negative powers in z_{n+1} . But it is not clear which denominator factors are contributing. Indeed there are plenty of cancelations that are easier to describe using the Omega notation. Now, the key part of our product can be rewritten in the form

$$\Omega[-Mz_{n+1}] \prod_{1 \leq k \leq n} \Omega[-Mz_{n+1}w_T(k)] = \Omega[-Mz_{n+1}(1 + w_T(1) + \dots + w_T(n))] = \Omega[-z_{n+1}MB_T],$$

where B_T is precisely as defined in 3.3. Now using 1.22 we may in turn rewrite this as

$$\Omega[z_{n+1}(x_0 + x_1 + \dots + x_m - u_0 - u_1 - \dots - u_m)] = \frac{\prod_{j=0}^m (1 - z_{n+1}u_j)}{\prod_{j=0}^m (1 - z_{n+1}x_j)},$$

where $u_0 = 1$, x_0, x_1, \dots, x_m and u_0, u_1, \dots, u_m are the corner weights of the shape of T . It results from this that the contributing factors in the denominators are $(1 - x_i z_{n+1})$ for x_i the weight of an outer corner of T . For each such corner cell c construct T' by adding $n+1$ to T at the cell c . We thus obtain

$$\mathcal{H}_n(z; q, t)F'(z) \prod_{1 \leq k \leq n} (1 - z_k w_T(k)) \Big|_{S_T} \Big|_{z_{n+1}^0} = \sum_{T'} \mathcal{H}_{n+1}(z; q, t)F_{n+1}(z) \prod_{1 \leq k \leq n+1} (1 - z_k w_{T'}(k)) \Big|_{S_{T'}}$$

where T' ranges over all SYT's obtained from T by adding $n+1$ to an outer corner.

This completes the proof.

We can clearly see that, by appropriately specializing $F_n(z)$, 3.21 gives I.14. To see that it contains 3.15 as well we need only give the right hand side of 3.21 a non recursive construction. To this end, note that from the proof, we see that $(1 - z_k w_T(k))$ cancels with one factor in the denominator. This is $\Omega[z_k w_T(k)]$ and becomes $\Omega[1]$ after the substitution $z_k^{-1} = w_T(k)$. The same substitution also produces an “-1” to cancel this “1” within the Omega bracket. However $\Omega[1]$ itself has no meaning and the cancelation $\Omega[1 - 1] = 1$ sometimes is not correct. For instance, we may have the following

$$\Omega[z - z^3] \Big|_{z=1} = \frac{1 - z^3}{1 - z} \Big|_{z=1} = 1 + z + z^2 \Big|_{z=1} = 3.$$

This is not a problem but rather only another manifestation of the 0/0 form in calculus limits. The general situation is guided by the following rule:

$$\Omega[z^{a_1} + \dots + z^{a_\ell} - z^{b_1} - \dots - z^{b_\ell}] \Big|_{z=1} = \prod_{i=1}^{\ell} \frac{(1 - z^{b_i})/(1 - z)}{(1 - z^{a_i})/(1 - z)} \Big|_{z=1} = \frac{b_1 \cdots b_\ell}{a_1 \cdots a_\ell}.$$

In our calculations, the a_i and b_i are all equal to 1, so we can simply cancel them. With this provision the result of the recursive use of the substitution set S_T can also be computed as follows

$$\begin{aligned} \left(-\frac{1}{M}\right)^n \prod_{1 \leq i \leq n} f(z_i) \prod_{1 \leq i > j \leq n} f(z_j/z_i) \prod_{1 \leq k \leq n} (1 - z_k w_T(k)) \Big|_{S_T} &= \\ &= \left(-\frac{1}{M}\right)^n \Omega \left[-M(z_1 + \dots + z_n) - M \sum_{1 \leq i < j \leq m} z_j/z_i - \sum_{1 \leq k \leq n} z_k w_T(k) \right] \Big|_{S_T} \\ &= \left(-\frac{1}{M}\right)^n \Omega \left[-M \sum_{1 \leq k \leq n} w_i(k)^{-1} - M \sum_{1 \leq i < j \leq m} w_T(i)/w_T(j) - n \right] \end{aligned}$$

Using this expression for the summand in 3.21 we clearly see that 3.15 is also a special case of 3.21.

As we mentioned in the introduction computer data suggests that the Tesler polynomial

$$P_{a_1, a_2, \dots, a_n}(q, t) = \left(-\frac{1}{M}\right)^n \sum_{A \in \mathcal{T}(a_1, a_2, \dots, a_n)} wt(A) \tag{3.22}$$

has non-negative integer coefficients when $a_1 \leq a_2 \leq \dots \leq a_n$. Further computer explorations prompted by the identity

$$P_{a_1, a_2, \dots, a_n}(q, t) = (-1)^n M \partial_{p_1} \nabla^{-a_n} \partial_{p_1} \nabla^{a_n - a_{n-1}} \dots \partial_{p_1} \nabla^{a_3 - a_2} \partial_{p_1} \nabla^{a_2 - a_1} \partial_{p_1} \nabla^{a_1} p_{n+1} \left[\frac{X}{M} \right] \tag{3.23}$$

revealed an even stronger fact. Namely, starting with $p_{n+1}[\frac{X}{M}]$, repeated applications of the operator $\partial_{p_1} \nabla^a$ (with $a \geq 0$) invariably yields a Schur positive symmetric function. Thus it seems plausible that all these positivities may have a Representation Theoretical proof based on Haiman's discovery [7] of the role of the operator ∇ in the Algebraic Geometry of the Hilbert Scheme. The authors have been trying to contact Mark Haiman regarding this matter, but so far without success.

At any rate it would be preferable to have a more elementary approach to proving these positivities. In the last few pages it will be instructive to see how this can be achieved by constant term methods.

To this end let us set

$$\mathcal{K}(a_1, \dots, a_n) := \frac{1}{(-M)^n x_1^{a_1} \cdots x_n^{a_n}} \prod_{1 \leq i \leq n} f(x_i) \prod_{1 \leq i < j \leq n} f(x_j/x_i) \Big|_{x^0}. \tag{3.24}$$

This given, we can form the generating function

$$G_n(y_1, \dots, y_n) = \sum_{a_i \geq 0} \mathcal{K}(a_1 + \dots + a_n + 1, a_2 + \dots + a_n + 1, \dots, a_n + 1) y_1^{a_1} \cdots y_n^{a_n} \tag{3.25}$$

and show that all coefficients of $G(y)$ in the y_i 's are polynomials in $\mathbf{N}[q, t]$.

We can obtain a constant term expression for $G_n(y)$ as follows. It is easy to see that

$$\sum_{a_i \geq 0} \frac{y_1^{a_1} \cdots y_n^{a_n}}{x_1^{a_1 + \cdots + a_n + 1} \cdots x_n^{a_n + 1}} = \frac{x_1^{-1} \cdots x_n^{-1}}{\left(1 - \frac{y_1}{x_1}\right) \left(1 - \frac{y_2}{x_1 x_2}\right) \cdots \left(1 - \frac{y_n}{x_1 x_2 \cdots x_n}\right)}. \quad 3.26$$

Thus it follows that

$$G_n(y) = \mathcal{G}_n(y; x) \Big|_{x^0}, \quad 3.27$$

where

$$\mathcal{G}_n(y; x) = \frac{x_1^{-1} \cdots x_n^{-1}}{\left(1 - \frac{y_1}{x_1}\right) \left(1 - \frac{y_2}{x_1 x_2}\right) \cdots \left(1 - \frac{y_n}{x_1 x_2 \cdots x_n}\right)} \frac{1}{(-M)^n} \prod_{1 \leq i \leq n} f(x_i) \prod_{1 \leq i < j \leq n} f(x_j/x_i).$$

Using this formula we can compute $G_n(y)$ for $n = 1, 2, 3$ quickly using Xin's Ell2 package. For example,

$$G_2(y) = \frac{1 + t + q - qty_1 + qt^2y_2 + q^2ty_2 + q^2t^2y_2 - qt^3y_1y_2 - q^2t^2y_1y_2 - q^3ty_1y_2 - q^2t^3y_1y_2 - q^3t^2y_1y_2}{(1 - ty_1)(1 - qy_1)(1 - qty_2)(1 - q^3y_2)(1 - t^3y_2)}.$$

In this form, showing the positivity does not appear promising. The following lemma is straightforward, but significantly reduces the complexity.

Lemma 3.1

Suppose $F(q, t) = \sum_{i,j} c_{i,j} q^i t^j$ is a polynomial with integer coefficients. Then the positivity of

$$\tilde{F}(q, t) = \sum_{i>j} (c_{i,j} - c_{j,i}) q^i t^j$$

implies the positivity of the divided difference

$$\partial_{q,t} F(q, t) = \frac{F(q, t) - F(t, q)}{q - t} = \partial_{q,t} \tilde{F}(q, t) \in \mathbf{N}[q, t].$$

Proof

Direct computation yields

$$\partial_{q,t} q^i t^j = \begin{cases} q^{i-1} t^j + q^{i-2} t^{j+1} + \cdots + q^j t^{i-1} & \text{if } i > j, \\ 0, & \text{if } i = j, \\ -\partial_{q,t} q^j t^i = -(q^{j-1} t^i + q^{j-2} t^{i+1} + \cdots + q^i t^{j-1}), & \text{if } i < j. \end{cases}$$

which clearly implies the lemma.

Remark 3.2

The lemma clearly extends by linearity for series with coefficients polynomials in q, t . Moreover, it is not hard to see that $\tilde{F}(q, t)$ can be computed by

$$\tilde{F}(q, t) = F(\alpha, \beta) \frac{1}{\left(1 - \frac{q}{\alpha}\right) \left(1 - \frac{qt}{\alpha\beta}\right)} \Big|_{\alpha^0 \beta^0} - F(\alpha, \beta) \frac{1}{\left(1 - \frac{q}{\beta}\right) \left(1 - \frac{qt}{\alpha\beta}\right)} \Big|_{\alpha^0 \beta^0}.$$

Theorem 3.2

The positivity conjecture holds for $n = 1, 2, 3$.

Proof

The $n = 1$ case is trivial since $G_1(y_1) = \frac{y_1}{(1-ty_1)(1-ty_1)}$.

For $n \geq 2$ we can show the positivity as follows. We first take constant term in x_1 . That is the constant term in x_1 of

$$\frac{x_1^{-1} \cdots x_n^{-1}}{(1 - \frac{y_1}{x_1})(1 - \frac{y_2}{x_1 x_2}) \cdots (1 - \frac{y_n}{x_1 x_2 \cdots x_n})} \frac{(1-x_1)(1-qt x_1)}{(1-qx_1)(1-tx_1)} \prod_{j=2}^n \frac{(1-x_j/x_1)(1-qt x_j/x_1)}{(1-qx_j/x_1)(1-tx_j/x_1)}.$$

Only the factors $(1 - qx_1)$ and $(1 - tx_1)$ are contributing, so we have

$$\mathcal{G}_n(y; x) \Big|_{x_1^0} = \mathcal{G}_n(y; x)(1 - qx_1) \Big|_{x_1=q^{-1}} + \mathcal{G}_n(y; x)(1 - tx_1) \Big|_{x_1=t^{-1}}.$$

Now observe that $\mathcal{G}_n(y; x)$ is symmetric in q and t . Thus if we set

$$F_n(y; q, t) = (q - t) \mathcal{G}_n(y; x)(1 - qx_1) \Big|_{x_1=q^{-1}} \Big|_{x_1^0},$$

then we shall have

$$G_n(y) = \partial_{q,t} F_n(y; q, t) = \frac{F_n(y; q, t) - F_n(y; t, q)}{q - t}.$$

The Ell2 package delivers $F_n(y; q, t)$ quickly for $n = 2, 3, 4$. The $n = 2$ case is simple:

$$F_2(y; q, t) = \frac{q(1+q)}{(1-ty_1)(1-qt y_2)(1-q^3 y_2)}.$$

Lemma 3.1 applies with $\tilde{F}_2(y; q, t) = F_2(y; q, t)$ and the positivity of $G_2(y)$ follows. For the $n = 3$ case, we obtain

$$F_3(y; q, t) = \frac{(a \text{ lengthy polynomial})}{(1-ty_1)(1-qt y_2)(1-q^3 y_2)(1-qt^3 y_3)(1-q^2 t^2 y_3)(1-q^3 t y_3)(1-q^6 y_3)}.$$

We can use Remark 3.2 to compute $\tilde{F}_3(y; q, t)$. By letting $E_3 = E_3(y; q, t) = \tilde{F}_3(y; q, t/q)$, we have

$$E_3(y; q, t) = \frac{(a \text{ lengthy polynomial})}{(1-ty_1)(1-ty_2)(1-t^2 y_3)(1+t^2 y_3)(1-q^2 t y_3)(1-q^3 y_2)(1-t^3 y_1^2 y_3)(1-q^6 y_3)}.$$

The positivity of $G_3(y)$ clearly follows from that of E_3 , which is obvious once we rewrite

$$\begin{aligned} E_3(y; q, t) &= \frac{(1+q)^2 q^2}{(1-ty_2)(1-ty_1)(1-t^2 y_3)(1-q^2 t y_3)} \\ &+ \frac{y_2 q^4 (1+2q+2q^2+q^3)}{(1-ty_2)(1-ty_1)(1-t^2 y_3)(1-q^3 y_2)(1-q^2 t y_3)} \\ &+ \frac{y_3 q^4 (t+q^3)(1+2q+2q^2+q^3)}{(1-ty_1)(1-t^2 y_3)(1-q^3 y_2)(1-q^2 t y_3)(1-q^6 y_3)} \\ &+ \frac{q(1+t+q+ty_1+qt^2 y_3+t^3 y_1 y_3+qt^2 y_1 y_3+qt^3 y_1 y_3)}{(1-ty_2)(1-ty_1)(1-q^2 t y_3)(1-t^3 y_1^2 y_3)(1-t^4 y_3^2)} \end{aligned}$$

The above form is obtained by using partial fraction decompositions and inspections. We first observe the positivity of $E_3(y; q, t)(1 - y_1q)(1 - y_3q^2t)$, whose partial fraction with respect to y_1 is reasonably simple:

$$E_3(y; q, t)(1 - y_1q)(1 - y_3q^2t) = Q + \frac{q(1 + t + q + ty_1 + qt^2y_3 + t^3y_1y_3 + qt^2y_1y_3 + qt^3y_1y_3)}{(1 - ty_2)(1 + t^2y_3)(1 - t^2y_3)(1 - t^3y_1^2y_3)},$$

where Q is a little complex and the second term is already positive. Now the partial fraction decomposition of Q with respect to y_2 is

$$Q = \frac{q^2(t + 2qt + q^2 + q^2t + q^3)}{(1 - ty_2)(1 - t^2y_3)(t - q^3)} - \frac{q^4(1 + 2q + 2q^2 + q^3)}{(t - q^3)(1 - q^3y_2)(1 - q^6y_3)}.$$

It is an easy exercise to obtain from the above the desired form.

Our method at the moment appears forbidding for $n = 4$. The formula for $\tilde{F}_4(y; q, t)$ is already significantly complex.

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