Some New Symmetric Function Tools
and
their Applications

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Abstract

We prove a technical identity involving the Δ operator from Macdonald polynomial theory, which allows us to transform expressions involving the Δ operator and the Hall scalar product into other such expressions. We show how our technical identity, although following easily from the well-known Koornwinder-Macdonald reciprocity theorem, contains as special cases several identities occuring in the literature, proved there by more complicated arguments. We also show how our identity can be used to obtain some new expressions for the q,t-Narayana numbers introduced by Dukes and Le Borgne, as well as new identities involving the Δ operator and the power sum symmetric function p_n.

1 Introduction

We assume the reader is familiar with the standard notation involving bases of the ring of symmetric functions, as used in the classic texts [19, Chapter 1] and [22, Chapter 7]; the Schur functions s_λ, the Hall scalar product $\langle , \rangle$ (with respect to which the s_λ are orthonormal), the monomial symmetric functions m_λ, the complete homogeneous symmetric functions h_n, elementary symmetric functions e_n, and power-sums $p_n = \sum_i x_i^n$. Note that $h_n = s_n$ and $e_n = s_{1^n} = m_{1^n}$.

Another important basis, the Macdonald symmetric function basis $J_\mu(X; q, t)$, was introduced by Macdonald in 1988 [18], [19, Chapter VI]. The J_μ depend on two parameters, q, t, and contain the various bases listed above, as well as several other popular bases, as limiting or special cases. Macdonald conjectured that the coefficients of the J_μ, when expanded in a certain basis connected to the Schur basis, were in $\mathbb{N}[q,t]$, which became known as the Macdonald Positivity Conjecture. Garsia and Haiman [10] refined this conjecture by suggesting that the Schur coefficients of a modified version of the Macdonald polynomial, denoted $\tilde{H}_\mu(X;q,t)$, have a representation theoretical interpretation which implies their positivity. Specifically, the $\tilde{H}_\mu$ are the Frobenius image of the bigraded character of a module $M_\mu \subset \mathbb{C}[X_n, Y_n]$ under the diagonal action of S_n, where $X_n = x_1, \ldots, x_n$ and $Y_n = y_1, \ldots, y_n$. This module is the linear span of derivatives of a determinant Δ_μ, a bialternant in X_n, Y_n. Haiman proved their refinement of the Macdonald Positivity Conjecture in 2001 using the geometry of the Hilbert scheme [12].

A module which is closely related to the Garsia-Haiman modules is the space of diagonal harmonics DH_n, defined as the space of polynomials in $\mathbb{C}[X_n, Y_n]$ that are killed by the partial differential operators

$$\sum_{i=1}^{n} \partial_{x_i}^r \partial_{y_i}^s,$$

for all nonnegative integers r, s satisfying $r+s>0$. For each $\mu \vdash n$ we have $M_\mu \subseteq \mathrm{DH}_n$. Haiman's famous formula for the Frobenius image of the bigraded character of the action on DH_n (conjectured in [9] and proved in [13]) is

$$\mathcal{F}(char_{q,t} DH_n) = \sum_{\mu \vdash n} \frac{MB_\mu \Pi_\mu}{w_\mu} T_\mu \, \widetilde{\mathrm{H}}_\mu \,. \tag{1}$$

Here T_μ is a certain monomial in q, t, which, together with the other factors on the right-hand-side of (1), is defined in the next section. Comparing (1) to the following known identity giving the expansion of the elementary symmetric function e_n in terms of the modified Macdonald basis,

$$e_n = \sum_{\mu \vdash n} \frac{MB_\mu \Pi_\mu}{w_\mu} \, \widetilde{\mathrm{H}}_\mu, \tag{2}$$

we see that the difference between the coefficient of $\tilde{H}_\mu$ in (1) and the coefficient in (2) is just the factor T_μ.

This similarity led Bergeron and Garsia [3] to introduce the operator ∇ which acts diagonally on the modified Macdonald basis by $\nabla \, \widetilde{\mathrm{H}}_\mu = T_\mu \, \widetilde{\mathrm{H}}_\mu$. A consequence of (1) is that $\nabla e_n = \mathcal{F}(char_{q,t} DH_n)$, and hence when ∇e_n is expanded in terms of Schur functions, the coefficients are in $\mathbb{N}[q,t]$, i.e. ∇e_n is Schur positive. From the representation theoretical side, this is clear, but is quite mysterious looking just at formula (2). Bergeron, Garsia, Haiman, and Tesler [4] subsequently introduced an important generalization of ∇, the Delta operator Δ_F indexed by a symmetric function F, which we define in the next section. The $\tilde{H}_\mu$ are also eigenfunctions of Δ_F for any F, and as operators on symmetric functions of homogeneous degree n, $\nabla = \Delta_{e_n}$.

In [13], Haiman proved that $\Delta_{s_\lambda e_n} e_n$ is Schur positive for any λ, and furthermore conjectured that $\Delta_{s_\lambda} e_n$ is Schur positive for any λ. There has been a lot of research over the past fifteen years devoted to trying to understand the combinatorial structure of the coefficients when $\Delta_{s_\lambda} e_n$ is expanded in terms of the monomial symmetric functions, for various choices of λ. (Ideally, we would like to understand the Schur coefficients, but so far this has not proved tractable). The original Shuffle Conjecture of Haglund, Haiman, Loehr, Remmel, and Ulyanov [16] gives a combinatorial interpretation for the expansion of $\Delta_{e_n} e_n$ in terms of monomials. Haglund, Remmel, and Wilson [17] recently introduced a generalization of this they call the Delta Conjecture, which gives a combinatorial interpretation for the expansion of $\Delta_{e_k} e_n$ into monomials for any $k, n \in \mathbb{N}$. The Shuffle Conjecture was proved by Carlsson and Mellit in 2015 [5] but the Delta Conjecture is still open.

In this article we introduce (Theorem 1) a new transformation identity which allows one to express certain coefficients involving the Hall scalar product and the Δ operator in terms of other such coefficients. In Section 2 we introduce notation and collect a few standard identities used later on. In Section 3.1 we prove our main result and show how special cases of this transformation reduce to other transformations which played a central role in [14], [6], and [7]. In Section 3.2 we use Theorem 1 to obtain some new transformations involving p_n. Section 4 contains applications to the study of Dukes and Le Borgne's q,t-Narayana numbers [8], as well as other combinatorial topics connected to the study of the bigraded character of DH_n and the Delta Conjecture.

2 Preliminaries

Given a partition μ drawn as a French diagram, and a cell $c \in \mu$, we let $l(c), a(c), l'(c)$, and $a'(c)$ be the number of cells in μ strictly North, East, South, and West of c, respectively. These give the leg, arm, coleg, and coarm of the cell c. We will use the notation

$$\Pi_\mu(q,t) = \prod_{c\in\mu/(0,0)} (1 - q^{a'(c)}t^{l'(c)}), \qquad w_\mu(q,t) = \prod_{c\in\mu}(q^{a(c)} - t^{l(c)+1})(t^{l(c)} - q^{a(c)+1}),$$

$$B_\mu(q,t) = \sum_{c\in\mu} q^{a'(c)}t^{l'(c)}, \qquad \text{and} \qquad M = (1-q)(1-t).$$

We will also need $n(\mu) = \sum_{i=1}^{\ell(\mu)}(i-1)\mu_i$ and $T_\mu(q,t) = q^{n(\mu')}t^{n(\mu)}$.

Throughout this article, square brackets denote plethystic substitution, as in $F[E]$, which denotes the plethystic substitution of the expression E into F. If E is a positive alphabet, then $F[E] = F(E)$, the usual symmetric function F evaluated at the alphabet E. For background and further examples of plethystic notation see [15, Chapter 1]. For any symmetric function $F[X]$, let Δ_F be the linear operator defined on the Macdonald basis $\widetilde{\mathrm{H}}_\mu$ via

$$\Delta_F \widetilde{\mathrm{H}}_\mu[X;q,t] = F[B_\mu]\widetilde{\mathrm{H}}_\mu[X;q,t]. \tag{3}$$

For example, $B_{(2,1)}(q,t) = 1+q+t$, and $\Delta_{h_2}\tilde{H}_{(2,1)} = h_2(1,q,t)\tilde{H}_{(2,1)} = (q^2+t^2+1+qt+q+t)\tilde{H}_{(2,1)}$.

One of the defining properties of modified Macdonald polynomials are the orthogonality relations under the $*$-scalar product. The $*$-scalar product is defined by setting

$$\langle p_\lambda, p_\mu\rangle_* = (-1)^{|\mu|-\ell(\mu)} z_\mu p_\mu[M]\chi(\lambda = \mu),$$

where $\chi(A)$ is 1 if A is true and 0 otherwise. We can rewrite the relation as

$$\langle p_\lambda, p_\mu\rangle_* = \langle p_\lambda, \omega(p_\mu)[MX]\rangle$$

This gives the relationship between the $*$-scalar product and the Hall scalar product. For any two symmetric functions F and G, we have

$$\langle F[X], G[X]\rangle_* = \langle F[X], \omega(G)[MX]\rangle.$$

The orthogonality relations for modified Macdonald polynomials are given by

$$\langle \widetilde{\mathrm{H}}_\lambda[X], \widetilde{\mathrm{H}}_\mu[X]\rangle_* = \langle \widetilde{\mathrm{H}}_\lambda[X], \omega(\widetilde{\mathrm{H}}_\mu)[MX]\rangle = w_\mu(q,t)\chi(\lambda=\mu). \tag{4}$$

If we let

$$\alpha_n = \frac{(-1)^{n-1}}{[n]_q[n]_t} = \frac{(-1)^{n-1}M}{(1-q^n)(1-t^n)},$$

then we can write the expansion of p_n in terms of the modified Macdonald basis as

$$\alpha_n p_n = \sum_{\mu\vdash n} \frac{M\Pi_\mu}{w_\mu}\widetilde{\mathrm{H}}_\mu[X]. \tag{5}$$

The summands for $\alpha_n p_n$ and e_n (from (2)) differ by a factor of $B_\mu = e_1[B_\mu]$. Thus

$$\Delta_{e_1}\alpha_n p_n = e_n. \tag{6}$$

The expansion for h_n is given [4] by

$$h_n = (-qt)^{1-n} \sum_{\mu \vdash n} \frac{M\Pi_\mu B_\mu(1/q, 1/t)}{w_\mu} \widetilde{\mathrm{H}}_\mu\,. \tag{7}$$

Using the fact that for $\mu \vdash n$, $B_\mu(1/q, 1/t)T_\mu = e_{n-1}[B_\mu]$ and $e_n[B_\mu] = T_\mu$, we can write

$$h_n = (-qt)^{1-n}\Delta_{e_n}^{-1}\Delta_{e_{n-1}}\alpha_n p_n, \tag{8}$$

where for any f, $\Delta_f^{-1}\tilde{H}_\mu = \tilde{H}_\mu / f[B_\mu]$.

For any two partitions α and β, Macdonald-Koorwinder reciprocity [19], [11] gives a fundamental relation between the Macdonald polynomial indexed by α and the one indexed by β. Expressed in terms of the $\tilde{H}_\mu$ it says

$$\frac{\widetilde{\mathrm{H}}_\alpha\left[1 + u\left(MB_\beta - 1\right)\right]}{\prod_{c\in\alpha} 1 - uq^{a'(c)}t^{l'(c)}} = \frac{\widetilde{\mathrm{H}}_\beta\left[1 + u\left(MB_\alpha - 1\right)\right]}{\prod_{c\in\beta} 1 - uq^{a'(c)}t^{l'(c)}}. \tag{9}$$

Multiplying both sides by $(1-u)$ and letting $u \to 1$, we get the form of Macdonald reciprocity we will make the most use of:

$$\frac{\widetilde{\mathrm{H}}_\alpha[MB_\beta]}{\Pi_\alpha} = \frac{\widetilde{\mathrm{H}}_\beta[MB_\alpha]}{\Pi_\beta}. \tag{10}$$

We should note that the expansions of e_n and p_n described above are a consequence of the orthogonality relations (4) and (9).

Suppose that for a given symmetric function F, there exists a symmetric function G so that

$$\langle \widetilde{\mathrm{H}}_\mu, F\rangle = G[B_\mu],$$

for all $\mu \vdash n$. Then since $\langle \widetilde{\mathrm{H}}_\mu, s_n\rangle = 1$, it follows that $\langle \widetilde{\mathrm{H}}_\mu, F\rangle = \langle \Delta_G \widetilde{\mathrm{H}}_\mu, s_n\rangle$. This will be useful in the case that $F = e_k h_{n-k}$ since it is known [19, p. 362] that $\langle \widetilde{\mathrm{H}}_\mu, e_k h_{n-k}\rangle = e_k[B_\mu]$. It follows that for any homogeneous symmetric function P of degree n, we have

$$\langle \Delta_{e_k} P, s_n\rangle = \langle P, e_k h_{n-k}\rangle. \tag{11}$$

3 Scalar product identities

3.1 A General transformation

In [14], identity (14) below played a key role, and further applications of (14) were derived in [6], [7] and [20]. The main result in this article is the following more general form of (14), which can be shown to be equivalent to Macdonald reciprocity. After proving it, we devote the rest of the article to exploring applications.

Theorem 1 *Let P be a homogeneous symmetric function of degree k, and let Q be a homogeneous symmetric function of degree n. Then*

$$\langle \Delta_{\omega(P)}\alpha_n p_n, Q\rangle = \langle \Delta_{\omega(Q)}\alpha_k p_k, P\rangle. \tag{12}$$

Proof. It suffices to check this equality over a basis. We will assume $P = \omega\,\widetilde{\mathrm{H}}_\alpha[MX]$ and $Q = \omega\,\widetilde{\mathrm{H}}_\beta[MX]$ for arbitrary partitions $\alpha \vdash k$ and $\beta \vdash n$. Since

$$\Delta_{\omega(P)}\alpha_n p_n = \sum_{\mu\vdash n} \frac{M\Pi_\mu \omega(P)[B_\mu]}{w_\mu}\,\widetilde{\mathrm{H}}_\mu[X],$$

we have

$$\begin{aligned}
\langle \Delta_{\omega(P)}\alpha_n p_n, Q\rangle &= \sum_{\mu\vdash n} \frac{M\Pi_\mu \omega(\omega\,\widetilde{\mathrm{H}}_\alpha[MX])[B_\mu]}{w_\mu}\langle \widetilde{\mathrm{H}}_\mu[X], \omega\,\widetilde{\mathrm{H}}_\beta[MX]\rangle \\
&= \sum_{\mu\vdash n} \frac{M\Pi_\mu\,\widetilde{\mathrm{H}}_\alpha[MB_\mu]}{w_\mu}\langle \widetilde{\mathrm{H}}_\mu, \widetilde{\mathrm{H}}_\beta\rangle_* \\
&= \frac{M\Pi_\beta\,\widetilde{\mathrm{H}}_\alpha[MB_\beta]}{w_\beta} w_\beta \quad = \quad M\Pi_\beta\,\widetilde{\mathrm{H}}_\alpha[MB_\beta].
\end{aligned}$$

A similar computation gives

$$\langle \Delta_{\omega(Q)}\alpha_k p_k, P\rangle = M\Pi_\alpha\,\widetilde{\mathrm{H}}_\beta[MB_\alpha]$$

Therefore, equality holds precisely when

$$\Pi_\alpha\,\widetilde{\mathrm{H}}_\beta[MB_\alpha] = \Pi_\beta\,\widetilde{\mathrm{H}}_\alpha[MB_\beta],$$

which is another way of writing Macdonald reciprocity (10). □

Remark 1 *Note that if $n = k$, so both P and Q are of homogeneous degree n, we can cancel the factor α_n on both sides of (12), leaving the more compact form*

$$\langle \Delta_{\omega(P)} p_n, Q\rangle = \langle \Delta_{\omega(Q)} p_n, P\rangle. \tag{13}$$

We now list a few significant special cases of Theorem 1, both of which have already appeared in the literature, but with longer and less insightful proofs.

Corollary 1 *[14, Corollary 2.9] For $k > 0$ and any homogeneous symmetric function Q of degree n, we have*

$$\langle \Delta_{e_{k-1}} e_n, Q\rangle = \langle \Delta_{\omega(Q)} e_k, s_k\rangle. \tag{14}$$

Proof. We have seen that

$$e_n = \Delta_{e_1}\alpha_n p_n.$$

Therefore, using Theorem 1 with $P = h_{k-1}e_1$ we have

$$\begin{aligned}
\langle \Delta_{e_{k-1}} e_n, Q\rangle &= \langle \Delta_{\omega(h_{k-1}e_1)}\alpha_n p_n, Q\rangle \\
&= \langle \Delta_{\omega(Q)}\alpha_k p_k, h_{k-1}e_1\rangle \\
\text{(by (11))} \quad &= \langle \Delta_{\omega(Q)e_1}\alpha_k p_k, s_k\rangle \\
&= \langle \Delta_{\omega(Q)} e_k, s_k\rangle.
\end{aligned}$$

□

Corollary 2 *[2, Proposition 4.2] For $k > 0$ and any homogeneous symmetric function Q of degree n, we have*

$$\langle \Delta_{h_{k-1}} e_n, Q\rangle = (-qt)^{k-1}\langle \Delta_{\omega Q} h_k, s_{1^k}\rangle. \tag{15}$$

Proof. Letting $n = k$ in (8) yields

$$h_k = (-qt)^{1-k}\Delta_{e_k}^{-1}\Delta_{e_{k-1}}\alpha_k p_k.$$

Plugging this into the right-hand side of (15) we get

$$\begin{aligned}
(-qt)^{k-1}\langle\Delta_{\omega(Q)}h_k, s_{1^k}\rangle &= (-qt)^{k-1}\langle(-qt)^{1-k}\Delta_{\omega(Q)}\Delta_{e_k}^{-1}\Delta_{e_{k-1}}\alpha_k p_k, e_k\rangle \\
\text{(by (11))} \quad &= \langle\Delta_{e_k}\Delta_{e_k}^{-1}\Delta_{\omega(Q)e_{k-1}}\alpha_k p_k, s_k\rangle \\
&= \langle\Delta_{\omega(Q)e_{k-1}}\alpha_k p_k, s_k\rangle \\
\text{(by (11))} \quad &= \langle\Delta_{\omega(Q)}\alpha_k p_k, e_{k-1}h_1\rangle \\
\text{(by Theorem 1)} \quad &= \langle\Delta_{h_{k-1}e_1}\alpha_n p_n, Q\rangle \\
&= \langle\Delta_{h_{k-1}}e_n, Q\rangle.
\end{aligned}$$

□

3.2 Some new identities involving $\Delta_{e_k}p_n$

In the introduction we mentioned some of the connections between the symmetric functions $\Delta_{s_\lambda}e_n$ to representation theory and geometry, and results like the Shuffle Theorem expressing some of these symmetric functions in terms of monomials. There are also many nice results involving $(-1)^{n-1}\nabla p_n$. In particular, Sergel [21] has proved the Square Paths Conjecture of Loehr and Warrington, which gives a combinatorial interpretation for the expansion of $(-1)^{n-1}\nabla p_n$ into monomials. One thing that is missing from this theory though is a DH_n-type representation-theoretic interpretation for $(-1)^{n-1}\nabla p_n$, and more generally for the symmetric functions $(-1)^{n-1}\nabla p_n$, $\Delta_{e_k}e_n$, $(-1)^{n-1}\Delta_{e_k}p_n$, $(-1)^{n-1}\Delta_{h_k}p_n$, which all appear to be Schur positive.

Suppose for a moment we have a module M_n for which $\mathcal{F}(char_{q,t}M_n) = (-1)^{n-1}\nabla p_n$. It can be shown that

$$(-1)^{n-1}e_1^{\perp}\nabla p_n = [n]_q[n]_t\nabla e_{n-1}, \tag{16}$$

where $^{\perp}$ is the operator which is adjoint to multiplication with respect to the Hall scalar product, i.e. for all f, g, h,

$$\langle g^{\perp}f, h\rangle = \langle f, gh\rangle. \tag{17}$$

Since $e_1^{\perp}\mathcal{F}(charM)$ gives the Frobenius image of the restriction from an S_n-action to an S_{n-1}-action, we see that $M_n \downarrow_{S_{n-1}}^{S_n}$ gives n^2 copies of diagonal harmonics. One may hope that M_n may be represented by some space of polynomials that are killed by certain partial differential operators. In eq. (18) below we will see that the same property holds for Δ_{e_k}. This together with other identities we will derive suggest a remarkable relation between the yet to be discovered-modules corresponding to $(-1)^{n-1}\Delta_{e_k}p_n$, $\Delta_{e_{k-1}}e_{n-1}$ and even $\Delta_{h_{k-1}}e_{n-1}$.

Corollary 3 *For positive integers k and n we have*

$$(-1)^{n-1}e_1^{\perp}\Delta_{e_k}p_n = [n]_q[n]_t\Delta_{e_{k-1}}e_{n-1}. \tag{18}$$

Proof. The statement is equivalent to saying $e_1^{\perp}\Delta_{e_k}\alpha_n p_n = \Delta_{e_{k-1}}e_{n-1}$. Now two symmetric functions F and G are equal if and only if $\langle F, P\rangle = \langle G, P\rangle$ for any choice of P. It is easy to see that by applying Theorem 1 and Corollary 1, we have for any homogeneous symmetric function P of degree n,

$$\begin{aligned}
\langle e_1^{\perp}\Delta_{e_k}\alpha_n p_n, P\rangle &= \langle\Delta_{\omega(h_k)}\alpha_n p_n, e_1 P\rangle \\
\text{(by Theorem 1)} \quad &= \langle\Delta_{\omega(P)e_1}\alpha_k p_k, s_k\rangle \\
&= \langle\Delta_{\omega(P)}e_k, s_k\rangle \\
\text{(by Corollary 1 with } Q \to P) \quad &= \langle\Delta_{e_{k-1}}e_n, P\rangle.
\end{aligned}$$

$\square$

Eq. (18) can be viewed as a special case of the following result.

Theorem 2 *For positive integers m, n, k we have*

$$h_m^{\perp}\Delta_{e_k}\alpha_{n+m}p_{n+m} = \Delta_{e_{k-m}h_m}\alpha_n p_n. \tag{19}$$

In particular, when $m = 1$ and n is replaced by $n-1$, we have

$$(-1)^{n-1}e_1^{\perp}\Delta_{e_k}p_n = [n]_q[n]_t\Delta_{e_{k-1}}e_{n-1}. \tag{20}$$

If instead, in (19), k is replaced by $k+1$ and m by k we have

$$(-1)^{n+k-1}h_k^{\perp}\Delta_{e_{k+1}}p_{n+k} = [n+k]_q[n+k]_t\Delta_{h_k}e_n \tag{21}$$

Proof. Let P be a homogeneous symmetric function of degree n. Then

$$\begin{aligned}
\langle h_m^{\perp}\Delta_{e_k}\alpha_{n+m}p_{n+m}, P\rangle &= \langle \Delta_{e_k}\alpha_{n+m}p_{n+m}, h_m P\rangle \\
\text{(by Theorem 1)} \quad &= \langle \Delta_{\omega(P)e_m}\alpha_k p_k, s_k\rangle \\
\text{(by (11))} \quad &= \langle \Delta_{\omega(P)}\alpha_k p_k, e_m h_{k-m}\rangle \\
&= \langle \Delta_{e_{k-m}h_m}\alpha_n p_n, P\rangle.
\end{aligned}$$

$\square$

We now show there is a similar identity for the symmetric function $h_m^{\perp}\Delta_{h_k}\alpha_{n+m}p_{n+m}$.

Theorem 3 *For positive integers m, n, k we have*

$$h_m^{\perp}\Delta_{h_k}\alpha_{n+m}p_{n+m} = h_k^{\perp}\Delta_{h_m e_{k-m}}\alpha_{n+k}p_{n+k}. \tag{22}$$

In particular, when $m = 1$ and n is replaced by $n-1$, we have

$$(-1)^{n-1}e_1^{\perp}\Delta_{h_k}p_n = [n]_t[n]_q h_k^{\perp}\Delta_{e_{k-1}}e_{n+k-1}. \tag{23}$$

When $m = k-1$ and n is replaced by $n+1$, we have

$$(-1)^{n+k-1}h_{k-1}^{\perp}\Delta_{h_k}p_{n+k} = [n+k]_t[n+k]_q h_k^{\perp}\Delta_{h_{k-1}}e_{n+k+1}. \tag{24}$$

Proof. Let P be a homogeneous symmteric function of degree n. Then

$$\begin{aligned}
\langle h_m^{\perp}\Delta_{h_k}\alpha_{n+m}p_{n+m}, P\rangle &= \langle \Delta_{h_k}\alpha_{n+m}p_{n+m}, h_m P\rangle \\
\text{(by Theorem 1)} \quad &= \langle \Delta_{\omega(P)e_m}\alpha_k p_k, e_k\rangle \\
\text{(by (11))} \quad &= \langle \Delta_{\omega(P)e_m e_k}\alpha_k p_k, s_k\rangle \\
\text{(by (11))} \quad &= \langle \Delta_{\omega(P)e_k}\alpha_k p_k, e_m h_{k-m}\rangle \\
\text{(by Theorem 1)} \quad &= \langle \Delta_{h_m e_{k-m}}\alpha_{n+k}p_{n+k}, h_k P\rangle \\
&= \langle h_k^{\perp}\Delta_{h_m e_{k-m}}\alpha_{n+k}p_{n+k}, P\rangle.
\end{aligned}$$

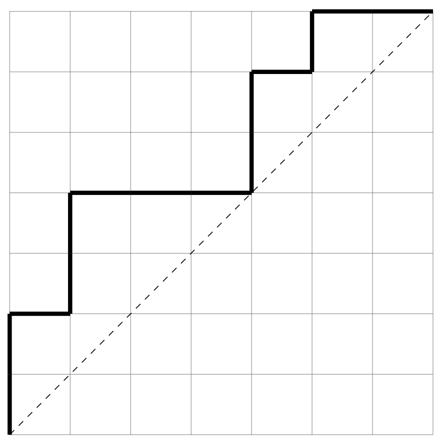
$\square$

4 Combinatorial Applications

The contents of Section 4.1 below are purely expository, giving background on the Delta Conjecture from [17]. We also include a discussion of results of D'Adderio and Wyngaerd [7], and D'Adderio and Iraci [6], which prove special cases of the Delta Conjecture, as well as a discussion of related work of Zabrocki [23]. We will refer to this material in Section 4.2 where we show how Theorem 1 and the combinatorics of the Delta Conjecture can be used to obtain some new results on the q,t-Narayana numbers introduced by Dukes and Le Borgne.

4.1 The Delta Conjecture

A Dyck path is a lattice path from $(0,0)$ to (n,n) consisting of unit North and East steps which never goes below the line $y=x$. As a running example, we take the Dyck path D below.



Given a Dyck path π let $a_i = a_i(\pi)$ denote the number of squares in the ith row (from the bottom) which are to the right of π and to the left of the diagonal $y=x$, where $1 \leq i \leq n$. In the case of D above,

$$(a_1, a_2, a_3, a_4, a_5, a_6, a_7) = (0, 1, 1, 2, 0, 1, 1).$$

We let $\operatorname{area}(\pi)$ denote the sum of the a_i. Furthermore let $\operatorname{dinv}(\pi)$ denote the number of pairs (i,j), $1 \leq i < j \leq n$, with either

$$a_i = a_j \text{ or } a_i = a_j + 1. \tag{25}$$

Next define the *reading order* of the rows of π to be the order in which the rows are listed by decreasing value of a_i, where if two rows have the same a_i-value, the row above is listed first. For the path D above, the reading order is

$$\text{row } 4, \text{ row } 7, \text{ row } 6, \text{ row } 3, \text{ row } 2, \text{ row } 5, \text{ row } 1. \tag{26}$$

Finally let $b_k = b_k(\pi)$ be the number of inversion pairs as in (25) which involve the kth row in the reading order and rows before it in the reading order. For D we have

$$b_1 = 0, b_2 = 1, b_3 = 2, b_4 = 2, b_5 = 3, b_6 = 2, b_7 = 1. \tag{27}$$

Note dinv is the sum of the b_k, and that values of i for which $b_i > b_{i-1}$ correspond to tops of columns in π (where we define $b_0 = -1$ so that $b_1 > b_0$).

One way of defining the q,t-Schröder polynomial from [14] is by setting

$$q^{\operatorname{dinv}(\pi)} \prod_{\substack{b_i > b_{i-1} \\ i \geq 2}} (1 + z/q^{b_i}). \tag{28}$$

Here $\mathcal{D}_n$ denotes the set of all Dycks paths from $(0,0)$ to (n,n). For example, the weight assigned to D in the right-hand-side of (28) is

$$t^6 q^{11}(1+z/q)(1+z/q^2)(1+z/q^3). \tag{29}$$

In particular, $C_n(q,t,0)$ is Garsia and Haiman's q,t-Catalan sequence [9].

Let

$$C_n(q,t,w,z) = \sum_{\pi} t^{\text{area}(\pi)} q^{\text{dinv}(\pi)} \prod_{\substack{b_i > b_{i-1} \\ i \geq 2}} (1+z/q^{b_i}) \prod_{\substack{a_i > a_{i-1} \\ i \geq 2}} (1+w/t^{a_i}). \tag{30}$$

This four variable Catalan polynomial appears in [17] under the context of the Delta Conjecture. Two different conjectured expressions for $C_n(q,t,z,w)$ are given there in terms of the Δ operator. One of these, eq. (31) below, was proved by Zabrocki in [23], and the other, (32) below, was proved by D'Adderio and Wyngaerd [7]. (Without the results in both [23] and [7], it is not known how to show the two Δ-operator expressions for $C_n(q,t,z,w)$ are equal).

Theorem 4 (Zabrocki [23], D'Adderio and Wyngaerd [7]) *For integers a,b,*

$$C_n(q,t,z,w)\Big|_{z^a w^b} = \langle \Delta_{h_a e_{n-a}} e_{n-a}, s_{b+1,1^{n-a-b-1}} \rangle \tag{31}$$

$$(1+z)(1+w)C_n(q,t,w,z)|_{z^a w^b} = \langle \Delta_{e_{n-b}} e_n, h_a e_{n-a} \rangle. \tag{32}$$

We also have

$$C_n(q,t,w,z) = C_n(t,q,w,z) = C_n(q,t,z,w). \tag{33}$$

Remark 2 *An equivalent form of (32) is*

$$(1+w)C_n(q,t,w,z)|_{z^a w^b} = \langle \Delta_{e_{n-b}} e_n, s_{a+1,a^{n-b-a-1}} \rangle. \tag{34}$$

We include here is a simple proof that the symmetry relations (33) follow from (32) using (11). *Proof.* Suppose we know that

$$(1+z)(1+w)C_n(q,t,w,z)|_{z^a w^b} = \langle \Delta_{e_{n-b}} e_n, h_a e_{n-a} \rangle.$$

The right-hand-side above equals

$$\langle \Delta_{e_{n-b}} e_n, h_a e_{n-a} \rangle = \langle \Delta_{e_{n-b} e_{n-a}} e_n, s_n \rangle,$$

which is clearly symmetric in a,b, so $C_n(q,t,z,w) = C_n(q,t,w,z)$. To see the symmetry in q,t, consider the expansion of

$$\Delta_{e_{n-a} e_{n-b}} e_n$$

in terms of the $\tilde{H}_\mu$, using (5) and (6), as a sum over partitions μ. One easily sees that the terms in this sum corresponding to μ and its conjugate are the same, after we interchange q and t, and hence their sum is symmetric in q,t. □

The Delta Conjecture gives a combinatorial description of $\Delta_{e_k} e_n$. Let P be a parking function, viewed as a placement of the integers 1 through n just to the right of the North steps of a Dyck path, with strict decrease down columns.

Conjecture 1 (Delta Conjecture [17]) *For any integer k, $0 \leq k \leq n$,*

$$\Delta_{e_{n-k}} e_n = \sum_{\pi} \sum_{P \in PF(\pi)} t^{area(\pi)} q^{dinv(P)} F_{des(read(P)^{-1})} \prod_{\substack{a_i > a_{i-1} \\ i \geq 1}} (1+w/t^{a_i})\Big|_{w^k}, \tag{35}$$

where $PF(\pi)$ is the set of parking functions with supporting path π from $(0,0)$ to (n,n); and F is the quasi-symmetric function weight attached to P. (See [15, Chapters 5 and 6] for a definition of $dinv(P)$, and also the quasi-symmetric function weight attached to P.)

The case $k = 0$ of (35) is the Shuffle Theorem of Carlsson and Mellit [5]). The results of D'Adderio and Wyngaerd [7] may be viewed as the Schröder case of the Delta Conjecture (it is common usage to refer to the coefficient of a Schur function of hook shape in symmetric functions defined via the Δ operator as the Schröder case, since in [14] it was first shown that the coefficient of a hook Schur function in ∇e_n can be expressed combinatorially in terms of Schröder lattice paths).

In related work D'Adderio and Iraci [6] were able to extend the work in [1] and give an interpretation of $\langle \Delta_{e_k} e_n, h_d h_{n-d} \rangle$ in terms of polyominoes, showing it agrees with a predicted combinatorial interpretation from the Delta Conjecture. We can rewrite this polynomial as

$$\begin{aligned}\langle \Delta_{e_k} e_n, h_d h_{n-d} \rangle &= \langle \Delta_{e_d e_{n-d}} e_{k+1}, s_{k+1} \rangle \\ &= \langle \Delta_{e_{n-d}} e_{k+1}, h_{k+1-d} e_d \rangle \\ &= (1+z)(1+w) C_{k+1}(q,t,w,z)\Big|_{z^{k+1-d} w^d},\end{aligned}$$

also relating these scalar-products with the polynomial $C_{k+1}(q,t,w,z)$. This transformation will play a role in the following section, where we connect Narayana numbers to the 4-variable Catalan polynomial using the scalar product identities.

4.2 The Narayana numbers

The Narayana number $N(n,k)$ counts the number of Dyck paths from $(0,0)$ to (n,n) with k columns, i.e. k pairs of consecutive NE steps. It is known that

$$N(n,k) = \frac{1}{n} \binom{n}{k-1} \binom{n}{k}, \tag{36}$$

and furthermore that $N(n,k) = h_{k-1}$, the kth element of the h-vector of the type A_{n-1} associahedron. The Narayana numbers can be decomposed into Kreweras numbers $\mathrm{Krew}(\lambda)$ which count the number of non-crossing partitions of $\{1,\dots,n\}$ whose blocks have sizes that rearrange to λ. For a partition λ of length k, we have

$$\mathrm{Krew}(\lambda) = \frac{1}{n+1} \binom{n+1}{m_1(\lambda),\dots,m_n(\lambda), n-k+1}, \tag{37}$$

where $m_i(\lambda)$ is the number of parts in λ of size i. This also counts (by the Cyclic Lemma) the number of Dyck paths from $(0,0)$ to (n,n) whose columns have lengths that rearrange to λ. This gives

$$N(n,k) = \sum_{\substack{\lambda \vdash n \\ \ell(\lambda)=k}} \mathrm{Krew}(\lambda). \tag{38}$$

Perhaps the most natural q-analog of (38) is given by

$$\begin{aligned} q^{(n+1-k)(n-k)} &\frac{1}{[n]_q} \begin{bmatrix} n \\ k-1 \end{bmatrix}_q \begin{bmatrix} n \\ k \end{bmatrix}_q \\ &= \sum_{\substack{\lambda \vdash n \\ \ell(\lambda)=k}} q^{(n)(n-\ell(\lambda))-c(\lambda)} \frac{1}{[n+1]_q} \begin{bmatrix} n+1 \\ m_1(\lambda),\dots,m_n(\lambda), n-k+1 \end{bmatrix}_q \end{aligned} \tag{39}$$

with $c(\lambda) = \sum \lambda'_j \lambda'_{j+1}$, providing as well a natural q-analog of Kreweras numbers.

There is a q,t version of the Narayana numbers $N(n,k,q,t)$ introduced by Dukes and LeBorgne [8], defined there as a weighted sum over parallelogram polyominoes. It was later shown by Aval, D'Adderio, Dukes, Hicks, and Le Borgne [1] that

$$N(n,k,q,t) = \langle \nabla e_{n-1}, h_{k-1} h_{n-k} \rangle. \tag{40}$$

They use the fact that a combinatorial description for the right-hand-side of (40) was proved in [14] (resolving a special case of the Shuffle Conjecture).

Now a basic result of Garsia and Haiman is that for any symmetric function F,

$$q^{\binom{n}{2}} \langle \nabla e_n[X], F[X] \rangle \Big|_{t=1/q} = \frac{1}{[n+1]_q} \omega(F) \, [[n+1]_q] \, . \tag{41}$$

It follows that

$$q^{\binom{n-1}{2}} N(n,k,q,1/q) = \frac{1}{[n]_q} e_{k-1} \, [[n]_q] \, e_{n-k} \, [[n]_q] \tag{42}$$

$$= q^{\binom{k-1}{2} + \binom{n-k}{2}} \frac{1}{[n]_q} \begin{bmatrix} n \\ k-1 \end{bmatrix}_q \begin{bmatrix} n \\ n-k \end{bmatrix}_q , \tag{43}$$

giving up to a scaling power of q the q-analog of $N(n,k)$ appearing on the left in (39).

At the Wachfest conference held at the University of Miami in January 2015, Vic Reiner of the University of Minnesota gave a talk involving representation theory and identities involving q-Narayana and q-Kreweras numbers for various types, including (39). In a subsequent private conversation with the first author, Reiner posed the question of finding a q,t-Kreweras number which when t is specialized to $1/q$ reduces to the q-Kreweras number appearing on the right in (39), in the same way that $N(n,k,q,t)$ reduces to the q-Narayana in (43). So far we have been unable to solve this problem, but this study has led to a new and perhaps better way of expressing $N(n,k,q,t)$ as a weighted sum over lattice paths. The expression (40) for $N(n,k,q,t)$ from [14] involves weighted lattice paths from $(0,0)$ to $(n-1,n-1)$ satisfying certain constraints, while in the new formulation below it is expressed as a weighted sum over lattice paths from $(0,0)$ to (n,n) with k columns, a formulation which is compatible (when $q=t=1$) with the definition of the Kreweras numbers and (38).

Using our scalar product identities, we are able to rewrite the q,t-Narayana numbers as follows:

$$\begin{aligned} N(n,k,q,t) &= \langle \Delta_{e_{n-1}} e_{n-1}, h_{k-1} h_{n-k} \rangle \\ &= \langle \Delta_{e_{k-1} e_{n-k}} e_n, s_n \rangle \quad \text{(by Corollary 1)} \\ &= \langle \Delta_{e_{k-1}} e_n, e_{n-k} h_k \rangle \quad \text{(by (11).} \end{aligned}$$

We could have also written this two other ways:

$$\begin{aligned} N(n,k,q,t) &= \langle \Delta_{e_{k-1} e_{n-k}} e_n, s_n \rangle \\ &= \langle \Delta_{e_{n-k}} e_n, e_{k-1} h_{n-k+1} \rangle \end{aligned}$$

and

$$\begin{aligned} N(n,k,q,t) &= \langle \Delta_{e_{k-1} e_{n-k} e_1} \alpha_n p_n, s_n \rangle \\ &= \langle \Delta_{e_{k-1} e_1} \alpha_n p_n, e_{n-k} h_k \rangle \\ &= \langle \Delta_{e_k h_{n-k}} \alpha_n p_n, h_{k-1} e_1 \rangle \quad \text{(by Theorem 1)} \\ &= \langle \Delta_{e_k h_{n-k} e_1} \alpha_k p_k, s_k \rangle \\ &= \langle \Delta_{h_{n-k} e_1} \alpha_k p_k, e_k \rangle \\ &= \langle \Delta_{h_{n-k}} e_k, e_k \rangle \quad \text{(by (6).} \end{aligned}$$

The first and second equalities give

$$N(n,k,q,t) = (1+z)(1+w) C_n(q,t,w,z) \Big|_{z^k w^{n-k+1}} = (1+z)(1+w) C_n(q,t,w,z) \Big|_{z^{n-k+1} w^k} .$$

We can then write $N(n,k,q,t)$ as

$$(1+z)(1+w) \sum_{\pi} t^{\operatorname{area}(\pi)} q^{\operatorname{dinv}(\pi)} \prod_{\substack{b_i > b_{i-1} \\ i \geq 2}} (1 + z/q^{b_i}) \prod_{\substack{a_i > a_{i-1} \\ i \geq 2}} (1 + w/t^{a_i}) \Big|_{z^k w^{n-k+1}} .$$

Recall that if $b_i > b_{i-1}$, then b_i corresponds to the top of a column. Thus, if we are taking the coefficient of z^k, we are asking that our path π contains at least k columns. By taking the coefficient of w^{n-k+1}, we are asking that our path contains at least $n-k$ North steps which are preceded by a North step, i.e. rows i for which $a_i > a_{i-1}$. But our path is from $(0,0)$ to (n,n), leaving only those paths which have precisely k columns. This answers the question of giving $N(n,k,q,t)$ as a sum over Dyck paths with exactly k columns. To be more precise, let $\overline{\text{area}}(\pi)$ be the sum over all a_i for which $a_i \leq a_{i-1}$, and let $\overline{\text{dinv}}(\pi)$ be the sum over all b_i for which $b_i \leq b_{i-1}$. In other words, $\overline{\text{area}}(\pi)$ is the area contributed by rows i whose North step is preceded by an East step. Then,

$$N(n,k,q,t) = \sum_{\pi \text{ with } k \text{ columns}} t^{\overline{\text{area}}(\pi)} q^{\overline{\text{dinv}}(\pi)}. \tag{44}$$

One can then potentially define a q,t-analog of the Kreweras number $\text{Krew}(\lambda)$ by restricting the sum (44) to paths π whose k columns have lengths which rearrange to λ. Unfortunately, examples for small n show that this way of defining a q,t-Kreweras number does not have the desired specialization at $t = 1/q$, and we must leave the question of whether this definition can be modified to obtain the desired $t = 1/q$ specialization as a question for future research.

The equality $N(n,k,q,t) = \langle \Delta_{h_{n-k}} e_k, e_k \rangle$, also found in [1], gives another natural way of decomposing $N(n,k,q,t)$. First note that

$$\Delta_{h_{n-k}} e_k = \sum_{\mu \vdash n-k} \Delta_{m_\mu} e_k.$$

If $\ell(\mu) > k$, we have $\Delta_{m_\mu} e_k = 0$. The partitions of n with k parts are in bijection with set of partitions of $n-k$ with at most k parts, simply by removing the first column. Let $\overline{\lambda}$ be the partition obtained from λ by removing the first column. Then

$$N(n,k,q,t) = \langle \Delta_{h_{n-k}} e_k, e_k \rangle = \sum_{\substack{\lambda \vdash n \\ \ell(\lambda) = k}} \langle \Delta_{m_{\overline{\lambda}}} e_k, e_k \rangle.$$

It can be shown by the methods in [20] that $\langle \Delta_{m_{\overline{\lambda}}} e_k, e_k \rangle|_{q=1}$ gives the number of Dyck paths whose columns have lengths that rearrange to λ. The statistic on t is given by $\overline{\text{area}}$ as is the case in $N(n,k,q,t)$. This means

Remark 3 *For a partition λ of length k, we have*

$$\langle \Delta_{m_{\overline{\lambda}}} e_k, e_k \rangle|_{q=t=1} = \text{Krew}(\lambda),$$

This gives another potential q,t analogue of the Kreweras number. However, at $t = 1/q$, it also does not give the desired q-Kreweras number appearing in (39).

Our last observation is that $N(n,k,q,t)$ also appears in the symmetric function $\Delta_{e_{n-k+1}} \alpha_n p_n$. Recall that (21) gives

$$\Delta_{h_{n-k}} e_k = h_{n-k}^{\perp} \Delta_{e_{n-k+1}} \alpha_n p_n.$$

This means

$$\langle \Delta_{h_{n-k}} e_k, e_k \rangle = \langle \Delta_{e_{n-k+1}} \alpha_n p_n, h_{n-k} e_k \rangle,$$

embedding the q,t-Narayana numbers in the symmetric function $\Delta_{e_{n-k+1}} \alpha_n p_n$. In other words,

$$(-1)^{n-1} \langle \Delta_{e_{n-k+1}} p_n, h_{n-k} e_k \rangle = [n]_q [n]_t N(n,k,q,t).$$

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