# $p$-Rook Numbers and Cycle Counting in $C_p \wr S_n$

James Haglund, Jeffrey Remmel and Meesue Yoo

**Abstract** Cycle-counting rook numbers were introduced by Chung and Graham [*J. Combin. Theory Ser. B* **65** (1995), 273–290]. Cycle-counting $q$-rook numbers were introduced by Ehrenborg, Haglund, and Readdy [unpublished] and cycle-counting $q$-hit numbers were introduced by Haglund [*Adv. Appl. Math.* **17** (1996), 408–459]. Briggs and Remmel [*J. Combin. Theory Ser. A* **113** (2006), 1138–1171] introduced the theory of $p$-rook and $p$-hit numbers which is a rook theory model where the rook numbers correspond to partial permutations in $C_p \wr S_n$, the wreath product of the cyclic group $C_p$ and the symmetric group $S_n$, and the hit numbers correspond to permutations in $C_p \wr S_n$. In this paper, we extend the cycle-counting $q$-rook numbers and cycle-counting $q$-hit numbers to the Briggs–Remmel model. In such a setting, we define a multivariable version of the cycle-counting $q$-rook numbers and cycle-counting $q$-hit numbers where we keep track of cycles of permutations and partial permutations of $C_p \wr S_n$ according to the signs of the cycles.

**Key words:** Rook numbers, hit numbers, cycle-counting rook numbers, cycle-counting hit numbers, wreath product.
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# 1 Introduction

We let $[n] = \{1,\ldots,n\}$. We let $\mathbb{N} = \{0,1,2,\ldots\}$ denote the natural numbers and $\mathbb{P} = \{1,2,\ldots\}$ denote the positive integers. A *board* is a subset of $\mathbb{P}\times\mathbb{P}$. We label the rows of $\mathbb{P}\times\mathbb{P}$ from bottom to top with $1,2,3,\ldots$, and the columns of $\mathbb{P}\times\mathbb{P}$ from left to right with $1,2,3,\ldots$, and $(i,j)$ denote the square in the $i$-th column and $j$-th row. Given $b_1,\ldots,b_n \in \mathbb{N}$, we let $F(b_1,\ldots,b_n)$ denote the board consisting of all the cells $\{(i,j) : 1 \le i \le n \text{ and } 1 \le j \le b_i\}$. If a board $B$ is of the form $B = F(b_1,\ldots,b_n)$, then we say that $B$ is *skyline board* and if, in addition, $b_1 \le b_2 \le \cdots \le b_n$, then we say that $B$ is a *Ferrers board*.

Given a board $B \subseteq [n]\times[n]$, we let $\mathcal{N}_k(B)$ denote the set of all placements of $k$ rooks in $B$ such that no two rooks lie in the same row or column. Elements of $\mathcal{N}_k(B)$ will be called *rook placements*. For $k = 1,\ldots,n$, we let $r_k(B) = |\mathcal{N}_k(B)|$. By convention, we set $r_0(B) = 1$. We refer to $r_k(B)$ as the **$k$-th rook number of $B$**.

Let $S_n$ denote the symmetric group of $n$ elements, i.e. the group of all permutations of $1,\ldots,n$ under composition. Given a permutation $\sigma = \sigma_1\cdots\sigma_n \in S_n$, we identify each $\sigma \in S_n$ with the rook placement $\{(i,\sigma_i) : i = 1,\ldots,n\}$ on $[n]\times[n]$. We let

$$H_{k,n}(B) = |\{\sigma \in S_n : |\sigma \cap B| = k\}|.$$

We shall refer to $H_{k,n}(B)$ as the **$k$-th hit number of $B$ relative to** $[n]\times[n]$.

Kaplansky and Riordan [20] proved the following fundamental relationship between the rook numbers and the hit numbers of a board $B \subseteq [n]\times[n]$.

**Theorem 12.1.** *For any board $B \subseteq [n]\times[n]$,*

$$\sum_{k=0}^{n} H_{k,n}(B)x^k = \sum_{k=0}^{n} r_k(B)(n-k)!(x-1)^k. \tag{12.1}$$

With each rook placement $P \in \mathcal{N}_k(B)$, we can associate a directed graph $G_P = ([n],E_P)$, where $E_P$ is the set of $(i,j)$ such that $P$ has a rook in cell $(i,j)$. We let $\mathrm{cyc}(P)$ denote the number of cycles in the graph of $P$. For example, in Figure 1, we picture a rook placement $P \in \mathcal{N}_5(B)$, where $B$ is the $6\times 6$ board such that $\mathrm{cyc}(P) = 2$.

For any board $B \subseteq [n]\times[n]$, we let

$$\begin{aligned} r_k(B,y) &= \sum_{P\in\mathcal{N}_k(B)} y^{\mathrm{cyc}(P)} \text{ and} \\ H_{k,n}(B,y) &= \sum_{\sigma\in S_n,\ |\sigma\cap B|=k} y^{\mathrm{cyc}(P)}. \end{aligned}$$

For $k \ge 1$, we let $(y)\uparrow_k = y(y+1)\cdots(y+k-1)$ and $(y)\downarrow_k = y(y-1)\cdots(y-k+1)$. We let $(y)\uparrow_0 = (y)\downarrow_0 = 1$. We then have the following analogue of Theorem 12.1.

**Theorem 12.2.** *For any board $B \subseteq [n]\times[n]$,*

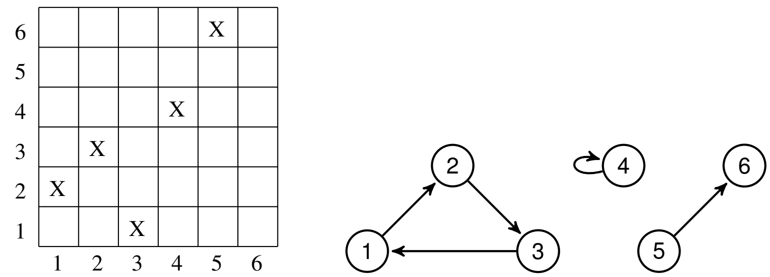


Fig. 1: The graph associated with a rook placement.

$$\sum_{k=0}^{n} H_{k,n}(B,y)x^k = \sum_{k=0}^{n} r_k(B,y)(y)\uparrow_{n-k} (x-1)^k. \tag{12.2}$$

*Proof.* First replace $x$ by $x+1$ in equation (12.2). Then we must prove

$$\sum_{k=0}^{n} H_{k,n}(B,y)(x+1)^k = \sum_{k=0}^{n} r_k(B,y)(y)\uparrow_{n-k} x^k. \tag{12.3}$$

For (12.3), we consider configurations $C$ which consist of a rook placement corresponding to a permutation $\sigma \in S_n$, where we circle some of the rooks that fall in $B \cap \sigma$. We let $\mathrm{cyc}(C)$ denote the number of cycles in the graph of the underlying permutation of $C$ and $\mathrm{circle}(C)$ denote the number of circled rooks in $C$. It is then easy to see that the left-hand side of (12.3) can be interpreted as counting $y^{\mathrm{cyc}(C)}x^{\mathrm{circle}(C)}$ over all such configurations. The right-hand side of (12.3) can be interpreted as follows. First pick the circled rooks which correspond to a placement $Q \in \mathscr{N}_k(B)$ for some $k$. Then we need to compute

$$A(Q,y) = \sum_{C} y^{\mathrm{cyc}(C)}, \tag{12.4}$$

where the sum runs over all configurations whose set of circled rooks equals $Q$. This sum is easy to compute. That is, let $i$ be the first column that does not contain a rook in $Q$. Then there are $n-k$ rows to place a rook in column $i$ that do not contain rooks in $Q$. We claim that there is exactly one row $r$ where placing a rook in cell $(i,r)$ completes a cycle in the graph of $Q$. That is, if there is no rook in $Q$ which is in row $i$, then $i$ is an isolated vertex in the graph of $Q$ so adding a rook in the cell $(i,i)$ will give a loop on vertex $i$ and hence increase the number of cycles by 1. Clearly in such a situation, placing a rook in cell $(i,j)$ for $j \neq i$ cannot complete a cycle. If there is a rook of $Q$ in row $i$, then there must be a maximal length path $p$ in the graph of $Q$ which ends in vertex $i$ since there are no edges coming out of the vertex $i$ in the graph of $Q$. If this path starts in vertex $j$, then there is no rook in row $j$ in $Q$. Hence if we add a rook to the cell $(i,j)$, then the edge corresponding to the added rook will complete a cycle. Clearly, adding a rook to any other row in column $i$ will not complete a cycle in this case. Thus the placement of a rook in column $i$ will contribute a factor $(y+n-k-1)$ to $A(Q,y)$. But then we can repeat the argument for every placement $Q'$ which arises from $Q$ by adding a rook in the next empty column, say column $i_1$. That is, for each such $Q'$, the addition of a rook in column $i_1$ will contribute a factor $(y+n-k-2)$ to $A(Q,y)$. Continuing on in this way, we see that

$$A(Q,y) = (y+n-k-1)(y+n-k-2)\cdots(y) = (y)\uparrow_{n-k}.$$

Thus another way to sum $y^{\mathrm{cyc}(C)}x^{\mathrm{circle}(C)}$ over all rook configurations is

$$\begin{aligned}
&\sum_{k=0}^{n} x^k \sum_{Q\in\mathcal{N}_k(B)} y^{\mathrm{cyc}(Q)} A(Q,y)\\
&= \sum_{k=0}^{n} x^k \sum_{Q\in\mathcal{N}_k(B)} y^{\mathrm{cyc}(Q)} (y)\uparrow_{n-k}\\
&= \sum_{k=0}^{n} x^k (y)\uparrow_{n-k} \sum_{Q\in\mathcal{N}_k} y^{\mathrm{cyc}(Q)}\\
&= \sum_{k=0}^{n} r_k(B,y)(y)\uparrow_{n-k} x^k.
\end{aligned}$$

Chung and Graham [8] proved that for any Ferrers boards $F(b_1,\ldots,b_n)\subseteq[n]\times[n]$, we have the following factorization theorem.

**Theorem 12.3.** *Let $B=F(b_1,\ldots,b_n)\subseteq[n]\times[n]$ be a Ferrers board. Then*

$$\prod_{i:b_i<i}(x+b_i-i+1)\prod_{i:b_i\geq i}(x+b_i-i+y) = \sum_{k=0}^{n} r_{n-k}(B,y)(x)\downarrow_k. \tag{12.5}$$

We let

$$\begin{aligned}
&[n]_q = \frac{q^n-1}{q-1} = 1+\cdots+q^{n-1},\\
&[n]_q! = [1]_q[2]_q\cdots[n]_q, \text{ and}\\
&\begin{bmatrix} n\\ k\end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}
\end{aligned}$$

be the usual $q$-analogues of $n$, $n!$, and $\binom{n}{k}$. In general, we let $[x]_q = \frac{q^x-1}{q-1}$. Then for $k\geq 1$, we let $[x]_q\uparrow_k = [x]_q[x+1]_q\cdots[x+k-1]_q$ and $[x]_q\downarrow_k = [x]_q[x-1]_q\cdots[x-(k-1)]_q$. We let $[x]_q\uparrow_0 = [x]_q\downarrow_0 = 1$.

In an unpublished paper, Ehrenborg, Haglund, and Readdy [10] defined a $q$-analogue of the cycle counting rook numbers $r_k(B,y,q)$ for Ferrers boards which generalized the $q$-analogue of the rook numbers for Ferrers boards introduced by Garsia and Remmel [12]. They proved the following generalization of Chung and Graham's theorem.

**Theorem 12.4.** *Let $B = F(b_1,\dots,b_n) \subseteq [n]\times[n]$ be a Ferrers board. Then*

$$\prod_{i:b_i<i}[x+b_i-i+1]_q \prod_{i:b_i\geq i}[x+b_i-i+y]_q = \sum_{k=0}^{n} r_{n-k}(B,y,q)[x]_q\downarrow_k . \tag{12.6}$$

Haglund [14] also extended the definition of the $q$-hit numbers of Garsia and Remmel [12] for Ferrers boards by defining $q,x,y$-hit numbers algebraically by the equation

$$\sum_{k=0}^{n} H_{k,n}(B,x,y,q)z^k = \sum_{k=0}^{n} r_{n-k}(B,y,q)[x]_q\uparrow_k z^k \prod_{i=k+1}^{n}(1-zq^{x+i-1}). \tag{12.7}$$

Haglund [14] developed several connections between formulas for the $q,x,y$-hit numbers and hypergeometric series. Later Butler [6] gave a combinatorial interpretation of $H_{k,n}(B,x,y,q)$ for Ferrers boards.

The main goal of this paper is to define analogues of cycle-counting rook numbers, cycle-counting hit numbers, and their $q$-analogues relative to the group $C_p \wr S_n$ which is the wreath product of the cyclic group $C_p$ of order $p$ with the symmetric group $S_n$. In particular, we extend the combinatorics of cycle-counting rook numbers and cycle-counting hit numbers to the rook theory model of Briggs and Remmel [5] where the rook placements correspond to partial permutations in $C_p \wr S_n$ and hit numbers correspond to permutations in $C_p \wr S_n$.

Let $\omega = e^{\frac{2\pi i}{p}}$. One can think of the group $C_p \wr S_n$ as the group of matrices under matrix multiplication where the underlying set is the set of matrices that one can form by starting with an $n\times n$ permutation matrix $M$ and replacing 1's by powers of $\omega$. Thus we can think of $C_p \wr S_n$ as the group of $p^n n!$ signed permutations where there are $p$ signs, $\omega^0 = 1$, $\omega$, $\omega^2,\dots,\omega^{p-1}$. We will usually write the signed permutations in either one-line notation or in disjoint cycle form. For example, if $\sigma \in C_3 \wr S_8$ is the map sending $1\to\omega 5$, $2\to 8$, $3\to\omega^2 3$, $4\to\omega^2 1$, $5\to 4$, $6\to\omega^2 7$, $7\to\omega 2$, and $8\to\omega 6$, then in one-line notation,

$$\sigma = \omega 5 \;\; 8 \;\; \omega^2 3 \;\; \omega^2 1 \;\; 4 \;\; \omega^2 7 \;\; \omega 2 \;\; \omega 6,$$

whereas in disjoint cycle form,

$$\sigma = (\omega^2 1 \;\; \omega 5 \;\; 4)(\omega 2 \;\; 8 \;\; \omega 6 \;\; \omega^2 7)(\omega^2 3).$$

In other words, in disjoint cycle form, to determine where $i$ is being mapped, we ignore the sign on $i$ and only consider the sign on the element to which it is mapped. Whenever we have an $r$-cycle $C = (\omega^{a_0}c_0,\dots,\omega^{a_{r-1}}c_{r-1})$ in a signed permutation in $C_p \wr S_n$, we define $sgn(C) = \prod_{i=0}^{r-1}\omega^{a_i}$. Thus in our example,

$$\begin{aligned}
&sgn((\omega^2 1 \;\; \omega 5 \;\; 4)) = 1,\\
&sgn((\omega 2 \;\; 8 \;\; \omega 6 \;\; \omega^2 7)) = \omega, \text{ and}\\
&sgn((\omega^2 3)) = \omega^2.
\end{aligned}$$

Given $\sigma \in C_p \wr S_n$ we will write $\sigma(i)$ as $\varepsilon_i \sigma_i$, where $\sigma_i \in [n] = \{1, \ldots, n\}$, and where $\varepsilon_i = sgn(\sigma_i) \in \{1, \omega, \omega^2, \ldots, \omega^{p-1}\}$ is called the *sign* of $\sigma_i$. For each $1 \leq i \leq n$, we define $|\varepsilon_i \sigma_i| = \sigma_i$ and call this the *absolute value* of $\sigma(i)$.

Next we shall describe the rook model due to Briggs and Remmel [5] where the rook numbers correspond to partial permutations in $C_p \wr S_n$ and the hit numbers correspond to permutations in $C_p \wr S_n$.

The idea of Briggs and Remmel was to start with the $[n] \times [n]$ board and subdivide each row into $p$ subrows. We will denote the resulting board by $B_n^p$. For example, if $n = 6$ and $p = 3$, then $B_6^3$ is pictured in Figure 2. We shall refer to the rows of the original $[n] \times [n]$ board as levels and label the levels with $1, \ldots, n$ from bottom to top. We label the columns with $1, \ldots, n$ from left to right. Finally, within each level, we label the sublevels from bottom to top with $1, \omega, \omega^2, \ldots, \omega^{p-1}$. We let $(i, j, k)$ denote the square in the $i$-th column, the $j$-th level, and in the sublevel labelled with $\omega^k$.

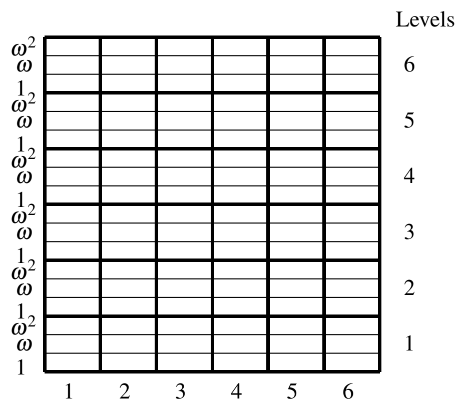


Fig. 2: The board $B_6^3$.

In the Briggs–Remmel model, a *board* is a subset of $B_n^p$. Given $b_1, \ldots, b_n \in [pn]$, we let $F(b_1, \ldots, b_n)$ denote the board consisting of all the cells $\{(i, j, k) : 1 \leq i \leq n \text{ and } 1 \leq pj + k \leq b_i\}$. If a board $B$ is of the form $B = F(b_1, \ldots, b_n)$, then we say that $B$ is a *skyline board* and if, in addition, $b_1 \leq b_2 \leq \cdots \leq b_n$, then we say that $B$ is a *Ferrers board*. If $B = F(b_1, \ldots, b_n)$ is a Ferrers board and $b_{i+1} \geq rp$ whenever $(r-1)p + 1 \leq b_i \leq rp$ , then we say that $B$ is a *singleton Ferrers board*. Here the last condition for a singleton Ferrers board in $B_n^p$ says that whenever there are cells in level $r$ in column $i$, column $i+1$ must contain all the cells in the level $r$. Finally, we shall say that a board $B$ is a *full board* whenever, if $B$ contains a cell $(i, j, k)$, then it must contain the cells $(i, j, r)$ for $r = 0, \ldots, p-1$. In other words, a Ferrers board $F(b_1, \ldots, b_n)$ is a full board if and only if $b_i$ is a multiple of $p$ for all $i = 1, \ldots, n$. We say that a full Ferrers board $B = F(b_1, \ldots, b_n) \subseteq B_p^n$ is *regular* if $b_i = p \cdot c_i$, where $c_i \geq i$ for $1 \leq i \leq n$.

Given a board $B \subseteq B_n^p$, we let $\mathscr{N}_k^p(B)$ denote the set of all placements of $k$ rooks in $B$ such that no two rooks lie in the same level or column. Elements of $\mathscr{N}_k^p(B)$ will be called *p-rook placements*. For $k = 1,\ldots,n$, we let $r_k^p(B) = |\mathscr{N}_k^p(B)|$. By convention, we set $r_0^p(B) = 1$. We refer to $r_k^p(B)$ as the **$k$-th $p$-rook number of $B$**. An alternative model for $r_k^p(B)$ was proposed by Wachs and Remmel [19]. In the case $p = 2$, Haglund and Remmel [16] gave yet another rook model for $r_k^p(B)$.

Given a signed permutation $\sigma = \omega^{a_1}\sigma_1 \cdots \omega^{a_n}\sigma_n \in C_p \wr S_n$, we identify $\sigma$ with the $p$-rook placement $\{(i,\sigma_i,a_i) : i = 1,\ldots,n\}$ on $B_n^p$. We let

$$H_{k,n}^p(B) = |\{\sigma \in C_p \wr S_n : |\sigma \cap B| = k\}|.$$

We shall refer to $H_{k,n}^p(B)$ as the **$k$-th $p$-hit number of $B$ relative to $B_n^p$** .

With each $p$-rook placement $P \in \mathscr{N}_k^p(B)$, we can associate a directed graph $G_P = ([n], E_P)$ with labelled edges, where $E_P$ is the set of $(i,j)$ such that $P$ has a rook in cell $(i,j,k)$ and we label the edge $(i,j)$ with $\omega^k$. For example, see Figure 3 for the graph associated with a 3-rook placement on $B_6^3$. For any $p$-rook placement, we let $\mathrm{cyc}_i(P)$ denote the number of cycles in the graph of $P$ such that product of labels on the cycle is $\omega^i$.

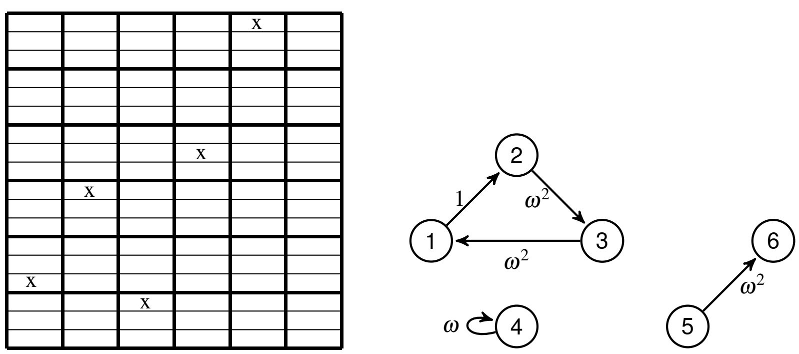


Fig. 3: The graph associated with a 3-rook placement in $\mathscr{N}_5^3(B_6^3)$.

For any board $B \subseteq B_n^p$, we let

$$r_k^p(B, y_0,\ldots,y_{p-1}) = \sum_{P \in \mathscr{N}_k^p(B)} \prod_{i=0}^{p-1} y_i^{\mathrm{cyc}_i(P)} \text{ and}$$

$$H_{k,n}(B, y_0,\ldots,y_{p-1}) = \sum_{\substack{\sigma \in C_p \wr S_n,\\ |\sigma \cap B| = k}} \prod_{i=0}^{p-1} y_i^{\mathrm{cyc}_i(\sigma)}.$$

The outline of the paper is as follows. In Section 2, we shall prove the analogues of Theorem 12.2 and Theorem 12.3 as well as give an example of cycle-counting

$p$-Lah numbers. In Section 3, we shall define a $q$-analogue of the cycle-counting $p$-rook numbers and prove an analogue of the Ehrenborg, Haglund, and Readdy factorization theorem [10]. In Section 4, we shall define a $q$-analogue of the cycle counting $p$-hit numbers $H^p_{k,n}[B,q,y_0,\dots,y_{p-1}]$ for a full regular Ferrers board $B$. We will prove analogues of some results of Haglund [14] and Bulter [6] on the $q$-cycle-counting rook numbers and $q,x,y$-hit numbers for full regular Ferrers boards which will allow us to prove that $H^p_{k,n}[B,q,y_0,\dots,y_{p-1}]$ is always a polynomial in $q$ with non-negative coefficients when $y_0,\dots,y_{p-1}$ are non-negative integers. We will end Section 4 by giving a conjectured combinatorial interpretation of the $H^p_{k,n}[B,q,y_0,\dots,y_{p-1}]$'s.

## 2 Cycle-counting $p$-rook numbers and $p$-hit numbers.

We start this section by proving analogues of Theorem 12.2 and Theorem 12.3 for the cycle-counting $p$-rook and $p$-hit numbers.

Suppose that $p \geq 2$. Then for $k \geq 1$, we let $(y)\uparrow_{k,p}= y(y+p)\cdots(y+p(k-1))$ and $(y)\downarrow_{k,p}= y(y-p)\cdots(y-p(k-1))$. We also let $(y)\uparrow_{0,p}=(y)\downarrow_{0,p}=1$. We then have the following analogue of Theorem 12.2.

**Theorem 12.5.** *For any $p \geq 2$ and any board $B \subseteq B^p_n$,*

$$\begin{aligned}&\sum_{k=0}^{n} H^p_{k,n}(B,y_0,\dots,y_{p-1})x^k \\ &= \sum_{k=0}^{n} r^p_k(B,y_0,\dots,y_{p-1})(y_0+\cdots+y_{p-1})\uparrow_{n-k,p}(x-1)^k.\end{aligned} \tag{12.8}$$

*Proof.* Fix $p \geq 2$. First replace $x$ by $x+1$ in equations (12.8). Thus we must prove

$$\begin{aligned}&\sum_{k=0}^{n} H^p_{k,n}(B,y_0,\dots,y_{p-1})(x+1)^k \\ &\qquad = \sum_{k=0}^{n} r^p_k(B,y_0,\dots,y_{p-1})(y_0+\cdots+y_{p-1})\uparrow_{n-k,p} x^k.\end{aligned} \tag{12.9}$$

For (12.9), we consider configurations $C$ which consist of a rook placement corresponding to a permutation $\sigma \in C_k \wr S_n$, where we circle some of the rooks that fall in $B \cap \sigma$. We then let $\mathrm{cyc}_i(C)$ denote the number of cycles of sign $\omega^i$ in the graph of the underlying permutation of $C$ and $\mathrm{circle}(C)$ denote the number of circled rooks in $C$. It is then easy to see that the left-hand side of (12.9) can be interpreted as counting $x^{\mathrm{circle}(C)}\prod_{i=0}^{p-1} y_i^{\mathrm{cyc}_i(C)}$ over all such configurations. The right-hand side of (12.9) can be interpreted as follows. First pick the circled rooks which correspond to a placement $Q \in \mathscr{N}^p_k(B)$ for some $k$. Then we need to compute

$$A(Q, y_0, \ldots, y_{p-1}) = \sum_C \prod_{i=0}^{p-1} y_i^{\mathrm{cyc}_i(C)}, \tag{12.10}$$

where the sum runs over all configurations whose set of circled rooks equals $Q$. Again this sum is easy to compute. Let $i$ be the first column that does not contain a rook in $Q$. Then there are $n-k$ levels in which to place a rook in column $i$ that do not contain rooks in $Q$. We claim that there is exactly one level $r$ where placing a rook in the cell $(i,r,k)$ for any $k$, $0 \le k \le p-1$, completes a cycle in the graph of $Q$. That is, if there is no rook in $Q$ which is in level $i$, then $i$ is an isolated vertex in the graph of $Q$ so adding a rook in cell $(i,i,k)$ will give a loop on vertex $i$ with label $\omega^k$ and hence increases the number of cycles with sign $\omega^k$ by 1. Clearly in such a situation, placing a rook in cell $(i,j,k)$ for $j \neq i$ and $0 \le k \le p-1$ cannot complete a cycle. If there is a rook of $Q$ in level $i$, then there must be a path $p$ of the maximal length in the graph of $Q$ which ends in vertex $i$ since there are no edges coming out of the vertex $i$ in the graph of $Q$. If this path starts in vertex $j$, then there is no rook in level $j$ in $Q$. Hence if we add a rook to cell $(i,j,k)$ for any $0 \le k \le p-1$, then this will complete a cycle. No matter what the labels are on the edges of the path from $j$ to $i$ in the graph corresponding to $Q$, there will be exactly one choice of $k$ which results in the completed cycle having sign $\omega^i$ for any given $i \in \{0,\ldots,p-1\}$. Clearly, adding a rook to any other level in column $i$ will not complete a cycle in this case. Thus the placement of a rook in column $i$ will contribute a factor $(y_0 + \cdots + y_{p-1} + p(n-k-1))$ to $A(Q, y_0, \ldots, y_{p-1})$. But then we can repeat the argument for every placement $Q'$ which arises from $Q$ by adding a rook in the next empty column, say column $i_1$. That is, for each such $Q'$, the addition of a rook in column $i_1$ will contribute a factor $(y_0 + \cdots + y_{p-1} + p(n-k-2))$ to $A(Q, y_0, \ldots, y_{p-1})$. Continuing on in this way, we see that $A(Q, y_0, \ldots, y_{p-1})$ equals

$$\begin{aligned}(y_0+\cdots+y_{p-1}+p(n-k-1))(y_0+\cdots+y_{p-1}+p(n-k-2))\cdots(y_0+\cdots+y_{p-1})\\ = (y_0+\cdots+y_{p-1})\uparrow_{n-k,p}.\end{aligned}$$

Thus another way to sum $x^{\mathrm{circle}(C)} \prod_{i=0}^{p-1} y_i^{\mathrm{cyc}_i(C)}$ over all configurations is

$$\begin{aligned}&\sum_{k=0}^{n} x^k \sum_{Q \in \mathcal{N}_k^p(B)} \prod_{i=0}^{p-1} y_i^{\mathrm{cyc}_i(Q)} A(Q, y_0, \ldots, y_{p-1})\\ &= \sum_{k=0}^{n} x^k \sum_{Q \in \mathcal{N}_k^p(B)} \prod_{i=0}^{p-1} y_i^{\mathrm{cyc}_i(Q)} (y_0+\cdots+y_{p-1})\uparrow_{n-k,p}\\ &= \sum_{k=0}^{n} x^k (y_0+\cdots+y_{p-1})\uparrow_{n-k,p} \sum_{Q \in \mathcal{N}_k^p(B)} \prod_{i=0}^{p-1} y_i^{\mathrm{cyc}_i(Q)}\\ &= \sum_{k=0}^{n} r_k^p(B, y_0, \ldots y_{p-1})(y_0+\cdots+y_{p-1})\uparrow_{n-k,p} x^k.\end{aligned}$$

Next we shall prove a factorization theorem for cycle counting $p$-rook numbers for full Ferrers boards $B \subseteq B_n^p$.

**Theorem 12.6.** *Let $p \geq 2$ and $B = F(b_1,\dots,b_n)$ be a full Ferrers board contained in $B_n^p$. Then we have*

$$\prod_{i:b_i<pi}(x+b_i-p(i-1))\prod_{i:b_i\geq pi}(x+b_i-pi+y_0+\cdots+y_{p-1}) \\ = \sum_{k=0}^{n} r_{n-k}^p(B,y_0,\dots,y_{p-1})(x)\downarrow_{k,p}. \tag{12.11}$$

*Proof.* The assumption that $B$ is a full board implies that $b_i$ is divisible by $p$ for all $i$. Since both sides of (12.11) are polynomials in $x$ of degree $n$, it is enough to prove that (12.11) holds for infinitely many integers.

First we shall show that (12.11) holds for infinitely many integers $px$, where $x \in \mathbb{P}$. Given $x \in \mathbb{P}$, we let $B_x$ denote the board which results by adding $x$-levels of length $n$ below $B$. For example, if $p = 3$, $B = (3,6,6,6,9,9)$, and $x = 6$, then the board $B_x$ is pictured in Figure 4. We call the boundary between $B$ and the $x$-levels that we added below $B$ the *bar*.

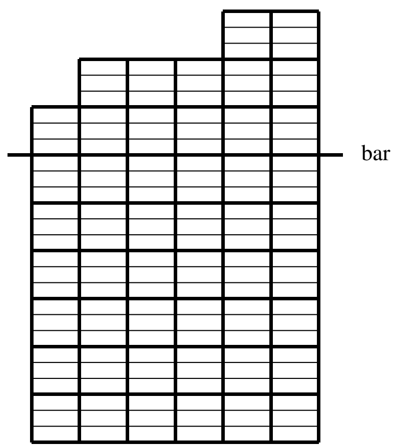


Fig. 4: The board $B_x$.

We let $\mathcal{N}_k^p(B_x)$ denote the set of all placements of $k$ rooks in $B_x$ such that there is at most one rook in each level and each column. Given a placement $P \in \mathcal{N}_n^p(B_x)$, we let $wt(P) = \prod_{i=0}^{p-1} y_i^{\mathrm{cyc}_i(P\cap B)}$. Then we claim that (12.11) where $x$ is replaced by $px$ arises from two different ways of computing

$$S(B,y_0,\dots,y_{p-1}) = \sum_{P\in\mathcal{N}_n^p(B_x)} wt(P).$$

Next we prove a key lemma.

**Lemma 12.1.** *Suppose that $Q \in \mathcal{N}_t^p(B_x)$ is a p-rook placement of t rooks in the first $i-1$ columns of $B_x$. Let $D_i(Q)$ denote the set of all p-rook placements P that result from Q by adding a rook in column i. Then*

$$\sum_{P \in D_i(Q)} \prod_{l=0}^{p-1} y_l^{cyc_l(P)} =$$
$$\begin{cases} (b_i + px - p(t+1) + y_0 + \cdots + y_{p-1}) \prod_{l=0}^{p-1} y_l^{cyc_l(Q \cap B)} & \text{if } b_i \geq pi, \\ (b_i + px - pt) \prod_{l=0}^{p-1} y_l^{cyc_l(Q \cap B)} & \text{if } b_i < pi. \end{cases}$$

*Proof.* First we claim that there is exactly one level $j$ above the bar such that placing a rook in a cell $(i,j,k)$ will complete a cycle in the graph of $Q \cap B$ if $b_i \geq pi$ and there is no level $j$ above the bar such that placing a rook in a cell $(i,j,k)$ will complete a cycle in the graph of $Q \cap B$ if $b_i < pi$. That is, suppose that $b_i \geq pi$. If there is no rook in $Q \cap B$ which is in level $i$, then $i$ is an isolated vertex in the graph of $Q \cap B$ so adding a rook in cell $(i,i,k)$ will give a loop on vertex $i$ with label $\omega^k$ and hence increase the number of cycles with sign $\omega^k$ by 1. Clearly in such a situation, placing a rook in cell $(i,j,k)$ for $j \neq i$ and $0 \leq k \leq p-1$ cannot complete a cycle. If there is a rook in $Q \cap B$ in row $i$, then there must be a maximal length path $p$ in the graph of $Q \cap B$ which ends in vertex $i$ since there are no edges coming out of $i$ in the graph of $Q \cap B$. If this path starts in vertex $j$, then $j \leq i \leq b_i/p$ and there is no rook in level $j$ in $Q \cap B$ above the bar. Hence if we add a rook to cell $(i,j,k)$ for any $0 \leq k \leq p-1$, then it will complete a cycle. No matter what the labels are on the edges of the path from $j$ to $i$ in the graph corresponding to $Q$, there will be exactly one choice for $k$ which results in the completed cycle having sign $\omega^i$ for any given $i \in \{0,\ldots,p-1\}$. In such a situation, we will call the level $j$ such that adding a rook in a cell $(i,j,k)$ completes a cycle the *special level relative to Q*. It easily follows that in this case

$$\sum_{P \in D_i(Q)} \prod_{l=0}^{p-1} y_l^{\mathrm{cyc}_l(P)} = (b_i + px - p(t+1) + y_0 + \cdots + y_{p-1}) \prod_{l=0}^{p-1} y_l^{\mathrm{cyc}_l(Q \cap B)}.$$

Alternatively, if $b_i < pi$, then we must have that $b_1 \leq \cdots \leq b_{i-1} \leq p(i-1)$ since we are assuming that $B$ is a full Ferrers board. This implies that there can be no edge which ends in the vertex $i$ in the graph of $Q \cap B$. Hence $i$ is an isolated vertex in the graph $Q \cap B$. Thus placing a rook in the cell $(i,j,k)$ where $j < i$ cannot create a new cycle. It easily follows that in this case

$$\sum_{P \in D_i(Q)} \prod_{l=0}^{p-1} y_l^{\mathrm{cyc}_l(P)} = (b_i + px - pt) \prod_{l=0}^{p-1} y_l^{\mathrm{cyc}_l(Q \cap B)}.$$

Now think of adding rooks column by column starting from the left to form an element $P \in \mathcal{N}_n^p(B_x)$. In the first column, we have $b_1 + px$ choices. If $b_1 \geq p$, then if we add a rook in cell $(1,1,k)$ then we create a cycle of sign $\omega^k$ and we do not create a cycle otherwise. Thus the first column will contribute a factor $(px+b_1-p+y_0+\cdots+y_{p-1})$ if $b_1 \geq p$ or a factor $(px+b_1)$ otherwise. Next if we start with a placement $Q \in \mathcal{N}^p_{i-1}(B_x)$ of $i-1$ rooks in the first $i-1$ columns of $B_x$, then we will have $px+b_i-p(i-1)$ cells to add a rook in column $i$. By Lemma 12.1, our choices for placing a rook in these $px+b_i-p(i-1)$ cells will contribute a factor $(px+b_i-pi+y_0+\cdots+y_{p-1})$ if $b_i \geq pi$ and will contribute a factor $(px+b_i-p(i-1))$ otherwise. Thus it follows that

$$S(B,y_0,\ldots,y_{p-1}) = \prod_{i:b_i<pi}(px+b_i-p(i-1))\prod_{i:b_i\geq pi}(px+b_i-pi+y_0+\cdots+y_{p-1}).$$

On the other hand, suppose that we fix a $p$-rook placement $Q \in \mathcal{N}^p_{n-k}(B)$ of $n-k$ rooks above the bar. Then we want to compute

$$B_Q = \sum_{P\in\mathcal{N}^p_n(B_x):P\cap B=Q} wt(P). \tag{12.12}$$

In this case, there will be $k$ columns below the bar which do not contain rooks in $Q$. If those columns are $1 \leq i_1 < \cdots < i_k \leq n$, then we have $px$ choices to place a rook below the bar in column $i_1$. Once we have placed a rook in column $i_1$ below the bar, we will have $px-p$ choices to add a rook below the bar in column $i_2$. Continuing on in this way it is easy to see that we have $(px)(px-p)\cdots(px-p(k-1)) = (px)\downarrow_{k,p}$ ways to extend $Q$ to a placement in $\mathcal{N}^p_n(B_x)$. By definition, the weight of any such placement $P$ is $\prod_{i=0}^{p-1} y_i^{\mathrm{cyc}_i(Q)}$. Thus

$$\begin{aligned} S(B,y_0,\ldots,y_{p-1}) &= \sum_{k=0}^{n}\sum_{Q\in\mathcal{N}^p_{n-k}(B)}\prod_{i=0}^{p-1} y_i^{\mathrm{cyc}_i(Q)}(px)\downarrow_{k,p} \\ &= \sum_{k=0}^{n}(px)\downarrow_{k,p}\sum_{Q\in\mathcal{N}^p_{n-k}(B)}\prod_{i=0}^{p-1} y_i^{\mathrm{cyc}_i(Q)} \\ &= \sum_{k=0}^{n} r^p_{n-k}(B,y_0,\ldots,y_{p-1})(px)\downarrow_{k,p}. \end{aligned}$$

A natural question here would be whether there is a similar result for singleton Ferrers boards or Ferrers boards. In the case where we set $y_i = 1$ for $i = 0,\ldots,p-1$, Briggs and Remmel [5] proved a factorization theorem for the $p$-rook numbers for singleton Ferrers boards and Barrese, Loehr, Remmel and Sagan [1] proved a factorization theorem for $p$-rook numbers for all Ferrers boards.

As an example of an application of Theorem 12.6, we give the cycle counting $p$-rook analogue of the Lah numbers. The Lah numbers $L_{n,k}$ are defined by the equation

$$(x)\uparrow_n = \sum_{k=1}^{n} L_{n,k}(x)\downarrow_k .$$

They can also be defined by the following recursion

$$L_{n+1,k} = L_{n,k-1} + (n+k)L_{n,k}, \tag{12.13}$$

with initial conditions $L_{0,0} = 1$ and $L_{n,k} = 0$ if $k < 0$ or $k > n$. The $L_{n,k}$'s have a nice rook theory interpretation, that is, $L_{n,k} = r_{n-k}(\mathscr{L}_n)$, where $\mathscr{L}_n$ is the Ferrers board consisting of $n$ columns of height $n-1$, see [12]. From this interpretation, it is easy to see that

$$L_{n,k} = \frac{(n-1)!}{(k-1)!}\binom{n}{k}. \tag{12.14}$$

That is, to create a rook placement of $n-k$ rooks in $\mathscr{L}_n$, we first pick the $n-k$ columns that will contain the rooks. We can do this in $\binom{n}{n-k} = \binom{n}{k}$ ways. Then we have to place the rooks in these columns starting from the left. We clearly have $n-1$ choices where to put a rook in the left most column, then $n-1-1$ ways to place a rook in the next column, etc. Thus we will have $(n-1)\downarrow_{n-k} = \frac{(n-1)!}{(k-1)!}$ ways to place the rooks in the $n-k$ columns that we chose.

For the obvious cycle-counting analogue of the $L_{n,k}$'s for $C_p \wr S_n$, consider the Ferrers board $\mathscr{L}_n^p$ which consists of $n$ columns of height $p(n-1)$. We let

$$L^p_{n,k}(y_0,\ldots y_{p-1}) = r^p_{n-k}(\mathscr{L}^p_n, y_0,\ldots,y_{p-1}). \tag{12.15}$$

In this case, (12.11) becomes

$$x(x+y_0+\cdots+y_{p-1})\uparrow_{n-1,p} = \sum_{k=1}^{n} L^p_{n,k}(y_0,\ldots y_{p-1})(x)\downarrow_{k,p}. \tag{12.16}$$

Note that

$$\begin{aligned}
&\sum_{k=1}^{n+1} L^p_{n+1,k}(y_0,\ldots y_{p-1})(x)\downarrow_{k,p} \\
&\qquad = x(x+y_0+\cdots+y_{p-1})\uparrow_{n,p} \\
&\qquad = (x+y_0+\cdots+y_{p-1}+p(n-1))x(x+y_0+\cdots+y_{p-1})\uparrow_{n-1,p} \\
&\qquad = (x+y_0+\cdots+y_{p-1}+p(n-1))\sum_{k=1}^{n} L^p_{n,k}(y_0,\ldots y_{p-1})(x)\downarrow_{k,p} \\
&\qquad = \sum_{k=1}^{n} L^p_{n,k}(y_0,\ldots y_{p-1})(x)\downarrow_{k,p}\,(x-kp+y_0+\cdots+y_{p-1}+p(n+k-1)) \\
&\qquad = \sum_{k=1}^{n} L^p_{n,k}(y_0,\ldots y_{p-1})(x)\downarrow_{k+1,p} \\
&\qquad\qquad + \sum_{k=1}^{n} L^p_{n,k}(y_0,\ldots y_{p-1})(x)\downarrow_{k,p}\,(y_0+\cdots+y_{p-1}+p(n+k-1)).
\end{aligned}$$

It thus follows that

$$\begin{aligned}&L^p_{n+1,k}(y_0,\dots y_{p-1})\\&=L^p_{n,k-1}(y_0,\dots y_{p-1})+(y_0+\cdots+y_{p-1}+p(n+k-1))L^p_{n,k}(y_0,\dots y_{p-1}).\end{aligned} \quad (12.17)$$

We also have an analogue of (12.14) in this case. That is, we want to compute

$$\sum_{P\in\mathcal{N}^p_{n-k}(\mathcal{L}^p_n)}\prod_{i=0}^{p-1}y_i^{\mathrm{cyc}_i(P)}.$$

We divide the $p$-rook placements in $\mathcal{N}^p_{n-k}(\mathcal{L}^p_n)$ into two sets, $N_1$ consisting of those $p$-rook placements with no rook in the last column and $N_2$ consisting of those $p$-rook placements that have a rook in the last column. For $N_1$, there are $\binom{n-1}{n-k}=\binom{n-1}{k-1}$ ways to choose the $n-k$ columns in which we are going to place the rooks. If $i\leq n-1$, then the height of the $i$-th column is greater than or equal to $pi$. Hence, we can use Lemma 12.1 to argue that as we place the rooks in the columns from left to right, the sum of $\prod_{i=0}^{p-1}y_i^{\mathrm{cyc}_i(P)}$ over the possible placements in the $n-k$ columns that we choose is

$$\begin{aligned}&(p(n-2)+y_0+\cdots+y_{p-1})(p(n-3)+y_0+\cdots+y_{p-1})\\&\qquad\qquad\cdots(p(k-1)+y_0+\cdots+y_{p-1}).\end{aligned}$$

Thus

$$\sum_{P\in N_1}\prod_{i=0}^{p-1}y_i^{\mathrm{cyc}_i(P)}=\binom{n-1}{k-1}(p(k-1)+y_0+\cdots+y_{p-1})\uparrow_{n-k,p}.$$

For $N_2$, there are $\binom{n-1}{n-k-1}=\binom{n-1}{k}$ ways to choose the columns in which we are going to place the rooks in the first $n-1$ columns. As above, the sum of $\prod_{i=0}^{p-1}y_i^{\mathrm{cyc}_i(P)}$ over the possible placements in the $n-k$ columns that we choose is

$$(p(n-2)+y_0+\cdots+y_{p-1})(p(n-3)+y_0+\cdots+y_{p-1})\cdots(pk+y_0+\cdots+y_{p-1}).$$

Once we place these rooks, we still have to place a rook in the last column. However, the height of the last column in $\mathcal{L}^p_n$ is $(n-1)p<np$. Thus by Lemma 12.1, the factor contributed by placing the rook in the last column in the $n-1-(n-k-1)=k$ levels which are possible is $pk$. Thus

$$\sum_{P\in N_2}\prod_{i=0}^{p-1}y_i^{\mathrm{cyc}_i(P)}=\binom{n-1}{k}(pk)(pk+y_0+\cdots+y_{p-1})\uparrow_{n-k-1,p}.$$

Hence,

$$\begin{aligned} L^p_{n,k} &= \binom{n-1}{k-1}(p(k-1)+y_0+\cdots+y_{p-1})\uparrow_{n-k,p} \\ &\quad+\binom{n-1}{k}(pk)(pk+y_0+\cdots+y_{p-1})\uparrow_{n-k-1,p} \\ &= \binom{n-1}{k-1}(pk+y_0+\cdots+y_{p-1})\uparrow_{n-k,p} . \end{aligned}$$

## 3 $Q$-analogues of cycle counting $p$-rook numbers

In this section, we shall define $q$-analogues of cycle counting $p$-rook numbers and prove a factorization theorem for such $q$-analogues for full Ferrers boards.

First we shall recall the definitions of the $q$-analogues of the $p$-rook and $p$-hit numbers as defined by Briggs and Remmel [5]. Let $B = F(b_1,\ldots,b_n)$ be a Ferrers board contained in $B^p_n$. A rook in cell $(i,j,k)$ is said to *rook cancel* all cells in level $j$ that lie strictly its right, and all cells that lie directly below it. Then for any given $P \in \mathcal{N}^p_k(B)$, we let $inv_B(P)$ be the number of uncancelled cells in $B-P$. For example, in Figure 5 we have pictured a placement in $B = F(6,9,12,12,15,15) \subseteq B^3_6$ and we have put dots in cells which are rook cancelled by rooks in $P$. Thus $inv_B(P) = 30$ as there is a total of 30 squares which are not rook cancelled by rooks in $P$.

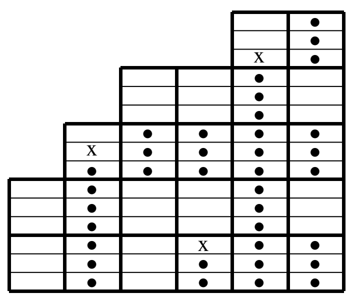

Fig. 5: An example of rook cancellation.

Suppose that $p \geq 2$. Then for $k \geq 1$, we let

$$\begin{aligned} [y]_q\uparrow_{k,p} &= [y]_q[y+p]_q\cdots[y+p(k-1)]_q \text{ and} \\ [y]_q\downarrow_{k,p} &= [y]_q[y-p]_q\cdots[y-p(k-1)]_q. \end{aligned}$$

We let $[y]_q\uparrow_{0,p} = [y]_q\downarrow_{0,p} = 1$. Then for any Ferrers board $B = F(b_1,\ldots,b_n) \subseteq B^p_n$, Briggs and Remmel defined $r^p_k(B,q)$ by

$$r^p_k(B,q) = \sum_{P\in\mathcal{N}^p_k} q^{inv(P)} \tag{12.18}$$

and $H^p_{k,n}(B,q)$ algebraically by

$$\sum_{k=0}^{n} H^p_{k,n}(B,q)x^k = \sum_{k=0}^{n} r^p_k(B,q)[p(n-k)]_q \downarrow_{n-k,p} \prod_{\ell=n-k+1}^{n}(x-q^{p\ell}). \tag{12.19}$$

Briggs and Remmel [5] then proved the following two theorems.

**Theorem 12.7.** *Let $B = F(b_1,\dots,b_n) \subseteq B^p_n$ be a Ferrers board. Then*

$$\prod_{i=1}^{n}[x+b_i-p(i-1)] = \sum_{k=0}^{n} r^p_{n-k}(B,q)[px]\downarrow_{k,p}. \tag{12.20}$$

**Theorem 12.8.** *Let $B = F(b_1,\dots,b_n) \subseteq B^p_n$ be a Ferrers board. Then $H_{n,k}(B,q)$ is a polynomial in $q$ with non-negative integer coefficients for all $k = 0,\dots,n$.*

In fact, Briggs and Remmel proved $p,q$-analogues of Theorems 12.7 and 12.8 but we shall not concern ourselves with $p,q$-analogues in this paper.

We define the $q$-analogue of the cycle-counting $p$-rook number by

$$r^p_k(B,q,y_0,\dots,y_{p-1}) = \sum_{P\in\mathscr{N}^p_k(B)} \left(\prod_{j=0}^{p-1}[y_j]_q^{\mathrm{cyc}_j(P)}\right) q^{\mathrm{inv}(P)+\sum_{j=0}^{p-1}(y_j-1)E_j(P)}, \tag{12.21}$$

where

$\mathrm{inv}(P)$ is the number of uncancelled cells (considering one sub level as one cell) when a rook cancels all the cells below it and all the cells to the right in the same level with the rook, and

$E_j(P)$ is the number of $i$'s such that $b_i \geq pi$ and there is no rook from $P$ in column $i$ on or above $s^j_i(P)$, where $s^j_i(P)$ is the unique sub level which, considering only rooks from $P$ in column 1 through $i-1$ of $B$, completes a $\omega_j$ cycle.

Then we have the following $q$-analogue of the factorization theorem.

**Theorem 12.9.** *Let $B = F(b_1,\dots,b_n)$ be a full Ferrers board contained in $B^p_n$.*

$$\prod_{i:b_i<pi}[px+b_i-p(i-1)]_q \prod_{i:b_i\geq pi}[px+b_i-pi+y_0+\cdots+y_{p-1}]_q$$
$$= \sum_{k=0}^{n} r^p_{n-k}(B,q,y_0,\dots,y_{p-1})[px]_q\downarrow_{k,p}. \tag{12.22}$$

*Proof.* It is not difficult to show that it is enough to prove (12.22) holds whenever $x,y_0,\dots,y_{p-1}$ are positive integers. The proof is similar to the proof of Theorem 12.6. Given $x\in\mathbb{P}$, we consider the extended board $B_x$ by adding $x$-levels of length $n$ below $B$. Then suppose that $y_0,\dots,y_{p-1}$ are fixed elements of $\mathbb{P}$. For a given $P\in\mathscr{N}^p_n(B_x)$, we let

$$wt(P) = \left(\prod_{j=0}^{p-1} [y_j]_q^{\mathrm{cyc}_j(P\cap B)}\right) q^{\mathrm{inv}(P)+\sum_{j=0}^{p-1}(y_j-1)E_j(P\cap B)}.$$

Then we claim that (12.22) arises by calculating

$$S(B,q,y_0,\ldots,y_{p-1}) = \sum_{P\in\mathcal{N}_n^p(B_x)} wt(P)$$

in two different ways.

First, we fix a $p$-rook placement $Q\in\mathcal{N}_{n-k}^p(B)$ of $n-k$ rooks in $B$. Then we want to compute

$$A_Q = \sum_{P\in\mathcal{N}_n(B_x),\ P\cap B=Q} wt(P).$$

In this case, there are $k$ columns below the bar which do not contain rooks in $Q$. First consider the contribution that comes from placing a rook below the bar in the first available column, reading from left to right. If we place a rook in the top cell of the first available column, then it would contribute $q^0$ to the weight of the rook placement. If we place that rook one cell below, then it would give $q^1$, and so on. Thus, our choices for placing a rook in this column contributes the weight sum

$$q^0+q^1+\cdots+q^{px-1} = [px]_q$$

to $A_Q$. Once we place a rook in the first available column, then we can use the same argument to show that our choices of placing a rook below the bar in the next available column contributes a factor $[px-p]_q$ to $A_Q$. By continuing in this way, we get

$$A_Q = \left(\prod_{j=0}^{p-1} [y_j]_q^{\mathrm{cyc}_j(Q)}\right) q^{\mathrm{inv}(Q)+\sum_{j=0}^{p-1}(y_j-1)E_j(Q)}[px]_q[px-p]_q\cdots[px-p(k-1)]_q.$$

Thus

$$\begin{aligned} S(B,q,y_0,\ldots,y_{p-1}) &= \sum_{k=0}^{n}\sum_{Q\in\mathcal{N}_{n-k}^p(B)} A_Q \\ &= \sum_{k=0}^{n}[px]_q\downarrow_{k,p}\sum_{Q\in\mathcal{N}_{n-k}^p(B)}\left(\prod_{j=0}^{p-1}[y_j]_q^{\mathrm{cyc}_j(Q)}\right) q^{\mathrm{inv}(Q)+\sum_{j=0}^{p-1}(y_j-1)E_j(Q)} \\ &= \sum_{k=0}^{n} r_{n-k}^p(B,q,y_0,\ldots,y_{p-1})[px]_q\downarrow_{k,p} \end{aligned}$$

which is the left-hand side of (12.22).

On the other hand, we can calculate $S(B,q,y_0,\ldots,y_{p-1})$ by adding rooks column by column, starting from left to right. To do this, we need an analogue of Lemma 12.1, which we state and prove separately subsequent to the current proof; see Lemma 12.2.

If we start with a placement $Q \in \mathcal{N}_{i-1}^{p}(B_x)$ of $i-1$ rooks in the first $i-1$ columns of $B_x$, then the $i$ th column will contribute the factor $[px+b_i-pi+y_0+\cdots+y_{p-1}]_q$ for placing a rook in the column $i$ if $b_i \geq pi$ and will contribute a factor $[px+b_i-p(i-1)]_q$ if $b_i < pi$. Thus,

$$S(B,q,y_0,\ldots,y_{p-1}) \\ = \prod_{i:b_i<pi}[px+b_i-p(i-1)]_q \prod_{i:b_i\geq pi}[px+b_i-pi+y_0+\cdots+y_{p-1}]_q,$$

which is the right-hand side of (12.22).

**Lemma 12.2.** *Suppose that $Q \in \mathcal{N}_t^{p}(B_x)$ is a p-rook placement of t rooks in the first $i-1$ columns of $B_x$. Let $D_i(Q)$ denote the set of all p-rook placements P that result from Q by adding a rook in column i. Then*

$$\sum_{P\in D_i(Q)} wt(P) = \begin{cases} [b_i+px-p(t+1)+y_0+\cdots+y_{p-1}]_q wt(Q), & \text{if } b_i \geq pi, \\ [b_i+px-pt]_q wt(Q), & \text{if } b_i < pi. \end{cases} \tag{12.23}$$

*Proof.* The proof is similar to the proof of Lemma 12.1. That is, if $b_i < pi$, then any placement of a rook in column $i$ will not contribute to $E_j(P\cap B)$ for any $j$. Now there are $px+b_i-pt$ uncancelled squares in the $i$-th column. If we place a rook $r_i$ in the $j$-th uncancelled cell from the top in column $i$, then $r_i$ will contribute a factor $q^{j-1}$ to $wt(P)$ as the contribution to $inv(P)$ from $r_i$ will be $j-1$. Thus in this case, the placement of $r_i$ will contribute a factor

$$wt(Q)\sum_{j=1}^{px+b_i-pt} q^{j-1} = wt(Q)[px+b_i-pt]_q$$

to $\sum_{P\in D_i(Q)} wt(P)$.

If $b_i \geq pi$, then there is a level $l_i \leq i$ such that placing a rook $r_i$ in level $\ell_i$ in column $i$ will complete a cycle relative to the rooks in $Q$. Assume that if we place a rook in cell $(i,l_i,s)$, then we complete a cycle of sign $\omega^{u_s}$. Thus $\omega^{u_0},\ldots,\omega^{u_{p-1}}$ must be a rearrangement of $1,\omega,\ldots,\omega^{p-1}$. In addition, assume that there are $pt_i$ uncancelled cells above level $\ell_i$ in column $i$. Then as before, placing a rook in $j$-th uncancelled cell from the top, where $j \leq pt_i$, will give a factor $q^{j-1}$ to $\sum_{P\in D_i(Q)} wt(P)$. Thus the placements of a rook in the top $pt_i$ cells will give a factor

$$wt(Q)(1+q+\cdots+q^{pt_i-1}) = wt(Q)[pt_i]_q$$

to $\sum_{P\in D_i(Q)} wt(P)$.

Now consider the effect of placing a rook $r_i$ in the cell $(i,l_i,p-1)$. Then $r_i$ would contribute a factor

$$[y_{u_{p-1}}]_q q^{pt_i} = q^{pt_i}+\cdots+q^{pt_i+y_{u_{p-1}}-1}$$

to $wt(P)$. Here $[y_{u_{p-1}}]_q$ comes from the fact that we completed a cycle of sign $\omega^{u_{p-1}}$ and $q^{pt_i}$ comes from the contribution of $r_i$ to $inv(P)$. Note that $r_i$ makes no contribution to $E_j(P)$ for any $j$ in this case. Next consider the effect of placing a rook $r_i$ in the cell $(i,l_i,p-2)$. Then $r_i$ would contribute

$$[y_{u_{p-2}}]_q q^{pt_i+1} q^{y_{u_{p-1}}-1} = q^{pt_i+y_{u_{p-1}}} + \cdots + q^{pt_i+y_{u_{p-1}}+y_{u_{p-2}}-1}$$

to $wt(P)$. Here $[y_{u_{p-2}}]_q$ comes from the fact that we completed a cycle of sign $\omega^{u_{p-2}}$, $q^{pt_i+1}$ comes from the contribution of $r_i$ to $inv(P)$, and $q^{y_{u_{p-1}}-1}$ comes from the fact that the placement of $r_i$ contributes 1 to $E_{u_{p-1}}(P)$. Next consider the effect of placing a rook $r_i$ in the cell $(i,l_i,p-3)$. Then $r_i$ would contribute

$$[y_{u_{p-3}}]_q q^{pt_i+2} q^{y_{u_{p-1}}-1+y_{u_{p-2}}-1} = q^{pt_i+y_{u_{p-1}}+y_{u_{p-2}}} + \cdots + q^{pt_i+y_{u_{p-1}}+y_{u_{p-2}}+y_{u_{p-3}}-1}$$

to $wt(P)$. Here $[y_{u_{p-3}}]_q$ comes from the fact that we completed a cycle of sign $\omega^{u_{p-3}}$, $q^{pt_i+2}$ comes from the contribution of $r_i$ to $inv(P)$, and $q^{y_{u_{p-1}}-1+y_{u_{p-2}}-1}$ comes from the fact that the placement of $r_i$ contributes 1 to both $E_{u_{p-2}}(P)$ and $E_{u_{p-1}}(P)$. Continuing on in this way, one can show that the contribution of all the possible placements of $r_i$ in level $\ell_i$ in column $i$ contribute a factor $wt(Q)q^{pt_i}[y_0+\cdots y_{p-1}]_q$ to $\sum_{P\in D_i(Q)} wt(P)$.

We have $px+b_i-pt-pt_i-p$ uncancelled cells below level $\ell_i$ in column $i$. If we place a rook $r_i$ in the $s$-th such cell reading from the top, then $r_i$ contributes $q^{pt_i+p+s-1}q^{\sum_{j=0}^{p-1} y_i-1} = q^{pt_i+y_0+\cdots+y_{p-1}+s-1}$ to $wt(P)$. Here $q^{pt_i+p+s-1}$ comes from $r_i$ contribution to $inv(P)$ and $q^{\sum_{j=0}^{p-1} y_i-1}$ comes from the fact that $r_i$ would contribute 1 to $E_j(P)$ for $j=0,\ldots,p-1$. It follows that contribution to $\sum_{P\in D_i(Q)} wt(P)$ over all possible placements of rooks in the remaining $px+b_i-pt-pt_i-p$ uncancelled cells is

$$wt(Q)q^{pt_i+y_0+\cdots+y_{p-1}}[px+b_i-pt-pt_i-p]_q.$$

Hence the total contribution to $\sum_{P\in D_i(Q)} wt(P)$ of the placements of rooks in the $i$-th column in the case where $b_i \geq pi$ is

$$\begin{aligned} wt(Q)([pt_i]_q + q^{pt_i}[\sum_{i=0}^{p-1} y_i]_q + q^{pt_i+\sum_{i=0}^{p-1} y_i}[px+b_i-pt-pt_i-p]_q) \\ = wt(Q)[px+b_i-p(t+1)+y_0+\cdots+y_{p-1}]_q, \end{aligned}$$

as desired.

*Example 12.1 (q-cycle counting Lah numbers).* We consider the $q$-analogue of cycle-counting Lah numbers $L^p_{n,k}(y_0,\ldots,y_{p-1})$ for $C_p \wr S_n$. We let

$$L^p_{n,k}(q,y_0,\ldots,y_{p-1}) = r^p_{n-k}(\mathscr{L}^p_n, q, y_0,\ldots,y_{p-1}), \tag{12.24}$$

where $\mathscr{L}^p_n$ is the Ferrers board which consists of $n$ columns of height $p(n-1)$. Then, by Theorem 12.9, we have

$$
\begin{aligned}
&[px]_q[px+y_0+\cdots+y_{p-1}]_q\uparrow_{n-1,p} \\
&\qquad = \sum_{k=1}^{n}[px]_q[p(x-1)]_q\cdots[p(x-k+1)]_q L^p_{n,k}(q,y_0,\ldots,y_{p-1}).
\end{aligned} \tag{12.25}
$$

Note that

$$
\begin{aligned}
&\sum_{k=1}^{n+1} L^p_{n+1,k}(q,y_0,\ldots,y_{p-1})[px]_q\downarrow_{k,p} \\
&\qquad = [px]_q[px+y_0+\cdots+y_{p-1}]_q\uparrow_{n,p} \\
&\qquad = [px]_q[px+y_0+\cdots+y_{p-1}]_q\uparrow_{n-1,p}\,[px+p(n-1)+y_0+\cdots+y_{p-1}]_q \\
&\qquad = \sum_{k=1}^{n} L^p_{n,k}(q,y_0,\ldots,y_{p-1})[px]_q\downarrow_{k,p}\,[px+p(n-1)+y_0+\cdots+y_{p-1}]_q \\
&\qquad = \sum_{k=1}^{n} L^p_{n,k}(q,y_0,\ldots,y_{p-1}) \\
&\qquad\qquad \times\ [px]_q\downarrow_{k,p}\,[p(x-k)+p(n+k-1)+y_0+\cdots+y_{p-1}]_q \\
&\qquad = \sum_{k=1}^{n} q^{p(n+k-1)+y_0+\cdots+y_{p-1}} L^p_{n,k}(q,y_0,\ldots,y_{p-1})[px]_q\downarrow_{k+1,p} \\
&\qquad\quad + \sum_{k=1}^{n} L^p_{n,k}(q,y_0,\ldots,y_{p-1})[px]_q\downarrow_{k,p}\,[p(n+k-1)+y_0+\cdots+y_{p-1}]_q.
\end{aligned}
$$

Thus we get the recurrence relation

$$
\begin{aligned}
L^p_{n+1,k}(q,y_0,\ldots,y_{p-1}) &= q^{p(n+k-1)+y_0+\cdots+y_{p-1}} L^p_{n,k-1}(q,y_0,\ldots,y_{p-1}) \\
&\quad + L^p_{n,k}(q,y_0,\ldots,y_{p-1})[p(n+k-1)+y_0+\cdots+y_{p-1}]_q.
\end{aligned} \tag{12.26}
$$

Using this recursion, we can also prove the following closed form expression

$$
\begin{aligned}
&L^p_{n,k}(q,y_0,\ldots,y_{p-1}) \\
&\qquad = q^{k(k-1)p+(k-1)(y_0+\cdots+y_{p-1})} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_{q^p} [pk+y_0+\cdots y_{p-1}]_q\uparrow_{n-k,p}\,.
\end{aligned} \tag{12.27}
$$

## 4 $Q$-analogues of cycle counting $p$-hit numbers

Recall that a full Ferrers board $B = F(b_1,\ldots,b_n) \subseteq B^n_p$ is *regular* if $b_i = p\cdot c_i$, where $c_i \geq i$ for $1 \leq i \leq n$. The goal of this section is to define a $q$-analogue of cycle counting $p$-hit numbers for full regular Ferrers boards and to give a conjectured combinatorial interpretation for them. Before we start, we introduce an alternate notation for a Ferrers board. Given a Ferrers board $B = F(b_1,b_2,\ldots,b_n) \subseteq B^p_n$, we will also use the notation $B = B(h_1^p, d_1; \ldots; h_t^p, d_t)$ which uses the step heights and depths as pictured in Figure 6.

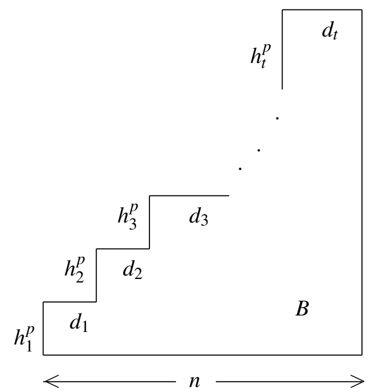


Fig. 6: The Ferrers board $B = B(h_1^p, d_1; \ldots; h_t^p, d_t)$

Now if $B = F(pc_1, \ldots, pc_n)$ is a regular full Ferrers board contained in $B_n^p$, then, in the notation $B = B(h_1^p, d_1; \ldots; h_t^p, d_t)$, $h_j^p = p \cdot h_j$ where $h_j$'s are the number of levels of the corresponding step. Then by Theorem 12.9,

$$\sum_{k=0}^{n} r_{n-k}^p(B, q, y_0, \ldots, y_{p-1})[px]_q\downarrow_{k,p} = \prod_{i=1}^{n}[px + p(c_i - i) + y_0 + \cdots + y_{p-1}]_q. \quad (12.28)$$

We let the right-hand side of (12.28) be

$$\mathrm{PR}[B, x, y_0, \ldots, y_{p-1}] := \prod_{i=1}^{n}[px + p(c_i - i) + y_0 + \cdots + y_{p-1}]_q.$$

We define our $q$-analogue $H_{k,n}^p(B, q, y_0, \ldots, y_{p-1})$ of the cycle counting $p$-hit numbers by

$$\sum_{k=0}^{n} r_{n-k}^p(B, q, y_0, \ldots, y_{p-1})[y_0 + \cdots + y_{p-1}]_q\uparrow_{k,p} z^k \times \prod_{i=k+1}^{n}(1 - zq^{y_0 + \cdots + y_{p-1} + p(i-1)}) = \sum_{k=0}^{n} H_{k,n}^p(B, q, y_0, \ldots, y_{p-1}) z^k. \quad (12.29)$$

Note that when $q = 1$, by changing $z$ to $z^{-1}$ and multiplying $z^n$ on both sides, we can transform (12.29) to

$$\sum_{k=0}^{n} H^p_{n-k,n}(B,1,y_0,\ldots,y_{p-1})z^k$$
$$= \sum_{k=0}^{n} r^p_k(B,1,y_0,\ldots,y_{p-1})(y_0+\cdots+y_{p-1})\uparrow_{n-k,p}(z-1)^k.$$

By comparing it to the result of Theorem 12.5, we can see that

$$H^p_{k,n}(B,1,y_0,\ldots,y_{p-1}) = H^p_{n-k,n}(B,y_0,\ldots,y_{p-1}).$$

Our first goal is to give a recursion for the $H^p_{k,n}(B,q,y_0,\ldots,y_{p-1})$'s which will show that $H^p_{k,n}(B,q,y_0,\ldots,y_{p-1})$ is a polynomial in $q$ with non-negative coefficients when $y_0,\ldots,y_{p-1}$ are non-negative integers. To derive our desired recursion of $H^p_{k,n}(B,q,y_0,\ldots,y_{p-1})$, we define a more general version of it. That is, we define $H^p_{k,n}(B,x,q,y_0,\ldots,y_{p-1})$ by

$$\sum_{k=0}^{n} H^p_{k,n}(B,x,q,y_0,\ldots,y_{p-1})z^k$$
$$= \sum_{k=0}^{n} r^p_{n-k}(B,q,y_0,\ldots,y_{p-1})[px]_q\uparrow_{k,p} z^k \prod_{i=k+1}^{n}(1-zq^{px+p(i-1)}).$$

*Remark 12.1.* We note that

$$H^p_{k,n}(B,q,y_0,\ldots,y_{p-1}) = H^p_{k,n}(B,x,q,y_0,\ldots,y_{p-1})\Big|_{x=\frac{y_0+\cdots+y_{p-1}}{p}},$$

and $H^p_{k,n}(B,x,q,y_0,\ldots,y_{p-1})$ is a generalization of $H_k(x,y,B)$ as defined by Haglund in [14] and used by Butler in [6].

The following two propositions are the generalizations of the result of Haglund in [14, Lemma 5.1, Lemma 5.7].

**Proposition 12.1.** *Suppose $B = F(pc_1,\ldots,pc_n)$ is a regular full Ferrers board contained in $B^p_n$. Then we have*

$$H^p_{k,n}(B,x,q,y_0,\ldots,y_{p-1})$$
$$= \sum_{j=0}^{k} \begin{bmatrix} n+x \\ k-j \end{bmatrix}_{q^p} \begin{bmatrix} x+j-1 \\ j \end{bmatrix}_{q^p} (-1)^{k-j} q^{p\binom{k-j}{2}} PR[B,j,y_0,\ldots,y_{p-1}], \quad (12.30)$$

*where $PR[B,j,y_0,\ldots,y_{p-1}] = \prod_{i=1}^{n}[pj+p(c_i-i)+y_0+\cdots+y_{p-1}]_q$.*

*Proof.* In the proof, we use the following short-hand notation

$$([x]_{q^p})_j = [x]_{q^p}[x+1]_{q^p}\cdots[x+j-1]_{q^p}.$$

The right-hand side of (12.30) is

$$
\begin{aligned}
&\sum_{j=0}^{k} \begin{bmatrix} n+x \\ k-j \end{bmatrix}_{q^p} (-1)^{k-j} q^{p\binom{k-j}{2}} \begin{bmatrix} x+j-1 \\ j \end{bmatrix}_{q^p} \\
&\qquad \times \sum_{s=0}^{n} [p]_q^s [j]_{q^p} [j-1]_{q^p} \cdots [j-s+1]_{q^p} r_{n-s}^p(B,q,y_0,\ldots,y_{p-1}) \\
&= \sum_{s=0}^{n} [p]_q^s r_{n-s}^p(B,q,y_0,\ldots,y_{p-1}) \\
&\qquad \times \sum_{j\geq s} \begin{bmatrix} n+x \\ k-j \end{bmatrix}_{q^p} (-1)^{k-j} q^{p\binom{k-j}{2}} \frac{([x]_{q^p})_j}{([1]_{q^p})_j} [j]_{q^p} \cdots [j-s+1]_{q^p} \\
&= \sum_{s=0}^{n} [p]_q^s r_{n-s}^p(B,q,y_0,\ldots,y_{p-1}) \\
&\qquad \times \sum_{\substack{u\geq 0 \\ j=u+s}} \begin{bmatrix} n+x \\ k-u-s \end{bmatrix}_{q^p} (-1)^{k-u-s} q^{p\binom{k-u-s}{2}} \frac{([x]_{q^p})_{u+s}}{([1]_{q^p})_{u+s}} ([u+1]_{q^p})_s \\
&= \sum_{s=0}^{n} [p]_q^s r_{n-s}^p(B,q,y_0,\ldots,y_{p-1}) \sum_{u\geq 0} \begin{bmatrix} n+x \\ k-s \end{bmatrix}_{q^p} (-1)^{k-s} q^{p\binom{k-s}{2}} \\
&\qquad \times \frac{([s-k]_{q^p})_u}{([n+x-k+s+1]_{q^p})_u} \frac{([x]_{q^p})_s([x+s]_{q^p})_u}{([s+1]_{q^p})_u} \begin{bmatrix} u+s \\ u \end{bmatrix}_{q^p} \\
&= \sum_{s=0}^{n} [p]_q^s r_{n-s}^p(B,q,y_0,\ldots,y_{p-1}) \sum_{u\geq 0} \begin{bmatrix} n+x \\ k-s \end{bmatrix}_{q^p} (-1)^{k-s} q^{p\binom{k-s}{2}} \\
&\qquad \times ([x]_{q^p})_s \sum_{u\geq 0} \frac{([-k+s]_{q^p})_s([x+s]_{q^p})_u}{([1]_{q^p})_u([n+x-k+s+1]_{q^p})_u} \\
&= \sum_{s=0}^{n} [p]_q^s r_{n-s}^p(B,q,y_0,\ldots,y_{p-1}) \\
&\qquad \times \begin{bmatrix} n+x \\ k-s \end{bmatrix}_{q^p} (-1)^{k-s} q^{p\binom{k-s}{2}} ([x]_{q^p})_s \frac{([n-k+1]_{q^p})_{k-s}}{([n+x-k+s+1]_{q^p})_{k-s}} \\
&= \sum_{s=0}^{n} [p]_q^s ([x]_{q^p})_s r_{n-s}^p(B,q,y_0,\ldots,y_{p-1}) \begin{bmatrix} n-s \\ k-s \end{bmatrix}_{q^p} (-1)^{k-s} q^{p\binom{k-s}{2}} \\
&= H_{k,n}^p(B,x,y_0,\ldots,y_{p-1}).
\end{aligned}
$$

**Proposition 12.2.** *Suppose $B = B(h_1^p, d_1; h_2^p, d_2; \ldots, ; h_t^p, d_t)$, where $h_i^p = ph_i$ for nonnegarive integer $h_i$, $i = 1, \ldots, t$, is regular full Ferrers board contained in $B_n^p$. Let $H_i := h_1 + \cdots + h_i$, $D_i := d_1 + \cdots + d_i$, and the notation $B - h_i - d_j$ refer to the board obtained from $B$ by decreasing $h_i$ and $d_j$ by one each and leaving the other parameters fixed. (For an example of the board $B - h_i - d_j$, refer to Figure 7). Then we have the following recursion for $H_{k,n}^p(B, x, q, y_0, \ldots, y_{p-1})$.*

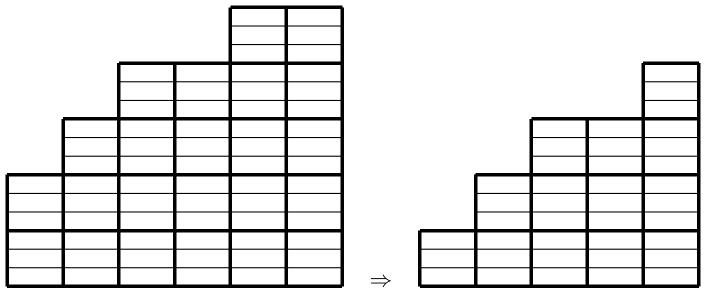

Fig. 7: $B = B(6,1;3,1;3,2;3,2)$ and $B - h_1 - d_4 = B(3,1;3,1;3,2;3,1)$, for $p = 3$.

$$\begin{aligned}
&H^p_{k,n}(B,x,q,y_0,\ldots,y_{p-1})\\
&\qquad = [p]_q \left[k + H_l - D_l + d_l - 1 + \frac{y_0+\cdots+y_{p-1}}{p}\right]_{q^p}\\
&\qquad\qquad \times H^p_{k,n-1}(B-h_l-d_l,x,y_0,\ldots,y_{p-1})\\
&\qquad + [p]_q \Bigg(-q^{p(n+x-1)}\left[k+H_l-D_l+d_l-1+\frac{y_0+\cdots+y_{p-1}}{p}\right]_{q^p}\\
&\qquad\qquad + q^{p(k+H_l-D_l+d_l-2+\frac{y_0+\cdots+y_{p-1}}{p})}[n+x]_{q^p}\Bigg)\\
&\qquad\qquad \times H^p_{k-1,n-1}(B-h_l-d_l,x,q,y_0,\ldots,y_{p-1}).
\end{aligned}$$

*Proof.* We have

$$\begin{aligned}
&H^p_{k,n}(B,x,q,y_0,\ldots,y_{p-1})\\
&= \sum_{s=0}^{k} \begin{bmatrix} n+x \\ k-s \end{bmatrix}_{q^p} \begin{bmatrix} x+s-1 \\ s \end{bmatrix}_{q^p} (-1)^{k-s} q^{p\binom{k-s}{2}}\\
&\qquad \times \prod_{i=1}^{n} [ps + p(b_i - i) + y_0 + \cdots + y_{p-1}]_q\\
&= \sum_{s=0}^{k} \begin{bmatrix} n+x \\ k-s \end{bmatrix}_{q^p} \begin{bmatrix} x+s-1 \\ s \end{bmatrix}_{q^p} (-1)^{k-s} q^{p\binom{k-s}{2}}\\
&\quad \times [ps + p(H_l - D_l + d_l - 1) + y_0 + \cdots + y_{p-1}]_q \mathrm{PR}[B-h_l-d_l,s,y_0,\ldots,y_{p-1}]\\
&= \sum_{s=0}^{k} \begin{bmatrix} n+x \\ k-s \end{bmatrix}_{q^p} \begin{bmatrix} x+s-1 \\ s \end{bmatrix}_{q^p} (-1)^{k-s} q^{p\binom{k-s}{2}}\\
&\qquad \times \mathrm{PR}[B-h_l-d_l,s,y_0,\ldots,y_{p-1}]\\
&\qquad \times \Bigg\{ [p] \left[k + H_l - D_l + d_l - 1 + \frac{y_0+\cdots+y_{p-1}}{p}\right]_{q^p}
\end{aligned}$$

$$
\left. - q^{p(s+H_l-D_l+d_l-1+\frac{y_0+\cdots+y_{p-1}}{p})}[p][k-s]_{q^p} \right\}
$$

$$
\begin{aligned}
&= [p]_q \left[ k+H_l-D_l+d_l-1+\frac{y_0+\cdots+y_{p-1}}{p} \right]_{q^p} \\
&\times \sum_{s=0}^{k} \begin{bmatrix} n+x \\ k-s \end{bmatrix}_{q^p} \begin{bmatrix} x+s-1 \\ s \end{bmatrix}_{q^p} (-1)^{k-s} q^{p\binom{k-s}{2}} \mathrm{PR}[B-h_l-d_l,s,y_0,\ldots,y_{p-1}] \\
&\quad - [p]_q q^{p(s+H_l-D_l+d_l-1+\frac{y_0+\cdots+y_{p-1}}{p})} \sum_{s=0}^{k} [n+x]_{q^p} \begin{bmatrix} n+x-1 \\ k-s-1 \end{bmatrix}_{q^p} \\
&\qquad \times \begin{bmatrix} x+s-1 \\ s \end{bmatrix}_{q^p} (-1)^{k-s} q^{p\left(\binom{k-s}{2}+s\right)} \mathrm{PR}[B-h_l-d_l,s,y_0,\ldots,y_{p-1}] \\
&= [p]_q \left[ k+H_l-D_l+d_l-1+\frac{y_0+\cdots+y_{p-1}}{p} \right]_{q^p} \sum_{s=0}^{k} \begin{bmatrix} x+s-1 \\ s \end{bmatrix}_{q^p} (-1)^{k-s} \\
&\times \left\{ \begin{bmatrix} n+x-1 \\ k-s \end{bmatrix}_{q^p} q^{p\binom{k-s}{2}} + \begin{bmatrix} n+x-1 \\ k-s-1 \end{bmatrix}_{q^p} q^{p\left(\binom{k-s-1}{2}+n+x-1\right)} \right\} \\
&\qquad\qquad \times \mathrm{PR}[B-h_l-d_l,s,y_0,\ldots,y_{p-1}] \\
&- [p]_q q^{p(s+H_l-D_l+d_l-1+\frac{y_0+\cdots+y_{p-1}}{p})} \sum_{s=0}^{k-1} [n+x]_{q^p} \begin{bmatrix} n+x-1 \\ k-s-1 \end{bmatrix}_{q^p} \\
&\qquad \times \begin{bmatrix} x+s-1 \\ s \end{bmatrix}_{q^p} (-1)^{k-s} q^{p\left(\binom{k-s-1}{2}+k-1\right)} \mathrm{PR}[B-h_l-d_l,s,y_0,\ldots,y_{p-1}] \\
&= [p]_q \left[ k+H_l-D_l+d_l-1+\frac{y_0+\cdots+y_{p-1}}{p} \right]_{q^p} \\
&\qquad \times H^p_{k,n-1}(B-h_l-d_l,x,q,y_0,\ldots,y_{p-1}) \\
&+ [p]_q H^p_{k-1,n-1}(B-h_l-d_l,x,q,y_0,\ldots,y_{p-1}) \\
&\qquad\qquad \times \left\{ q^{p(k+H_l-D_l+d_l-2+\frac{y_0+\cdots+y_{p-1}}{p})}[n+x]_{q^p} \right. \\
&\qquad\qquad \left. - q^{p(n+x-1)} \left[ k+H_l-D_l+d_l-1+\frac{y_0+\cdots+y_{p-1}}{p} \right]_{q^p} \right\} \\
&= [p]_q \left[ k+H_l-D_l+d_l-1+\frac{y_0+\cdots+y_{p-1}}{p} \right]_{q^p} \\
&\qquad \times H^p_{k,n-1}(B-h_l-d_l,x,q,y_0,\ldots,y_{p-1}) \\
&+ [p]_q \left[ n+x-k-H_l+D_l-d_l+1-\frac{y_0+\cdots+y_{p-1}}{p} \right]_{q^p} \\
&\qquad \times q^{p(k+H_l-D_l+d_l-2+\frac{y_0+\cdots+y_{p-1}}{p})} H^p_{k-1,n-1}(B-h_l-d_l,x,q,y_0,\ldots,y_{p-1}).
\end{aligned}
$$

**Proposition 12.3.** *If $B_j$ is the board*

$$B(h_1^p, d_1; \cdots; h_{l-1}^p, d_{l-1}; h_l^p - pj, d_l - j; h_{l+1}^p, d_{l+1}; \ldots; h_t^p, d_t)$$

*obtained from a regular Ferrers board B by decreasing $h_l^p$ by $pj$ and $d_l$ by $j$ (here we assume that $j \le h_l, d_l$, where $h_l^p = ph_l$), then*

$$H_{k,n}^p(B,x,q,y_0,\ldots,y_{p-1}) = [p]_q^j [j]_{q^p}! \sum_{s=k-j}^{k} H_{s,n}^p(B_j,x,q,y_0,\ldots,y_{p-1}) \times \begin{bmatrix} T_l - 1 + s \\ j-k+s \end{bmatrix}_{q^p} \begin{bmatrix} n - T_l + x - s \\ k-s \end{bmatrix}_{q^p} q^{p(k-s)(T_l+k-j-1)}, \quad (12.31)$$

*where $T_l = H_l - D_{l-1} + \frac{y_0+\cdots+y_{p-1}}{p}$.*

*Proof.* The proof can be done by induction on $j$ and by using the recursion in Proposition 12.2. The proof is similar to the proof of [14, Theorem 4.1, Theorem 5.8], hence we omit the details.

By using Proposition 12.2, we can derive the recursion for $H_{k,n}^p(B,q,y_0,\ldots,y_{p-1})$.

**Theorem 12.10.** *Suppose $B = B(h_1^p, d_1; h_2^p, d_2; \ldots,; h_t^p, d_t)$, where $h_i^p = ph_i$, is regular full Ferrers board contained in $B_n^p$. Let $H_i := h_1 + \cdots + h_i$, $D_i := d_1 + \cdots + d_i$, and the notation $B - h_i - d_j$ refers to the board obtained from B by decreasing $h_i$ and $d_j$ by one each and leaving the other parameters fixed. Then we have the following recursion for $H_{k,n}^p(B,q,y_0,\ldots,y_{p-1})$:*

$$\begin{aligned} &H_{k,n}^p(B,q,y_0,\ldots,y_{p-1}) \\ &\quad = [p]_q \left[\frac{y_0+\cdots+y_{p-1}}{p} + k + d_t - 1\right]_{q^p} H_{k,n-1}^p(B - h_t - d_t, q, y_0,\ldots,y_{p-1}) \\ &\qquad + [p]_q q^{p\left(\frac{y_0+\cdots+y_{p-1}}{p}+k+d_t-2\right)} [n-k-d_t+1]_{q^p} \\ &\qquad\qquad \times H_{k-1,n-1}^p(B - h_t - d_t, q, y_0,\ldots,y_{p-1}), \end{aligned} \quad (12.32)$$

*where $h_t$ and $d_t$ are the height (number of levels) and the depth of the last step of B.*

We note that it follows from Theorem 12.10 that if $B = F(pc_1,\ldots,pc_n)$ is a regular full Ferrers board in $B_n^p$ and $y_0,\ldots,y_{p-1}$ are non-negative integers, then $H_{k,n}^p(B,q,y_0,\ldots,y_{p-1})$ is a polynomial with non-negative coefficients in $q$. Here are some small examples.

*Example 12.2.* When $B_1$ has only one square (level) with $p$ sublevels, i.e., $B_1 = F(p)$, then

$$
\begin{aligned}
&H^p_{0,1}(B_1, q, y_0, \ldots, y_{p-1}) \\
&= r^p_1 = \sum_{P \in \mathcal{N}^p_1(B_1)} \left[ \prod_{j=0}^{p-1} [y_j]_q^{\mathrm{cyc}_j(P)} q^{\mathrm{inv}(P) + \sum_{j=0}^{p-1}(y_j - 1)E_j(P)} \right] \\
&= q^0 [y_{p-1}]_q + q^{1+(y_{p-1}-1)} [y_{p-2}]_q + \cdots + q^{p-1+\sum_{j=1}^{p-1}(y_j-1)} [y_0]_q \\
&= q^0 [y_{p-1}]_q + q^{y_{p-1}} [y_{p-2}]_q + q^{y_{p-1}+y_{p-2}} [y_{p-3}]_q + \cdots + q^{\sum_{j=1}^{p-1} y_j} [y_0]_q \\
&= [y_0 + \cdots + y_{p-1}]_q,
\end{aligned}
$$

$H^p_{k,1}(B_1, q, y_0, \ldots, y_{p-1}) = 0,$ for $k > 0$.

We continue computing small examples for $n = 2$:

$$
\begin{aligned}
&H^p_{0,2}(\boxplus, q, y_0, \ldots, y_{p-1}) = [y_0 + \cdots + y_{p-1}]_q [y_0 + \cdots + y_{p-1} + p]_q, \\
&H^p_{k,2}(\boxplus, q, y_0, \ldots, y_{p-1}) = 0, \text{ for } k > 0.
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
&H^p_{0,2}(, q, y_0, \ldots, y_{p-1}) = [y_0 + \cdots + y_{p-1}]^2_q, \\
&H^p_{1,2}(, q, y_0, \ldots, y_{p-1}) = q^{(y_0 + \cdots + y_{p-1})} [p]_q [y_0 + \cdots + y_{p-1}]_q, \\
&H^p_{2,2}(, q, y_0, \ldots, y_{p-1}) = 0.
\end{aligned}
$$

Based on the $q$-statistics for the cycle counting hit numbers defined by Butler in [6], we conjecture a similar $q$-statistic for the cycle counting $p$-hit numbers. Before we make a precise statement, we need some definitions.

For a full regular Ferrers board $B \subseteq B^p_n$, let $\mathcal{N}^p(B) = \cup_{k=1}^n \mathcal{N}^p_k(B)$. For $p \in \mathcal{N}^p(B)$, note the Butler's statistic $s_{B,b}(P)$ [6] defined as the number of squares on $B^p_n$ which neither contain a rook from $P$ nor are cancelled, after applying the following cancellation scheme:

1. Each rook cancels all squares to the right in its row,
2. each rook on $B$ cancels all squares above it in its column (squares both on $B$ and strictly above $B$),
3. each rook on $B$ which also completes a cycle cancels all squares below it in its column as well,
4. each rook off $B$ cancels all squares below it but above $B$.

Define $\mathrm{cyc}_{\geq j}(P)$ by

$$
\mathrm{cyc}_{\geq j}(P) := \sum_{i=j}^{p-1} \mathrm{cyc}_i(P).
$$

Since $b_i \geq pi$, there exists a unique level, say $u$, in column $i$ such that considering only rooks from $P$ in column 1 through column $i-1$ of $B$, completes a cycle. At the $i^{\text{th}}$ column, define $\tilde{E}_i(P)$ by

$$\tilde{E}_i(P) = \begin{cases} p, & \text{if there is no rook from } P \text{ in column } i \text{ on or above the level } u, \\ 0, & \text{if there is a rook from } P \text{ in column } i \text{ above the level } u, \\ p-1-j, & \text{if there is a rook on the level } u \text{ completing a cycle of sign } \omega_j. \end{cases}$$

Then we conjecture the following combinatorial formula for $H^p_{k,n}(B,q,y_0,\ldots,y_{p-1})$.

*Conjecture 12.1.* Let $\mathscr{H}_{k,n}(B)$ be the set of all placements corresponding $\sigma \in C_p \wr S_n$ such that $|\sigma \cap B| = k$. Then, for a full regular Ferrers board $B \subseteq B^p_n$,

$$H^p_{k,n}(B,q,y_0,\ldots,y_{p-1}) = \sum_{P \in \mathscr{H}_{n-k,n}(B)} \left( \prod_{j=0}^{p-1} [y_j]_q^{\mathrm{cyc}_j(P)} \right) q^{s_{B,b}(P) + \sum_{i=1}^n \tilde{E}_i(P) + \sum_{j=0}^{p-1}((y_j-1)(n-\mathrm{cyc}_{\geq j}(P)))}. \quad (12.33)$$

An obvious approach to prove Conjecture 12.1 is to give a combinatorial proof that the recursion of $H^p_{k,n}(B,q,y_0,\ldots,y_{p-1})$ in (12.32) holds. We were not able to find a natural way to partition the rook placements in $\mathscr{N}_k(B)$ to account for the two terms on the right-hand side of (12.32). Our next example will show that while we can verify the recursion holds for $B = F(p,2p,3p,4p) \subset [4] \times [4p]$, the way that we can divide the partition the rook placements in $B$ to account for the two terms on the right-hand side of (12.32) is quite complicated. Thus we do not see how the recursion can be derived naturally by extending the rook placement corresponding to the permutations of $n-1$ numbers.

*Example 12.3.* We consider a staircase board $B = F(p,2p,3p,4p) \subset [4] \times [4p]$. Then $B - h_4 - d_4 = F(p,2p,3p)$ and the recursion (12.32) when $k=1$ is

$$\begin{aligned} H^p_{1,4}(B,q,y_0,\ldots,y_{p-1}) &= [y_0 + \cdots + y_{p-1} + p]_q H^p_{1,3}(B - h_4 - d_4, q, y_0, \ldots, y_{p-1}) \\ &\quad + q^{y_0+\cdots+y_{p-1}} [p]_q [3]_{q^p} H^p_{0,3}(B - h_4 - d_4, q, y_0, \ldots, y_{p-1}). \quad (12.34) \end{aligned}$$

For a rook placement $P \in \mathscr{H}_{n-k,n}(B)$, let

$$wt(P) = \left( \prod_{j=0}^{p-1} [y_j]_q^{\mathrm{cyc}_j(P)} \right) q^{s_{B,b}(P) + \sum_{i=1}^n \tilde{E}_i(P) + \sum_{j=0}^{p-1}[(y_j-1)(n-\mathrm{cyc}_{\geq j}(P))]}.$$

Then for $\sigma = (1)(2)(3) \in S_3$,

$$\begin{array}{|c|c|c|} \hline \bullet & \bullet & \mathrm{X} \\ \hline \bullet & \mathrm{X} & \bullet \\ \hline \mathrm{X} & \bullet & \bullet \\ \hline \end{array}$$

$$H^p_{0,3}(B - h_4 - d_4, q, y_0, \ldots, y_{p-1}) = \sum_{P \in C_p \wr \sigma} wt(P) = [y_0 + \cdots + y_{p-1}]_q^3.$$

This can be extended to a placement in $\mathscr{H}_{3,4}(B)$ as follows.

$$\begin{array}{|c|c|c|c|}\hline \text{X} & \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \text{X} & \bullet \\ \hline \bullet & \text{X} & \bullet & \bullet \\ \hline & \bullet & \bullet & \text{X} \\ \hline \end{array} \quad \sigma_1 = (14)(2)(3), \sum_{P \in C_p \wr \sigma_1} wt(P) = [y_0 + \cdots + y_{p-1}]_q^3 q^{y_0+\cdots+y_{p-1}} [p]_q, \tag{12.35}$$

$$\begin{array}{|c|c|c|c|}\hline \bullet & \text{X} & \bullet & \bullet \\ \hline \bullet & \bullet & \text{X} & \bullet \\ \hline \bullet & & \bullet & \text{X} \\ \hline \text{X} & \bullet & \bullet & \bullet \\ \hline \end{array} \quad \sigma_2 = (1)(24)(3), \sum_{P \in C_p \wr \sigma_2} wt(P) = [y_0 + \cdots + y_{p-1}]_q^3 q^{y_0+\cdots+y_{p-1}} [p]_q, \tag{12.36}$$

$$\begin{array}{|c|c|c|c|}\hline \bullet & \bullet & \text{X} & \bullet \\ \hline \bullet & \bullet & & \text{X} \\ \hline \bullet & \text{X} & \bullet & \bullet \\ \hline \text{X} & \bullet & \bullet & \bullet \\ \hline \end{array} \quad \sigma_3 = (1)(2)(34), \sum_{P \in C_p \wr \sigma_3} wt(P) = [y_0 + \cdots + y_{p-1}]_q^3 q^{y_0+\cdots+y_{p-1}} [p]_q. \tag{12.37}$$

There are four permutations in $S_3$ which can be considered for $H_{1,3}^p(B - h_4 - d_4, q, y_0, \ldots, y_{p-1})$ and they can be extended to a placement in $\mathscr{H}_{3,4}$ as follows.

$$\begin{array}{|c|c|c|}\hline \bullet & \text{X} & \bullet \\ \hline \bullet & & \text{X} \\ \hline \text{X} & \bullet & \bullet \\ \hline \end{array} \quad \alpha = (1)(23), \sum_{C_p \wr \alpha} wt(P) = [y_0 + \cdots + y_{p-1}]_q^2 q^{y_0+\cdots+y_{p-1}} [p]_q$$

$\Rightarrow$

$$\begin{array}{|c|c|c|c|}\hline \bullet & & \bullet & \text{X} \\ \hline \bullet & \text{X} & \bullet & \bullet \\ \hline \bullet & & \text{X} & \bullet \\ \hline \text{X} & \bullet & \bullet & \bullet \\ \hline \end{array} \quad \alpha_1 = (1)(23)(4), \sum_{C_p \wr \alpha_1} wt(P) = [y_0 + \cdots + y_{p-1}]_q^3 q^{y_0+\cdots+y_{p-1}+p} [p]_q, \tag{12.38}$$

$$\begin{array}{|c|c|c|c|}\hline \bullet & \text{X} & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet & \text{X} \\ \hline \bullet & & \text{X} & \bullet \\ \hline \text{X} & \bullet & \bullet & \bullet \\ \hline \end{array} \quad \alpha_2 = (1)(243), \sum_{C_p \wr \alpha_2} wt(P) = [y_0 + \cdots + y_{p-1}]_q^2 q^{2(y_0+\cdots+y_{p-1})} [p]_q^2, \tag{12.39}$$

$$\begin{array}{|c|c|c|}\hline \text{X} & \bullet & \bullet \\ \hline \bullet & \text{X} & \bullet \\ \hline & \bullet & \text{X} \\ \hline \end{array} \quad \beta = (13)(2), \sum_{C_p \wr \beta} wt(P) = [y_0 + \cdots + y_{p-1}]_q^2 q^{y_0+\cdots+y_{p-1}} [p]_q$$

$\Rightarrow$

$$\begin{array}{|c|c|c|c|}\hline & \bullet & \bullet & X \\ \hline X & \bullet & \bullet & \bullet \\ \hline \bullet & X & \bullet & \bullet \\ \hline & \bullet & X & \bullet \\ \hline \end{array}$$

$$\beta_1 = (13)(2)(4), \sum_{C_p \wr \beta_1} wt(P) = [y_0 + \cdots + y_{p-1}]_q^3 q^{y_0 + \cdots + y_{p-1} + p}[p]_q, \tag{12.40}$$

$$\begin{array}{|c|c|c|c|}\hline X & \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet & X \\ \hline \bullet & X & \bullet & \bullet \\ \hline & \bullet & X & \bullet \\ \hline \end{array}$$

$$\beta_2 = (143)(2), \sum_{C_p \wr \beta_2} wt(P) = [y_0 + \cdots + y_{p-1}]_q^2 q^{2(y_0 + \cdots + y_{p-1})}[p]_q^2, \tag{12.41}$$

$$\begin{array}{|c|c|c|}\hline X & \bullet & \bullet \\ \hline \bullet & \bullet & X \\ \hline & X & \bullet \\ \hline \end{array}$$

$$\gamma = (132), \sum_{C_p \wr \gamma} wt(P) = [y_0 + \cdots + y_{p-1}]_q^2 q^{2(y_0 + \cdots + y_{p-1})}[p]_q^2$$

$\Rightarrow$

$$\begin{array}{|c|c|c|c|}\hline & \bullet & \bullet & X \\ \hline X & \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & X & \bullet \\ \hline & X & \bullet & \bullet \\ \hline \end{array}$$

$$\gamma_1 = (132)(4), \sum_{C_p \wr \gamma_1} wt(P) = [y_0 + \cdots + y_{p-1}]_q^2 q^{2(y_0 + \cdots + y_{p-1}) + p}[p]_q^2, \tag{12.42}$$

$$\begin{array}{|c|c|c|c|}\hline X & \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet & X \\ \hline \bullet & \bullet & X & \bullet \\ \hline & X & \bullet & \bullet \\ \hline \end{array}$$

$$\gamma_2 = (1432), \sum_{C_p \wr \gamma_2} wt(P) = [y_0 + \cdots + y_{p-1}]_q q^{3(y_0 + \cdots + y_{p-1})}[p]_q^3, \tag{12.43}$$

$$\begin{array}{|c|c|c|}\hline & \bullet & X \\ \hline X & \bullet & \bullet \\ \hline & X & \bullet \\ \hline \end{array}$$

$$\delta = (12)(3), \sum_{C_p \wr \delta} wt(P) = [y_0 + \cdots + y_{p-1}]_q^2 q^{y_0 + \cdots + y_{p-1} + p}[p]_q$$

$\Rightarrow$

$$\begin{array}{|c|c|c|c|}\hline & \bullet & \bullet & \mathrm{X} \\ \hline & \bullet & \mathrm{X} & \bullet \\ \hline \mathrm{X} & \bullet & \bullet & \bullet \\ \hline & \mathrm{X} & \bullet & \bullet \\ \hline \end{array} \quad \delta_1 = (12)(3)(4),\ \sum_{C_p \wr \delta_1} wt(P) = [y_0 + \cdots + y_{p-1}]_q^3 q^{y_0+\cdots+y_{p-1}+2p}[p]_q, \tag{12.44}$$

$$\begin{array}{|c|c|c|c|}\hline \mathrm{X} & \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \mathrm{X} & \bullet \\ \hline \bullet & \bullet & \bullet & \mathrm{X} \\ \hline & \mathrm{X} & \bullet & \bullet \\ \hline \end{array} \quad \delta_2 = (142)(3),\ \sum_{C_p \wr \delta_2} wt(P) = [y_0 + \cdots + y_{p-1}]_q^2 q^{2(y_0+\cdots+y_{p-1})}[p]_q^2. \tag{12.45}$$

$(12.35) + (12.40) + (12.44)$ has a common factor $q^{y_0+\cdots+y_{p-1}}[p]_q[3]_{q^p}$ which is the coefficient of $H^p_{0,3}(B - h_4 - d_4, q, y_0, \ldots, y_{p-1})$ in (12.34) and the rest makes $H^p_{0,3}(B - h_4 - d_4, q, y_0, \ldots, y_{p-1})$.

$$\begin{aligned}(12.35)+(12.40)+(12.44) &= [y_0 + \cdots + y_{p-1}]_q^3 q^{y_0+\cdots+y_{p-1}}[p]_q(1+q^p+q^{2p}) \\ &= q^{y_0+\cdots+y_{p-1}}[y_0 + \cdots + y_{p-1}]_q^3[p]_q[3]_{q^p} \\ &= q^{y_0+\cdots+y_{p-1}}[p]_q[3]_{q^p}H^p_{0,3}(B - h_4 - d_4, q, y_0, \ldots, y_{p-1}),\end{aligned}$$

Similarly, $((12.36) + (12.45))$, $((12.37) + (12.41))$, $((12.39) + (12.43))$ and $((12.38) + (12.42))$ have a common factor $[y_0 + \cdots + y_{p-1} + p]_q$ which is the coefficient of $H^p_{1,3}(B - h_4 - d_4, q, y_0, \ldots, y_{p-1})$ in (12.34).

$$\begin{aligned}&((12.36)+(12.45))+((12.37)+(12.41)) \\ &\qquad\qquad\qquad +((12.39)+(12.43))+((12.38)+(12.42)) \\ &= \big([y_0+\cdots+y_{p-1}]_q^2 q^{y_0+\cdots+y_{p-1}}[p]_q([y_0+\cdots+y_{p-1}]_q + q^{y_0+\cdots+y_{p-1}}[p]_q)\big) \\ &\quad + \big([y_0+\cdots+y_{p-1}]_q^2 q^{y_0+\cdots+y_{p-1}}[p]_q([y_0+\cdots+y_{p-1}]_q + q^{y_0+\cdots+y_{p-1}}[p]_q)\big) \\ &\quad + \Big([y_0+\cdots+y_{p-1}]_q q^{2(y_0+\cdots+y_{p-1})}[p]_q^2([y_0+\cdots+y_{p-1}]_q + q^{y_0+\cdots+y_{p-1}}[p]_q)\Big) \\ &\quad + \big([y_0+\cdots+y_{p-1}]_q^2 q^{y_0+\cdots+y_{p-1}+p}[p]_q([y_0+\cdots+y_{p-1}]_q + q^{y_0+\cdots+y_{p-1}}[p]_q)\big) \\ &= ([y_0+\cdots+y_{p-1}]_q + q^{y_0+\cdots+y_{p-1}}[p]_q) \\ &\qquad \times \big\{[y_0+\cdots+y_{p-1}]_q q^{y_0+\cdots+y_{p-1}}[p]([y_0+\cdots+y_{p-1}]_q + q^{y_0+\cdots+y_{p-1}}[p]) \\ &\qquad\quad + [y_0+\cdots+y_{p-1}]_q^2 q^{y_0+\cdots+y_{p-1}}[p](1+q^p)\big\} \\ &= [y_0+\cdots+y_{p-1}+p]_q q^{y_0+\cdots+y_{p-1}}[p]_q \\ &\quad \times [y_0+\cdots+y_{p-1}]_q\,([y_0+\cdots+y_{p-1}+p]_q + [2]_q[y_0+\cdots+y_{p-1}]_q) \\ &= [y_0+\cdots+y_{p-1}+p]_q H^p_{1,3}(B - h_4 - d_4, q, y_0, \ldots, y_{p-1}).\end{aligned}$$

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