

# Some conjectures on the zeros of approximates to the Riemann $\Xi$ -function and incomplete gamma functions

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## Abstract

Riemann conjectured that all the zeros of the Riemann  $\Xi$ -function are real, which is now known as the Riemann Hypothesis (RH). In this article we introduce the study of the zeros of the truncated sums  $\Xi_N(z)$  in Riemann's uniformly convergent infinite series expansion of  $\Xi(z)$  involving incomplete gamma functions. We conjecture that when the zeros of  $\Xi_N(z)$  in the first quadrant of the complex plane are listed by increasing real part, their imaginary parts are monotone nondecreasing. We show how this conjecture implies the RH, and discuss some computational evidence for this and other related conjectures.

## 1 Introduction

Following Riemann (as described in a copy of an English translation of his memoir contained in the appendix of [Edw01]), let

$$\Xi(z) = \frac{1}{2} \left(\frac{1}{2} + iz\right) \left(-\frac{1}{2} + iz\right) \pi^{-\frac{1}{2}(\frac{1}{2} + iz)} \Gamma\left(\frac{1}{2} \left(\frac{1}{2} + iz\right)\right) \zeta\left(\frac{1}{2} + iz\right). \quad (1)$$

$\Xi(z)$  is an even, entire function, and the famous Riemann Hypothesis (RH) says that all the zeros of  $\Xi$  are real. Let  $Q$  denote the first quadrant of the complex plane  $\Re(z) \geq 0, \Im(z) \geq 0$ . Since  $\Xi(z)$  is even and real on the real line, we can restate the RH as saying all zeros of  $\Xi(z)$  in  $Q$  are real. Since  $\zeta(s)$  is nonzero in  $\Re(s) > 1$ , it follows that all zeros of  $\Xi$  in  $Q$  satisfy  $\Im(z) \leq \frac{1}{2}$ . In 1914 Hardy (as reprinted in [BCRW08]) showed that  $\Xi(z)$  has infinitely many real zeros and in 1942 Selberg [Sel42] showed that a positive proportion of the zeros of  $\Xi(z)$  are real. More recent work of Conrey [Con89] has at least  $2/5$  of the zeros on the real line.

Riemann derived the following expression for  $\Xi(z)$ ;

$$\Xi(z) = \int_0^\infty \cos(zt) \phi(t) dt, \quad (2)$$

where  $\phi(t) = \sum_{n=1}^\infty \phi_n(t)$  with

$$\phi_n(t) = \exp(-n^2 \pi \exp(2t)) (8\pi^2 n^4 \exp(\frac{9}{2}t) - 12\pi n^2 \exp(\frac{5}{2}t)). \quad (3)$$

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The function  $\phi(t)$  is known to be an even function of  $t$ . Pólya [Pól26] investigated ways of approximating  $\phi(t)$  by simpler functions. He showed that if in (2) we replace  $\phi(t)$  by

$$\tilde{\phi}_1(t) = \exp(-\pi \cosh(2t))(8\pi^2 \cosh(\frac{9}{2}t) - 12\pi \cosh(\frac{5}{2}t)) \quad (4)$$

(obtained by replacing most of the exponentials in the definition of  $\phi_1(t)$  by hyperbolic cosines), then the resulting integral has only real zeros. Pólya also showed that if we replace  $\phi(t)$  by any function which is not an even function of  $t$ , then the resulting integral has only finitely many real zeros. Hejhal [Hej90] showed that if we replace  $\phi(t)$  in (2) by a ‘‘Pólya approximate’’, i.e. a finite sum of the form

$$\sum_{n=1}^N \exp(-n^2\pi \cosh(2t))(8\pi^2 n^4 \cosh(\frac{9}{2}t) - 12\pi n^2 \cosh(\frac{5}{2}t)), \quad (5)$$

then the resulting function asymptotically has 100% of its zeros on the real line (but, for  $N > 1$ , infinitely many zeros off the line). By 100% asymptotically we mean that the proportion of zeros in  $Q$  satisfying  $\Re(z) \leq m$  that are on the real line approaches 1 as  $m \rightarrow \infty$ .

The starting point for this investigation is the idea that perhaps it is not necessary for worthwhile approximates to have all their zeros on the real line. If a given family of approximates approach  $\Xi(z)$  uniformly, and if for each element in the family one could prove that within a certain sub-region of  $Q$  all the zeros are real, with the size of the sub-region expanding to eventually include all of  $Q$  as our approximates approach  $\Xi$ , then this would also imply RH. Thus it may be worth studying replacements for  $\phi(t)$  in (2) which are not even. With this in mind, a natural question to ask is what happens if we replace  $\phi(t)$  by  $\sum_{n=1}^N \phi_n(t)$ .

## 2 Preliminary calculations

Let  $G(z; a, b)$  denote the integral

$$G(z; a, b) = 4 \int_0^\infty \cos(2zu) \exp(2bu - a \exp(2u)) du, \quad (6)$$

where  $z \in \mathbb{C}$ ,  $a, b \in \mathbb{R}$  with  $a > 0$ . Making the change of variable  $t = a \exp(2u)$ , so  $dt = a \exp(2u) 2 du$ , and  $du = dt/2t$ , we get

$$G(z; a, b) = 4 \int_a^\infty \exp(b \log(t/a) - t) \cos(z \log(t/a)) \frac{dt}{2t} \quad (7)$$

$$= \int_a^\infty (t/a)^b \exp(-t) (\exp(iz \log(t/a)) + \exp(-iz \log(t/a))) \frac{dt}{t} \quad (8)$$

$$= \int_a^\infty \exp(-t) \left( (t/a)^{b+iz} + (t/a)^{b-iz} \right) \frac{dt}{t} \quad (9)$$

$$= \frac{\Gamma(b+iz, a)}{a^{b+iz}} + \frac{\Gamma(b-iz, a)}{a^{b-iz}}, \quad (10)$$

where

$$\Gamma(z, a) = \int_a^\infty \exp(-t) t^z \frac{dt}{t} \quad (11)$$

is the (upper) incomplete gamma function. For lack of a better name, we will refer to  $G(z; a, b)$  as a “hyperbolic gamma function”.

From (2) and (7) we have

$$\Xi(z) = \sum_{n=1}^{\infty} 2\pi^2 n^4 G(z/2; n^2\pi, 9/4) - 3\pi n^2 G(z/2; n^2\pi, 5/4), \quad (12)$$

the interchange in integration and summation being justified by the uniform convergence. For  $a \in \mathbb{R}$ ,  $a > 0$ , the function  $\Gamma(z, a)$  is entire (as a function of  $z$ ), and hence so is  $G(z; a, b)$ . There is a routine in Maple to compute  $\Gamma(z, a)$ , and (in the RootFinding package) a routine to compute the zeros, using the argument principle and Newton’s method, of a given analytic function in any rectangle of the complex plane. Using this, the author made several computations to compute the zeros of the  $\Xi$ -approximates

$$\Xi_N(z) := \sum_{n=1}^N \Phi_n(z), \quad (13)$$

where

$$\Phi_n(z) := 2\pi^2 n^4 G(z/2; n^2\pi, 9/4) - 3\pi n^2 G(z/2; n^2\pi, 5/4), \quad (14)$$

for various small values of  $N$ . Lists of zeros for some of these are contained in the Appendix. In these computer runs, the parameter “Digits” in Maple (which tells the computer to use this many significant digits in all calculations) was typically set to  $20N - 10$  or so, whatever number of digits was needed to compute the function in question over the specified rectangle accurately to 20 or so significant digits. After runs were made first with Digits equal to  $20N - 10$ , they were sometimes run again with Digits equal to  $20N$ , and the resulting zeros typically agreed to at least 16 decimal digits or so, which the author has taken to mean the computer generated zeros (for Digits equal to  $20N - 10$ ) agree with the actual ones to at least 10 decimal digits, although no attempt has been made to establish rigorous error bounds.

We say that a given function  $F(z)$  has *monotonic zeros* in a region  $D$  of the complex plane if, when we list the zeros of  $F$  in  $D$  by increasing real part, the imaginary parts of the zeros are monotone nondecreasing. Formally, if  $\{\alpha_1, \alpha_2, \dots\}$  are the zeros of  $F$  in  $D$  numbered so that  $\Re(\alpha_i) < \Re(\alpha_{i+1})$  for  $i \geq 1$ , then  $\Im(\alpha_i) \leq \Im(\alpha_{i+1})$ . (We assume  $F$  has at most one zero on the intersection of any vertical line with  $D$ .) The data in the Appendix and other computer runs support the following hypothesis.

**Conjecture 1** For  $N \in \mathbb{N}$ ,  $\Xi_N(z)$  has monotonic zeros in  $Q$ .

**Proposition 1** Conjecture 1 implies the Riemann Hypothesis.

*Proof.* This follows from the argument principle, combined with the simple fact that a function with infinitely many positive real zeros, and with monotonic zeros in  $Q$ , has only real zeros in  $Q$ . Assume the RH is false, and let  $\tau$  be the zero of  $\Xi(z)$  in  $Q$  with minimal real part, among those zeros with positive imaginary part, and let  $\tau = \sigma + it$ . By the argument principle, we have

$$\frac{1}{2\pi i} \oint_{C_\epsilon} \frac{\Xi'(z)}{\Xi(z)} dz = 1, \quad (15)$$

where the integral is taken counterclockwise around a circle  $C_\epsilon$  centered at  $\tau$ , of small radius  $\epsilon$ , so no other zeros of  $\Xi(z)$  are enclosed in  $C$ . Next choose  $N$  sufficiently large so that  $\Xi_N(z)$  has a real zero  $\gamma$  with  $\gamma > 2\sigma$ . We can do this since  $\Xi_N(z)$  converges uniformly to  $\Xi(z)$  on compacta in  $Q$ , both  $\Xi(z)$  and  $\Xi_N(z)$  are real on the real line, and since  $\Xi(z)$  has infinitely many positive real zeros. By assumption  $\Xi_N(z)$  has monotonic zeros in  $Q$ , hence has no non-real zeros in  $Q$  with real part less than  $2\sigma$ . This implies

$$\frac{1}{2\pi i} \oint_{C_\epsilon} \frac{\Xi'_N(z)}{\Xi_N(z)} dz = 0, \quad (16)$$

and so

$$1 = \frac{1}{2\pi i} \oint_{C_\epsilon} \frac{\Xi'(z)}{\Xi(z)} - \frac{\Xi'_N(z)}{\Xi_N(z)} dz \quad (17)$$

$$= \frac{1}{2\pi i} \oint_{C_\epsilon} \frac{\Xi'(z)\Xi_N(z) - \Xi'_N(z)\Xi(z)}{\Xi(z)\Xi_N(z)} dz. \quad (18)$$

On the closed and bounded set  $C_\epsilon$ ,  $|\Xi(z)|$  is nonzero and hence must assume an absolute minimum  $\delta > 0$ . Due to the uniform convergence, as  $N \rightarrow \infty$ , the minimum of  $|\Xi_N(z)|$  on  $C_\epsilon$  must eventually be greater than  $\delta/2$ . Hence for large  $N$  the modulus of the denominator of the integrand in (18) is bounded away from zero, but (since  $\Xi'_N(z)$  approaches  $\Xi'(z)$  uniformly) the numerator approaches zero, and so the integral will also approach zero, a contradiction.  $\square$

**Remark 1** *A weaker form of Conjecture 1, which still implies RH, is that there are no non-real zeros of  $\Xi_N(z)$  in  $Q$  whose real part is less than the largest real zero of  $\Xi_N(z)$ . Since  $\Xi_N(z)$  is real for real  $z$ , the real zeros can be found by looking at sign changes along the real line. Then the argument principle can be used via a numerical integration to obtain the total number of zeros of  $\Xi_N(z)$  with real part not greater than the largest real zero, and matched against the number of real zeros. Computations along these lines indicate this weaker form of Conjecture 1 is true at least for  $N \leq 10$ . Below we list the largest real zero and number of real zeros of  $\Xi_N(z)$  for  $N \leq 10$ .*

<u><math>N</math></u>	<u>largest real zero</u>	<u>number of real zeros</u>
1	14.0454395788	1
2	39.5324810798	7
3	65.0320737720	15
4	103.3679880094	31
5	149.0026994921	53
6	197.9575955732	79
7	258.5304836632	113
8	327.3794646017	155
9	406.8174206801	207
10	489.3900649445	263

### 3 Other Zeta Functions

Many other zeta functions which are conjectured to satisfy a Riemann Hypothesis can be approximated by sums of hyperbolic gamma functions. Let  $\tau(n)$  denote Ramanujan's  $\tau$ -function. Since the function

$$F(z) = \sum_{n=1}^{\infty} \tau(n) \exp(2\pi inz) \quad (19)$$

is a modular form of weight 12, we have

$$F(ix) = x^{-12} F(i/x), \quad (20)$$

which can be used to show

$$(2\pi)^{-s} \Gamma(s) \sum_{n=1}^{\infty} \frac{\tau(n)}{n^s} = \sum_{n=1}^{\infty} \tau(n) \left( \int_1^{\infty} x^{s-1} \exp(-2\pi nx) dx + \int_1^{\infty} x^{-s-1} x^{12} \exp(-2\pi nx) dx \right) \quad (21)$$

which implies

$$(2\pi)^{-6-iz} \Gamma(6+iz) \sum_{n=1}^{\infty} \frac{\tau(n)}{n^{6+iz}} = \sum_{n=1}^{\infty} \tau(n) G(z; 2\pi n, 6), \quad (22)$$

again a uniformly convergent sum of hyperbolic gamma functions. (It is known that  $|\tau(n)| = O(n^6)$  [Apo90].) The function defined by the left-hand-side of (22) is known as the Ramanujan  $\Xi$ -function, which we denote  $\Xi_{\Delta}(z)$ , and the approximate obtained by truncating the series on the right-hand-side of (22) after  $N$  steps we denote  $\Xi_{\Delta,N}(z)$ . Ramanujan conjectured that  $\Xi_{\Delta}(z)$  has only real zeros, which is still open. We mention that Ki [Ki08] has studied the zeros of different approximates to the Ramanujan  $\Xi$ -function.

**Conjecture 2** For  $N \in \mathbb{N}$ ,  $\Xi_{\Delta,N}(z)$  has monotonic zeros in  $Q$ .

Computations analogous to those described in Remark 1 indicate that the corresponding weaker form of Conjecture 2, that there are no nonreal zeros of  $\Xi_{\Delta,N}(z)$  in  $Q$  with real part smaller than the largest real zero of  $\Xi_{\Delta,N}(z)$ , is true at least for  $N \leq 10$ . Here is a small table of the number of real zeros and the largest real zero of  $\Xi_{\Delta,N}(z)$  for  $N \leq 10$ .

$N$	largest real zero	number of real zeros
1		0
2	9.1937689444922	1
3	13.885647964708	2
4	21.358047646119	5
5	25.047323063922	6
6	28.706422677689	8
7	33.529929734593	11
8	36.535376767485	12
9	40.190608700694	14
10	44.761812314903	17

More generally, we can start with any entire modular cusp form

$$f(z) = \sum_{n=1}^{\infty} c(n) \exp(2\pi iz) \quad (23)$$

of weight  $2k$ , and set

$$B(z) = (2\pi)^{-k-iz} \Gamma(k+iz) \sum_{n=1}^{\infty} \frac{c(n)}{n^{k+iz}}. \quad (24)$$

The modularity of  $f$  can be used to show [Apo90, pp. 137-138] that

$$B(z) = \int_1^{\infty} f(it) \left( t^{k+iz} + (-1)^k t^{k-iz} \right) dt/t \quad (25)$$

$$= \sum_{n=1}^{\infty} c(n) \int_{2\pi n}^{\infty} \exp(-u) \left( (u/2\pi n)^{k+iz} + (-1)^k (u/2\pi n)^{k-iz} \right) \frac{du}{u}, \quad (26)$$

where we have used the well-known bound  $c(n) = O(n^k)$  to justify the interchange of summation and integration. Thus we see that, at least for  $k$  even,  $B(z)$  is also a uniformly convergent (infinite) linear combination of hyperbolic gamma functions.

We can do a similar calculation for Dirichlet L-series  $L(s, \chi)$  with  $\chi$  a primitive character with modulus  $q$ . Assume for the moment that  $\chi(-1) = 1$ , and define

$$\Xi(z, \chi) = \pi^{-(1/4+iz/2)} \Gamma(1/4 + iz/2) q^{1/4+iz/2} L(1/2 + iz, \chi). \quad (27)$$

From [Dav00, pp.68-69] we have (using the fact that  $\chi(n) = \chi(-n)$ )

$$\Xi(z, \chi) = \sum_{n=1}^{\infty} \chi(n) \int_1^{\infty} \exp(-n^2 \pi t/q) t^{1/4+iz/2} \frac{dt}{t} \quad (28)$$

$$+ \sqrt{q} w(\chi) \sum_{n=1}^{\infty} \bar{\chi}(n) \int_1^{\infty} \exp(-n^2 \pi t/q) t^{1/4-iz/2} \frac{dt}{t} \quad (29)$$

$$= \sum_{n=1}^{\infty} \chi(n) \frac{\Gamma(1/4 + iz/2, \pi n^2/q)}{(\pi n^2/q)^{1/4+iz/2}} + \quad (30)$$

$$\sqrt{q} w(\chi) \sum_{n=1}^{\infty} \bar{\chi}(n) \frac{\Gamma(1/4 - iz/2, \pi n^2/q)}{(\pi n^2/q)^{1/4-iz/2}},$$

where  $\sqrt{q}w(\chi)$  is a certain complex number of modulus one and  $\bar{\chi}(n) = \overline{\chi(n)}$  is the conjugate character. Furthermore if  $\chi(-1) = -1$  we get

$$\pi^{-(3/4+iz/2)}\Gamma(3/4+iz/2)q^{3/4+iz/2}L(1/2+iz,\chi) \quad (31)$$

$$= \sum_{n=1}^{\infty} n\chi(n) \int_1^{\infty} \exp(-n^2\pi t/q)t^{3/4+iz/2} \frac{dt}{t} + \quad (32)$$

$$i\sqrt{q}w(\chi) \sum_{n=1}^{\infty} n\bar{\chi}(n) \int_1^{\infty} \exp(-n^2\pi t/q)t^{3/4-iz/2} \frac{dt}{t} \quad (33)$$

$$= \sum_{n=1}^{\infty} n\chi(n) \frac{\Gamma(3/4+iz/2, \pi n^2/q)}{(\pi n^2/q)^{3/4+iz/2}} + \quad (34)$$

$$i\sqrt{q}w(\chi) \sum_{n=1}^{\infty} n\bar{\chi}(n) \frac{\Gamma(3/4-iz/2, \pi n^2/q)}{(\pi n^2/q)^{3/4-iz/2}},$$

where again  $|i\sqrt{q}w(\chi)| = 1$ .

Let  $F(z)$  be the function defined by truncating the series on the right-hand-side of (26), or both of the series on the right-hand-side of either (30) or (34), after  $N$  steps. If Conjectures 1 and 2 are true, one might suspect that  $F(z)$  also has monotonic zeros in  $Q$ , although the author has not yet done any computations with these more general sums.

Another interesting question is where the zeros of  $\Gamma(z, a)$  are, for  $a$  a positive real number. Neilsen [Nie65] showed that  $\Gamma(z, a)$  has no zeros in  $\Re(z) < a$ , and Gronwall [Gro16] proved that  $\Gamma(z, a)$  has infinitely many zeros in  $Q$ . Mahler [Mah30] showed that, as  $a \rightarrow \infty$ , the zeros of  $\Gamma(az, a)$  cluster about the limiting curve

$$\Re(z \log z + 1 - z) = 0. \quad (35)$$

Tricomi and other authors have investigated the zeros of (the meromorphic continuation of)  $\Gamma(z, a)$  as a function of  $a$ , for fixed  $z$ . In summary, not much information seems to be known about the zeros of  $\Gamma(z, a)$ , for  $a$  a fixed positive real number (although the literature contains a number of detailed results on the zeros of the lower incomplete gamma function  $\Gamma(z) - \Gamma(z, a)$ ). In 1998 Gautschi [Gau98] published a nice survey of known results on incomplete gamma functions.

Computer calculations support the following.

**Conjecture 3** *For any fixed positive real number  $a$ , the incomplete gamma function  $\Gamma(z, a)$  has monotonic zeros in  $Q$ .*

Although some analog of Conjecture 3 may be true for hyperbolic gamma functions, in Section 5 we show that there exist some choices of  $a, b \in \mathbb{R}$ ,  $a > 0$  for which  $G(z; a, b)$  does not have monotonic zeros in  $Q$ .

**Remark 2** *To say a function has monotonic zeros in  $Q$  is equivalent to saying that the first differences of the imaginary parts of the zeros are all nonnegative. The zeros in  $Q$  of  $\Gamma(z, a)$ ,  $a > 0$  seem to satisfy the more general property that the  $k$ -th differences of the imaginary parts of the zeros are positive for  $k$  odd and negative for  $k$  even, for  $k \leq 7$  or 8. Thus these zero sets seem to have extra structure beyond being monotonic. For linear combinations of hyperbolic gamma functions the same phenomena seems to occur for  $x$  sufficiently large, which may be due to the main term in the asymptotics controlling the zeros.*

## 4 Asymptotics

Throughout this section  $z = x + iy \in Q$ ,  $x, y \geq 0$ ,  $\theta = \arg(z)$ ,  $0 \leq \theta \leq \pi/2$ ,  $a, b \in \mathbb{R}$ ,  $a > 0$ . We begin with Stirling's formula and some other known results:

$$\Gamma(z) = \sqrt{\frac{2\pi}{z}} \left(\frac{z}{e}\right)^z (1 + O(1/z)) \quad (36)$$

(where  $-\pi < \arg(z) \leq \pi$ )

$$\Gamma(z, a) = \Gamma(z) - \frac{a^z}{ze^a} \left(1 + \frac{a}{z+1} + \frac{a^2}{(z+1)(z+2)} + \dots + \frac{a^k}{(z+1)_k} + \dots\right) \quad (37)$$

([EMOT53, Vol. II, p. 135])

$$\frac{\Gamma(b+z)}{\Gamma(z)} = z^b (1 + O(1/z)) \quad (38)$$

([EMOT53, Vol. I, p. 47]), where in the big-Oh results we mean as  $|z| \rightarrow \infty$ . From (36) we see that

$$|\Gamma(z)| = \sqrt{\frac{2\pi}{|z|}} \exp(x \ln |z| - x - y\theta) (1 + O(1/|z|)). \quad (39)$$

Now  $\Gamma(z, a) = 0$  if and only if  $\Gamma(z, a)/\Gamma(z) = 0$  so by (37)  $\Gamma(z, a) = 0$ ,  $|z|$  large implies

$$\frac{a^z}{z\Gamma(z)e^a} \sim 1 \quad (40)$$

or

$$\exp(x \ln |z| - x - y\theta) \sqrt{2\pi|z|} \sim \exp(x \ln a - a). \quad (41)$$

If  $y$  remains bounded and  $x \rightarrow \infty$ , the left hand side above grows too fast, while if  $x$  remains bounded and  $y \rightarrow \infty$  it decreases too fast. Hence we need both  $x, y \rightarrow \infty$ . Furthermore,  $y\theta$  must be asymptotic to  $x \ln x$ , hence  $\theta \rightarrow \pi/2$  and the zeros of  $\Gamma(z, a)$  in  $Q$  satisfy  $y \sim \frac{2}{\pi} x \ln x$  as  $|z| \rightarrow \infty$ .

To perform the same analysis for  $G(z; a, b)$ , first note that

$$|\Gamma(iz)| = \sqrt{\frac{2\pi}{|z|}} \exp(-y \ln |z| + y - x(\theta + \pi/2)) (1 + O(1/|z|)) \quad (42)$$

$$|\Gamma(-iz)| = \sqrt{\frac{2\pi}{|z|}} \exp(y \ln |z| - y + x(\theta - \pi/2)) (1 + O(1/|z|)). \quad (43)$$

Thus using (38),

$$\frac{|\Gamma(b+iz)|}{|a^{b+iz}|} = \sqrt{\frac{2\pi}{|z|}} |z|^b \exp(-y \ln |z| + y - x(\theta + \pi/2) - (b-y) \ln a) (1 + O(1/|z|)) \quad (44)$$

$$\frac{|\Gamma(b-iz)|}{|a^{b-iz}|} = \sqrt{\frac{2\pi}{|z|}} |z|^b \exp(y \ln |z| - y + x(\theta - \pi/2) - (b+y) \ln a) (1 + O(1/|z|)). \quad (45)$$



From (37),

$$G(z; a, b) = \frac{\Gamma(b+iz)}{a^{b+iz}} + \frac{\Gamma(b-iz)}{a^{b-iz}} \quad (46)$$

$$- \frac{1}{(b+iz)e^a} \sum_{k=0}^{\infty} \frac{a^k}{(b+iz+1)_k} - \frac{1}{(b-iz)e^a} \sum_{k=0}^{\infty} \frac{a^k}{(b-iz+1)_k}.$$

One finds

$$- \frac{1}{(b+iz)e^a} \sum_{k=0}^{\infty} \frac{a^k}{(b+iz+1)_k} - \frac{1}{(b-iz)e^a} \sum_{k=0}^{\infty} \frac{a^k}{(b-iz+1)_k} \quad (47)$$

$$= \frac{2(a-b)}{e^a z^2} + \frac{2((b-a)^3 + 3a^2 - 3ab - a)}{e^a z^4} + O(1/z^6).$$

If  $y$  remains bounded and  $x$  doesn't, then the first two terms on the right-hand-side of (46) approach 0 like  $\exp(-x\pi/2)$ , so by (47), the expansion of  $G(z; a, b)$  in negative powers of  $z$  has a nonzero coefficient of  $z^{-2}$ , unless  $a = b$  in which case it has a nonzero coefficient of  $z^{-4}$ . In either case it cannot equal 0 for sufficiently large  $x$ . If  $x$  remains bounded and  $y$  doesn't, then the second term on the right-hand-side of (46) blows up, while the others approach 0. So as  $|z| \rightarrow \infty$ , for fixed  $a, b$ , if  $G(z; a, b) = 0$  we need both  $x, y \rightarrow \infty$ , and thus from (45)  $x(\theta - \pi/2) + y \ln |z|$  cannot approach positive or negative infinity too quickly. In fact the zeros of  $G(z; a, b)$  must satisfy  $x \sim \frac{2}{\pi} y \ln y$  as  $|z| \rightarrow \infty$ .

The argument above also applies to any function of the form

$$\sum_{k=1}^N u_k G(z; a_k, b_k) \quad (48)$$

$a_k, b_k \in \mathbb{R}, u_k \in \mathbb{C}, a_k > 0$ , i.e. any  $\mathbb{C}$ -linear combination of hyperbolic gamma functions. For if you have a linear combination of terms like (46), more than one of which is approaching  $\infty$ , the linear combination must also approach  $\infty$ , since by taking into account the contribution of  $a_k, b_k$ , no two such terms can approach  $\infty$  at the same rate. The other parts of the argument follow through similarly (the coefficient of  $1/z^{2j}$  in the appropriate version of (47) must be nonzero for some  $j$ , else (48) would essentially reduce to a linear combination of Gamma functions, which would not be entire) and thus the zeros of any function of the form (48) also satisfy  $x \sim \frac{2}{\pi} y \ln y$  as  $|z| \rightarrow \infty$ .

## 5 Linear Combinations

The examples in Section 3 involving Dirichlet characters motivate defining, for  $a, b \in \mathbb{R}, a > 0, w, \alpha \in \mathbb{C}, |w| = 1$ , the generalized hyperbolic gamma function as

$$G(z; a, b, \alpha, w) = \alpha \frac{\Gamma(b+iz, a)}{a^{b+iz}} + \bar{\alpha} \frac{\Gamma(b-iz, a)}{a^{b-iz}} w. \quad (49)$$

The author has made several hundred computer runs, calculating the zeros of various arbitrary  $\mathbb{C}$ -linear combinations of generalized hyperbolic gamma functions in different regions of  $\mathbb{Q}$ . Surprisingly, in all of these runs the zeros turned out to be monotonic. Perhaps this results from

a mysterious analytic principle, not yet understood, which causes generic sums of generalized hyperbolic gamma functions to have a high probability of having monotonic zeros. In this case the RH could be a consequence of this analytic principle, combined with Hardy's theorem that  $\Xi(z)$  has infinitely many real zeros.

It is not the case though that all linear combinations of generalized hyperbolic gamma functions have monotonic zeros in  $Q$ . For example, consider any function of the form  $t_1A(z) + t_2B(z)$ , where  $t_1, t_2$  are positive real numbers and  $A(z), B(z)$  are any two  $\Xi$ -functions, corresponding to different zeta functions from Section 3, which are real on the real line. Since  $A(z)$  and  $B(z)$  have infinitely many real zeros, as  $z \rightarrow \infty$  along the real line they each oscillate from positive to negative. Unless there is an unsuspected correlation between the two, any real linear combination of them will also oscillate and thus have infinitely many real zeros. But only for very special choices of  $A, B, t_1, t_2$  will this linear combination correspond to a zeta function with an Euler product, and without an underlying Euler product it is generally expected that such functions will have infinitely many non-real zeros as well. In particular,  $\Xi(2z) + \Xi_{\Delta,5}(z)$ , the sum of the Riemann  $\Xi$ -function and the fifth approximate to the Ramanujan  $\Xi$  function, has a non-real zero with real part between 19 and 23, and a real zero at  $z = 24.99871$ . (Here  $\Xi(z)$  is evaluated at  $2z$  to make the two functions compatible, since the expression (12) of  $\Xi(z)$  in terms of hyperbolic gamma functions involves  $z/2$ , while that of  $\Xi_{\Delta}$  involves  $z$ .)

More simply, one can create an example of non-monotonic zeros by considering what happens to  $\Xi_N(z)$  as  $z \rightarrow \infty$  along the positive real line. From (46) and (47) we get

$$G(x; a, b) = \frac{2(a-b)}{e^a} \frac{1}{x^2} + O\left(\frac{1}{x^4}\right). \quad (50)$$

Applying this to (13) yields

$$\Phi_n(x) = \frac{4n^4\pi^2(n^2\pi - 9/4) - 6n^2\pi(n^2\pi - 5/4)}{\exp(n^2\pi)} \frac{1}{x^2} + O\left(\frac{1}{x^4}\right), \quad (51)$$

which shows

$$\begin{aligned} \Phi_1(x) &= -.0197493826339/x^2 + O(1/x^4) \\ \Phi_2(x) &= .0197493413075/x^2 + O(1/x^4) \\ \Phi_3(x) &= .4132639781905 \times 10^{-7}/x^2 + O(1/x^4). \end{aligned} \quad (52)$$

It follows that  $t\Phi_1(x) + \Phi_2(x)$  is positive for sufficiently large  $x$  if  $t = 0$  and negative for sufficiently large  $x$  if  $t = 1$ , and so  $t\Phi_1(z) + \Phi_2(z)$  has a zero on the real line for some large  $x$  and some  $0 < t < 1$ . One finds in fact that there is a real zero between  $x = 10^5$  and  $x = 10^6$  when  $t = .999997907459$ , which occurs after many non-real zeros, giving an example of non-monotonic zeros. This real zero travels quickly left as  $t$  increases through real values, arriving at  $x = 39.53248$  (the largest real zero of  $\Xi_2(z)$ ) when  $t = 1$ .

Note that for  $k > 3$  the coefficient of  $1/x^2$  in  $\Phi_k(x)$  is positive. Also, by Riemann's definition of  $\Xi(z)$ , it is easy to see that  $|\Xi(x)|$  dies off exponentially as  $x$  approaches  $\infty$  along the positive real axis. (The gamma factor dies off exponentially, while the other factors grow at most polynomially - it is known that the modulus of  $\zeta(.5 + ix)$  is  $o(x)$  as  $x \rightarrow \infty$ .) Thus the coefficient of  $1/x^2$  in  $\Xi_N(x)$  approaches zero from below as  $N \rightarrow \infty$ .

One could hope that any series of the form

$$F(z) = \sum_{k=1}^N c_k \Phi_k(z), \quad c_k \in \mathbb{R}, c_k \geq 0 \quad (53)$$

has monotonic zeros, modulo the problem of the forced real zero described above, which could be avoided by say requiring  $c_1 = 1$ ,  $0 \leq c_i \leq 1$  for  $i > 1$ , since then

$$F(x) = \frac{c}{x^2} + O\left(\frac{1}{x^4}\right) \quad (54)$$

for some  $c > 0$ . A family  $\{f_1(z), \dots, f_k(z)\}$  of polynomials with real coefficients is called *compatible* if

$$\sum_{j=1}^k c_j f_j(z) \quad (55)$$

has only real zeros whenever  $c_j \in \mathbb{R}, c_j \geq 0$  for  $1 \leq j \leq k$ . It is called pairwise compatible if  $\{f_i, f_j\}$  forms a compatible family for each pair  $1 \leq i < j \leq k$ . Chudnovsky and Seymour [CS07] have shown that a family of polynomials whose members have only real zeros and positive leading coefficients is compatible if and only if it is pairwise compatible. Does a similar statement hold if we replace real polynomials with positive leading coefficients and only real zeros, by even, real, entire functions having monotonic zeros in  $Q$ ?

Eq. (50) also leads to an example of a single hyperbolic gamma function with non-monotonic zeros. For very large  $x$  clearly  $G(x; a, 2a) < 0$  and  $G(x; a, 0) > 0$ , so there must be some value of  $b$ ,  $0 < b < 2a$  for which  $G(x; a, b) = 0$  for some very large  $x$ , which will thus have non-monotonic zeros in  $Q$ .

Another interesting phenomena occurs when we consider the zeros of  $\Xi_k(z) + t \Phi_{k+1}(z)$ . As we let  $t$  vary continuously from 0 to 1, computations indicate that the imaginary parts of the non-real zeros decrease monotonically (i.e. continuously), in a very regular manner, until, for high enough  $k$ , they collide with the corresponding zero (from Schwartz reflection) in the fourth quadrant, and arrive on the real line, where they remain.

**Conjecture 4** *For  $k \geq 1$ , the imaginary part of each non-real zero of  $\Xi_k(z) + t \Phi_{k+1}(z)$  decreases monotonically (i.e. continuously) as  $t$  goes from 0 to 1.*

## 6 The modulus on vertical rays

It is known that the RH is equivalent to the statement that the modulus of  $\Xi(z)$  is monotone increasing along any vertical ray which starts at a point  $x \geq 0$  on the nonnegative real line and travels straight upward to  $x + i\infty$ . (Clearly if the RH is false the statement is false. On the other hand, if the RH is true, start with Hadamard's factorization theorem

$$|\Xi(z)| = |\Xi(0)| \prod_i |1 - z^2/\alpha_i^2|, \quad (56)$$

where the  $\alpha_i$  are the positive real zeros of  $\Xi(z)$ , and take the partial derivative with respect to  $y$ , where  $z = x + iy$ . This is easily seen to be positive for positive  $y$ .)

For a given function  $F(z)$ , analytic in  $Q$ , and  $\alpha$  a nonnegative real number, let  $M(F, \alpha)$  denote the value of  $y \geq 0$  where the function  $|F(\alpha + iy)|$  is minimal, i.e. the height where the minimum of the modulus of  $F$  occurs on the vertical ray starting at  $\alpha$  and going straight up. (If this minimum occurs at more than one  $y$ , let  $M(F, \alpha)$  denote the lim inf of such  $y$ .) Finally call the set of pairs  $\{(\alpha, M(F, \alpha))\}$  in  $Q$  the ‘‘M-curve’’ of  $F$ .

Examples of these curves are given in Figures 1, 2, and 3 below. To calculate  $M(F, \alpha)$  for a given pair  $F$  and  $\alpha$  the author simply calculated the modulus of  $F(\alpha + iy)$  at many closely spaced grid points  $y$  and then chose the  $y$  which gave the minimum of these numbers (from the asymptotics, the modulus of a sum of hyperbolic gamma functions increases quite rapidly when  $y$  increases beyond a certain point). This same procedure was followed for several closely spaced grid points  $\alpha$ , and the pairs  $(\alpha, M(F, \alpha))$  then plotted on a grid using Maple, with the result looking something like a continuous curve. They seem to have the property that the non-real zeros of  $F$  in  $Q$  occur at the same places as the local maxima of the M-curve. If so, this would give another (rather informal) method for calculating the zeros of  $\Xi_N(z)$  and other sums of hyperbolic gamma functions which doesn’t depend on the argument principle or Newton’s method.

If we make the simplifying assumption that the modulus of  $F(a + iy)$ ,  $y \geq 0$ , is monotone decreasing in  $y$  until it reaches a certain minimum and then is monotone increasing after that, the Cauchy-Riemann equations show why the local maximums of the  $M$ -curve are linked to the zeros of  $F$ . For under this assumption, setting the partial derivative of  $|F(a + iy)|^2$  with respect to  $y$  equal to 0 gives

$$uu_y + vv_y = 0, \tag{57}$$

where  $F = u(x, y) + iv(x, y)$ . Assume further that the Implicit Function Theorem holds here, which shows (57) implicitly defines  $y$  as a function of  $x$ . Now taking the derivative with respect to  $x$  of (57) we get

$$u_x u_y + uu_{yx} + (u_y u_y + uu_{yy})y' + v_x v_y + vv_{yx} + (v_y v_y + vv_{yy})y' = 0. \tag{58}$$

If  $u^2 + v^2 = 0$ , then  $u = v = 0$  and (58) implies

$$u_x u_y + v_x v_y + (u_y^2 + v_y^2)y' = 0. \tag{59}$$

The Cauchy-Riemann equations now give

$$(u_y^2 + v_y^2)y' = 0, \tag{60}$$

or  $y' = 0$ , since otherwise  $u_y = v_y = 0$  and  $F$  would be constant.

## 6.1 Some Notes on the Computations

The list of zeros of  $\Xi_1(z)$  in the Appendix was calculated using the argument principle and Newton’s method by a call to the function analytic in Maple, as were the zeros of  $\Xi_{\Delta,1}(z)$ ,  $\Xi_{\Delta,2}(z)$ , and the zeros of the sum of two generalized hyperbolic gamma functions. This procedure failed however to compute the zeros of  $\Xi_2(z)$ , as the program never finished even after running for over two days on the Sun system at the University of Pennsylvania. Note that  $\Xi_k(z)$  is a sum of  $4k$  incomplete gamma functions, which may explain why the computation of the zeros

of  $\Xi_k(z)$  quickly becomes difficult. The author was able to find the zeros of  $\Xi_2(z)$  by starting with the zeros of  $\Xi_1(z)$  and using Newton's method to find the zeros of  $\Xi_1(z) + t\Phi_2(z)$ , for  $t$  a small positive number, then recursively using these new zeros and Newton's method to find the zeros for a slightly larger value of  $t$ , slowly increasing  $t$  until  $t$  equaled 1. The author then tried to compute the zeros of  $\Xi_3(z)$  in the same way, by starting with the zeros of  $\Xi_2(z)$  and using Newton's method to compute the zeros of  $\Xi_2(z) + t\Phi_3(z)$  for small  $t$ , and gradually increasing  $t$  as before. This worked well until  $t$  became very close to 1, about  $t = .99$ , at which point Newton's method no longer converged. If the property discussed in the previous paragraph holds though, we can see from Figure 2 where the non-real zeros of  $\Xi_3(z)$  in the range  $0 \leq \Re(z) \leq 120$  are.

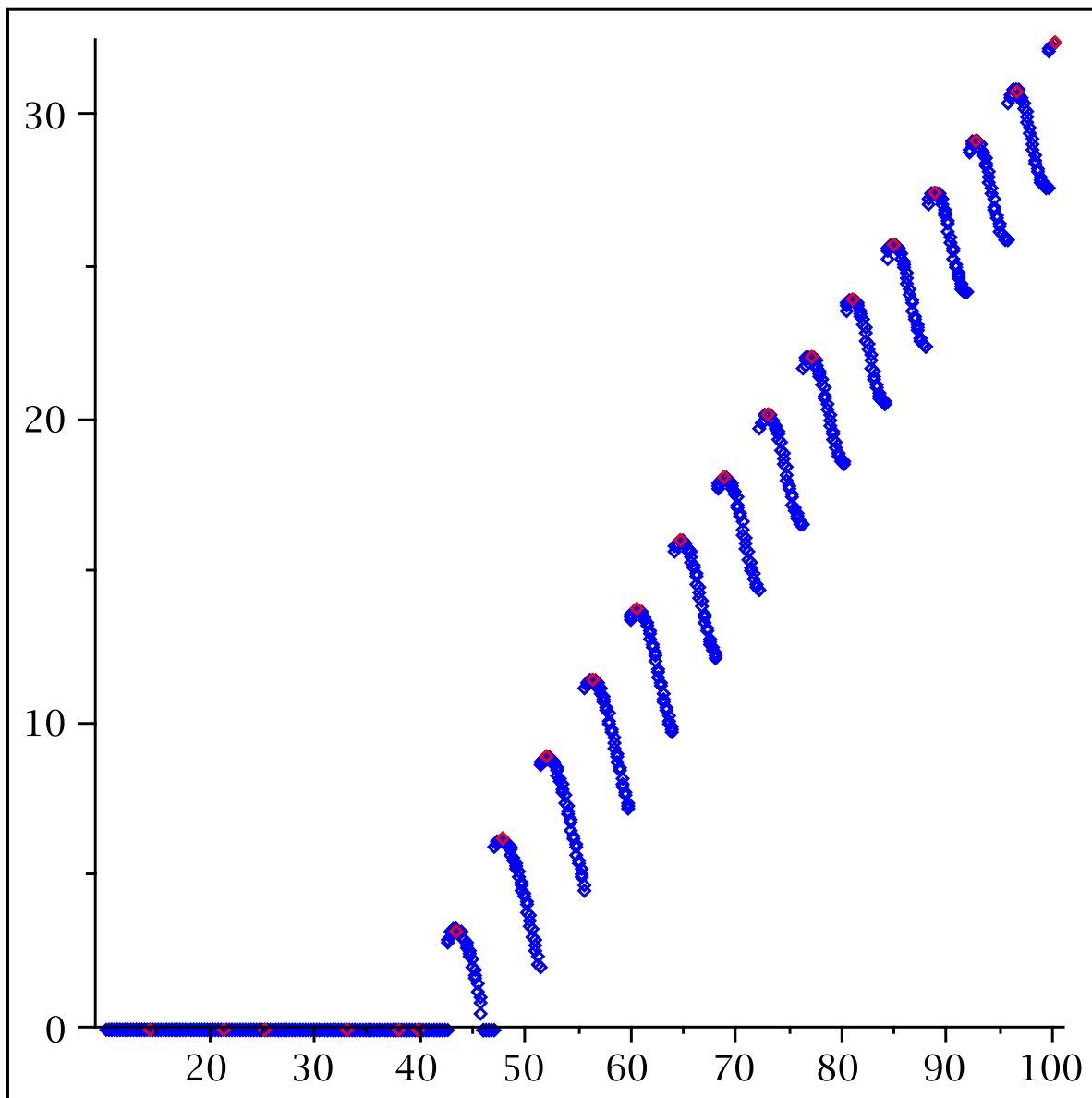


Figure 1: The M-curve (in blue) for  $\Xi_2(z)$ ,  $0 \leq \Re(z) \leq 100$ . The zeros are overlaid in red; note they occur at the local maxima of the M-curve.

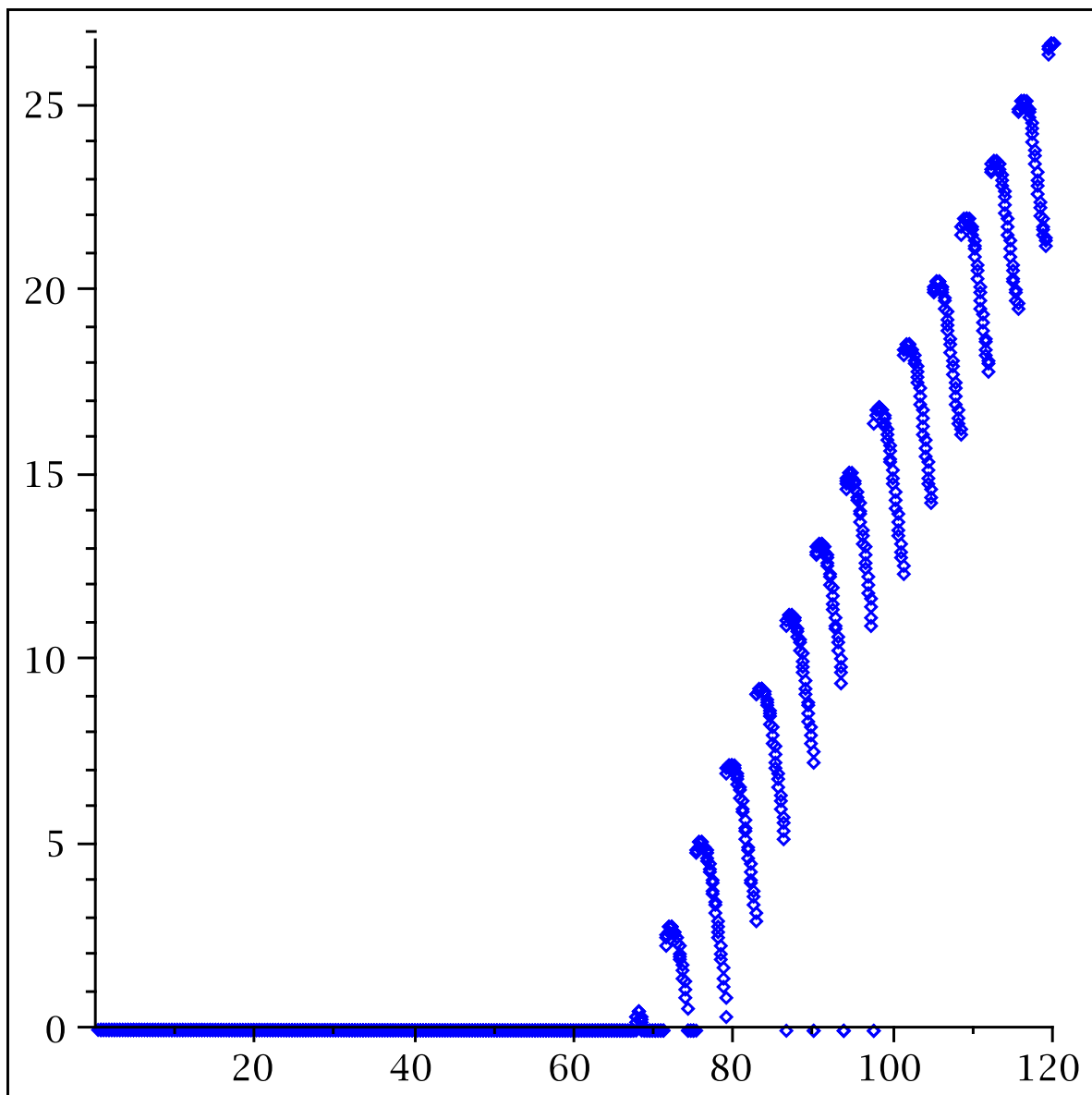


Figure 2: The M-curve for  $\Xi_3(z)$ ,  $0 \leq \Re(z) \leq 120$ .

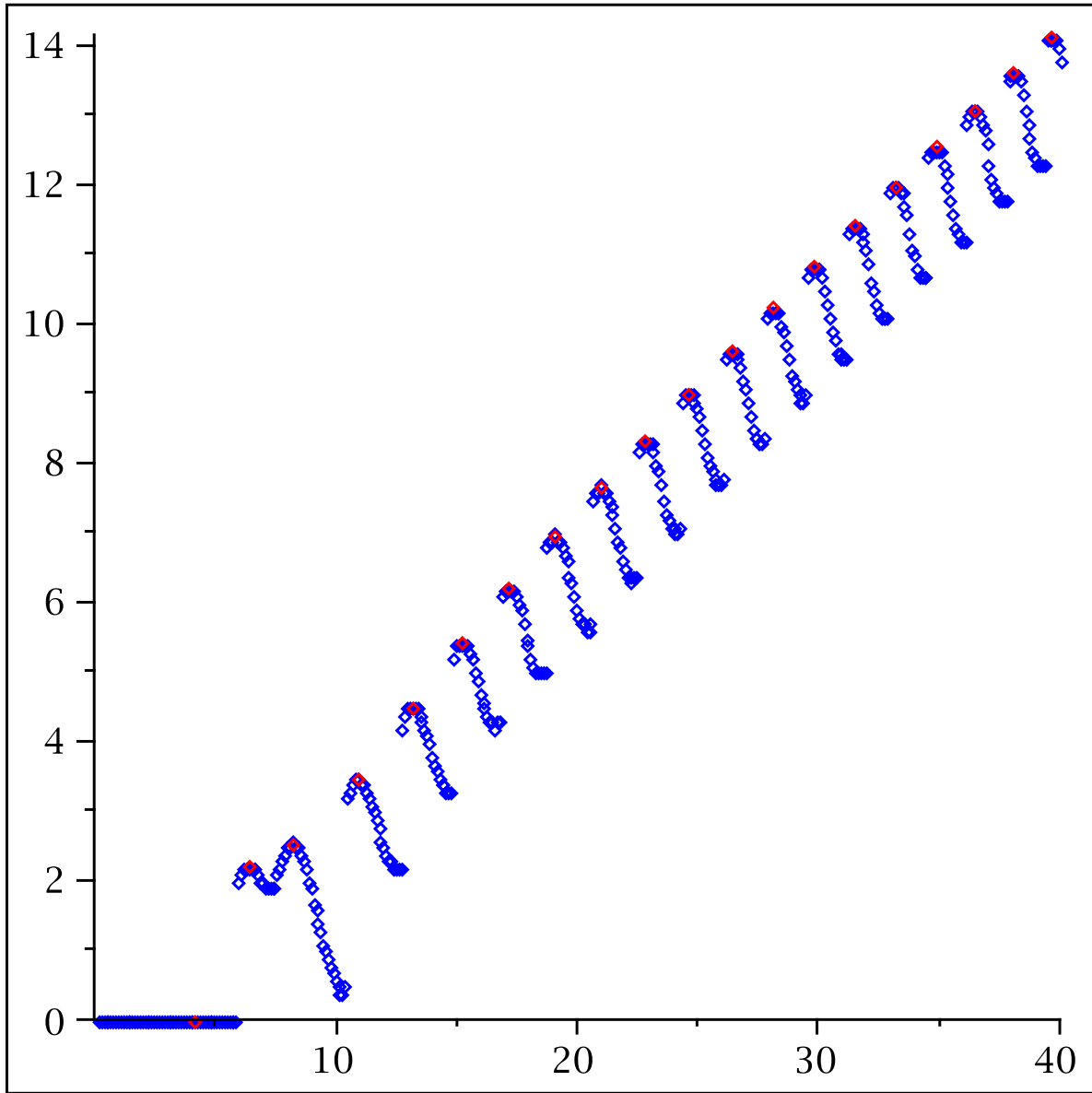


Figure 3: The M-curve for a sum of 8 arbitrarily chosen generalized hyperbolic gamma functions. The zeros (of the sum of these 8) are overlaid in red.



## 7 Appendix

*Digits* := 25, zeros of first  $\Xi$ -approximate  $\Xi_1(z)$  in rectangle with corners  $(0, -.1), (100, 500)$ .

14.04543957882981756479858,  
20.62534600592171760132974 + 2.697151842339519632505712 I  
26.05616693357829946749575 + 7.125359707612690330897455 I  
31.50143137824977099308422 + 10.72915037105496782822450 I  
36.72702276874255239918647 + 13.75961410603683555833019 I  
41.73703479849622101486046 + 16.44012737324329251859479 I  
46.56622866997881255099908 + 18.88186965378958902053812 I  
51.24456582311629453468990 + 21.14750420601374895347492 I  
55.79525368022472028456165 + 23.27625685820891335493023 I  
60.23621426525993802296865 + 25.29458549895993860216014 I  
64.58150497097301796850798 + 27.22133555778112035831075 I  
68.84235653395121330563843 + 29.07049609150585601287785 I  
73.02789933182939276748060 + 30.85279227139366634464017 I  
77.14567324003250763696303 + 32.57666324204392832752644 I  
81.20199121212953110713480 + 34.24889253114939152723783 I  
85.20220345212231662722890 + 35.87503096670553315342957 I  
89.15089297349449318064800 + 37.45968995259880236581690 I  
93.05202292717284600209187 + 39.00675061925213970478000 I  
96.90904939663401491219210 + 40.51951660155401879741380 I

1st differences of imaginary parts of zeros

2.697151842339519632505712  
4.428207865273170698391743  
3.603790663442277497327045  
3.03046373498186773010569  
2.68051326720645696026460  
2.44174228054629650194333  
2.26563455222415993293680  
2.12875265219516440145531  
2.01832864075102524722991  
1.92675005882118175615061  
1.84916053372473565456710  
1.78229617988781033176232  
1.72387097065026198288627  
1.67222928910546319971139  
1.62613843555614162619174  
1.58465898589326921238733  
1.54706066665333733896310  
1.51276598230187909263380

*Digits* := 35, zeros of  $\Xi_2(z)$  in rectangle with corners  $(0, -.1), (102, 100)$ .

14.1347251016150223590867934323428320  
21.0220425550989420016995644399118565

25.0108186655252580507941448128443705  
 30.4267684045343046403275534114141879  
 32.9244008910878391337889771099093884  
 37.8603410698702633339745855804483258  
 39.5324810797879508571153350051479630  
 43.1389080680988950956236929709951507 + 3.28097100306881350163884998294539870 I  
 47.5227563037011494043923449534796642 + 6.25250911174970354129046143001000600 I  
 51.8283147335875594556295444210165891 + 8.95857384762622450769949019637180856 I  
 56.1120029199774070873341435705058778 + 11.4796085175258061625014962442075566 I  
 60.3502919249623580499827793888173447 + 13.8416168548275384327656057308155260 I  
 64.5390675686875954205140494065924195 + 16.0705629583418306170962131727661418 I  
 68.6761126675575950561373116041091163 + 18.1868197094425056650011185673044960 I  
 72.7617836933667997393215687715081074 + 20.2066697023948159796720721861483021 I  
 76.7975312791721943222372025862518267 + 22.1432248556299027619859319905393103 I  
 80.7854051988095067379835000872591797 + 24.0071081808438939881307280663316996 I  
 84.7277019646143359935370107950650673 + 25.8070157111458796320542608491379890 I  
 88.6267772741056992788936274593068783 + 27.5501321650503597200806915781312903 I  
 92.4849402660525200867715711075727066 + 29.2424518834035459512085274002314648 I  
 96.3043979005927443711172685790765488 + 30.8890225459009695084228517057586202 I  
 100.087228624602886284768301012877784 + 32.4941324878154139135000560357642324 I

*Digits* := 35, zeros of  $\Xi_3(z)$  in rectangle with corners  $(0, -.1), (66, 67)$ .

14.13472514173469376946955013374227983392383419451107438  
 21.02203963877155659023693964971327708279243073624430253  
 25.01085758014566673273814860967472222032345345332059477  
 30.42487612586063748020316156144982639562443306193294638  
 32.93506158773265354499645423632051649764195395468931578  
 37.58617815896445021695716617231461056957864550061946742  
 40.91871901006637538924149719014594063254432154184503222  
 43.32707329088964789958258005109768490700030014005496697  
 48.00515051035397786165873118175586051353275737166717210  
 49.77383399904302558365984997979234680103752864503720614  
 52.97031190983202078103080161894128897412215226172440818  
 56.44637829985894482385431963634767652784719134486888209  
 59.34511184966886217051112646891854833040742035322093409  
 60.83672336038805375320166674337805885068490701004798353  
 65.03207377198913910137679883482704065081967504834437506

*Digits* := 35, zeros of first approximate  $\Xi_{\Delta,1}$  to the Ramanujan  $\Xi$ -function in rectangle with corners  $(0, -.1), (100, 100)$ .

9.5489635091412543125672429455388951 + 1.7119172216212706912355445109132207 I  
 13.789433051893611052566335275629933 + 5.2848734079809646491896114233647142 I  
 17.427358570320731571246782104695096 + 7.7540563400814520087583759352815352 I  
 20.757098189027862070211629799284288 + 9.7974743824676388131629735365802915 I  
 23.879056232416349736892490077064465 + 11.592456287772395127573044664358961 I  
 26.847356442625470267126707245453301 + 13.219199923922411333605849186386965 I

29.695457505212671889597523331405240 + 14.722240962151850502838429485181816 I  
 32.445858241734786458856784110868772 + 16.129361168830359821304709531815352 I  
 35.114596915766277030958889001310311 + 17.459270918804088444729979963005156 I  
 37.713610437352047890893961093475018 + 18.725280350575037459960230440611499 I  
 40.252083817737620602530239123250025 + 19.937256671472885578681886805140642 I  
 42.737274291251073747956340698712113 + 21.102754510914081337479910411119165 I  
 45.175041128226087752248534272004155 + 22.227709943351466358153291572887016 I  
 47.570200654826011921382328419781717 + 23.316887844914435394477534028073124 I  
 49.926772425743961667810425669964510 + 24.374181909750589494277749519061023 I  
 52.248154899971609917404551354946461 + 25.402822596146688919661995414148711 I  
 54.537253919041181821883861911330465 + 26.405525330914909309258552834547793 I  
 56.796578677359823555284912818063663 + 27.384598689419083619944995509463388 I  
 59.028314745351122620881979673545968 + 28.342025013215928897705357026998109 I  
 61.234380542300284878652304849348298 + 29.279521587539252797178004730959225 I  
 63.416471643946175987038219068629433 + 30.198587816232713426722974287579177 I  
 65.576095995892802432660080685311982 + 31.100542121402673258701892990637098 I  
 67.714602225166643613493231218811692 + 31.986551176775085187674532167087986 I  
 69.833202642000750529879098123320031 + 32.857653335480587110184136136077900 I  
 71.932992106046633644353345531169636 + 33.714777601864923783681632805009935 I  
 74.014963635252309537091144416299462 + 34.558759141214355169362183745447748 I  
 76.080021422732136425762615907263059 + 35.390352069528146990609600601997921 I  
 78.128991771589170231983397657306861 + 36.210240084528180967081058119582790 I  
 80.162632342789627094434403795275856 + 37.019045367211828466413017905373453 I  
 82.181640025246587241041487951836980 + 37.817336085881271861478883874617446 I  
 84.186657672248860977039557949370316 + 38.605632761831562562091441356062965 I  
 86.178279898670686132064566105956960 + 39.384413700925457309936759214348925 I  
 88.157058095044852280187866298051226 + 40.154119653347769812443058137516104 I  
 90.123504784721498632008498428129335 + 40.915157831527436633928427478875093 I  
 92.078097426893282987651528394221783 + 41.667905391109706656241503132525470 I  
 94.021281749721926032933713327889188 + 42.412712460188349826344885668577328 I  
 95.953474683021781433616191339406534 + 43.149904786473380482490406058490112 I  
 97.875066948097497309259057547588008 + 43.879786059713960896674731680251557 I  
 99.786425352756449975929052839495573 + 44.602639956801403127156000977181747 I

*Digits* := 25, zeros of second approximate  $\Xi_{\Delta,N}$  to the Ramanujan  $\Xi$ -function in rectangle with corners  $(0, -.1), (100, 200)$ .

9.19376894449217893  
 13.8348516743546257 + 1.54300060155520050 I  
 17.0006907368319637 + 4.58047657550007841 I  
 20.1893418670461022 + 6.99896197324829350 I  
 23.2665587059084961 + 9.04437482423736081 I  
 26.2157755564237380 + 10.8464634984827865 I  
 29.0559366154717340 + 12.4806210190923921 I  
 31.8034718883051467 + 13.9910040730670684 I  
 34.4718383897903041 + 15.4050720323654861 I  
 37.0718504676122273 + 16.7414677662025767 I

39.6120959833238977 + 18.0135340186718953 I  
 42.0994917987503797 + 19.2311725419877171 I  
 44.5397026159453686 + 20.4019721238147549 I  
 46.9374285351558464 + 21.5319037938006434 I  
 49.2966167316589794 + 22.6257645644359874 I  
 51.6206201858570109 + 23.6874764176583670 I  
 53.9123161924908609 + 24.7202947836257462 I  
 56.1741960636482404 + 25.7269571656683657 I  
 58.4084343572848694 + 26.7097915965161903 I  
 60.6169430187982438 + 27.6707975724897949 I  
 62.8014141732518389 + 28.6117075511920988 I  
 64.9633543292524033 + 29.5340343986557084 I  
 67.1041120187127688 + 30.4391085162350689 I  
 69.2249003477575950 + 31.3281072698702158 I  
 71.3268155548021774 + 32.2020785869307989 I  
 73.4108524066057312 + 33.0619600720511948 I  
 75.4779170679704262 + 33.9085946393025650 I  
 77.5288379348185691 + 34.7427434066455078 I  
 79.5643748117146020 + 35.5650964167873912 I  
 81.5852267334967588 + 36.3762816159807758 I  
 83.5920386687224333 + 37.1768724246086591 I  
 85.5854072948756678 + 37.9673941604083026 I  
 87.5658859982298918 + 38.7483295199556847 I  
 89.5339892223226212 + 39.5201232818472100 I  
 91.4901962662154965 + 40.2831863625189748 I  
 93.4349546156253617 + 41.0378993303884662 I  
 95.3686828755524769 + 41.7846154642047194 I  
 97.2917733614022269 + 42.5236634258506564 I  
 99.2045943961865493 + 43.2553496053959862 I

*Digits* := 25, zeros, in rectangle with corners (10000.0, 500), (10025.8, 10000), of linear combination of two generalized hyperbolic gamma functions

$$\sum_{k=1}^2 \beta_k G(z; A_k, B_k, \alpha_k, w) \quad (61)$$

where

$$\beta_1 = 1.0 + .3i, \beta_2 = -3 - i, \alpha_1 = 1.0 + .3i, \alpha_2 = -3 - i, \quad (62)$$

$$w = 0.7648421872844884262558600 + 0.6442176872376910536726144I \quad (63)$$

$$B_1 = 1.0, B_2 = 2.2, A_1 = \pi, A_2 = 7.853981633974483096156608 \quad (64)$$

10000.08899697917243762827 + 1943.699378898319010297001 I,  
 10000.84424695361683674126 + 1943.828178641310676925944 I,  
 10001.59949024766823939791 + 1943.956976210289663002380 I,  
 10002.35472686200573704240 + 1944.085771605493580928470 I,  
 10003.10995679730829337130 + 1944.214564827159996509445 I,

10003.86518005425474436858 + 1944.343355875526428966768 I,  
 10004.62039663352379834061 + 1944.472144750830350951281 I,  
 10005.37560653579403595119 + 1944.600931453309188556355 I,  
 10006.13080976174391025650 + 1944.729715983200321331031 I,  
 10006.88600631205174674006 + 1944.858498340741082293155 I,  
 10007.64119618739574334772 + 1944.987278526168757942515 I,  
 10008.39637938845397052261 + 1945.116056539720588273964 I,  
 10009.15155591590437124002 + 1945.244832381633766790543 I,  
 10009.90672577042476104237 + 1945.373606052145440516603 I,  
 10010.66188895269282807412 + 1945.502377551492710010906 I,  
 10011.41704546338613311658 + 1945.631146879912629379767 I,  
 10012.17219530318210962295 + 1945.759914037642206290096 I,  
 10012.92733847275806375302 + 1945.888679024918401982558 I,  
 10013.68247497279117440814 + 1946.017441841978131284633 I,  
 10014.43760480395849326601 + 1946.146202489058262623716 I,  
 10015.19272796693694481556 + 1946.274960966395618040195 I,  
 10015.94784446240332639171 + 1946.403717274226973200536 I,  
 10016.70295429103430821021 + 1946.532471412789057410354 I,  
 10017.45805745350643340246 + 1946.661223382318553627526 I,  
 10018.21315395049611805021 + 1946.789973183052098475038 I,  
 10018.96824378267965122043 + 1946.918720815226282254485 I,  
 10019.72332695073319499999 + 1947.047466279077648958683 I,  
 10020.47840345533278453047 + 1947.176209574842696284945 I,  
 10021.23347329715432804280 + 1947.304950702757875648075 I,  
 10021.98853647687360689210 + 1947.433689663059592193412 I,  
 10022.74359299516627559224 + 1947.562426455984204809882 I,  
 10023.49864285270786185070 + 1947.691161081768026142988 I,  
 10024.25368605017376660312 + 1947.819893540647322607883 I,  
 10025.00872258823926404804 + 1947.948623832858314402362 I,  
 10025.76375246757950168151 + 1948.077351958637175519890 I,

1st differences of imaginary parts

0.128799742991666628943  
 0.128797568978986076436  
 0.128795395203917926090  
 0.128793221666415580975  
 0.128791048366432457323  
 0.128788875303921984513  
 0.128786702478837605074  
 0.128784529891132774676  
 0.128782357540760962124  
 0.128780185427675649360  
 0.128778013551830331449  
 0.128775841913178516579  
 0.128773670511673726060  
 0.128771499347269494303

0.128769328419919368861  
0.128767157729576910329  
0.128764987276195692462  
0.128762817059729302075  
0.128760647080131339083  
0.128758477337355416479  
0.128756307831355160341  
0.128754138562084209818  
0.128751969529496217172  
0.128749800733544847512  
0.128747632174183779447  
0.128745463851366704198  
0.128743295765047326262  
0.128741127915179363130  
0.128738960301716545337  
0.128736792924612616470  
0.128734625783821333106  
0.128732458879296464895  
0.128730292210991794479  
0.128728125778861117528

2nd differences of imaginary parts of zeros (times minus 1)

0.2174012680552507  $\times 10^{-5}$   
0.2173775068150346  $\times 10^{-5}$   
0.2173537502345115  $\times 10^{-5}$   
0.2173299983123652  $\times 10^{-5}$   
0.2173062510472810  $\times 10^{-5}$   
0.2172825084379439  $\times 10^{-5}$   
0.2172587704830398  $\times 10^{-5}$   
0.2172350371812552  $\times 10^{-5}$   
0.2172113085312764  $\times 10^{-5}$   
0.2171875845317911  $\times 10^{-5}$   
0.2171638651814870  $\times 10^{-5}$   
0.2171401504790519  $\times 10^{-5}$   
0.2171164404231757  $\times 10^{-5}$   
0.2170927350125442  $\times 10^{-5}$   
0.2170690342458532  $\times 10^{-5}$   
0.2170453381217867  $\times 10^{-5}$   
0.2170216466390387  $\times 10^{-5}$   
0.2169979597962992  $\times 10^{-5}$   
0.2169742775922604  $\times 10^{-5}$   
0.2169506000256138  $\times 10^{-5}$   
0.2169269270950523  $\times 10^{-5}$   
0.2169032587992646  $\times 10^{-5}$   
0.2168795951369660  $\times 10^{-5}$   
0.2168559361068065  $\times 10^{-5}$

0.2168322817075249  $\times 10^{-5}$   
0.2168086319377936  $\times 10^{-5}$   
0.2167849867963132  $\times 10^{-5}$   
0.2167613462817793  $\times 10^{-5}$   
0.2167377103928867  $\times 10^{-5}$   
0.2167140791283364  $\times 10^{-5}$   
0.2166904524868211  $\times 10^{-5}$   
0.2166668304670416  $\times 10^{-5}$   
0.2166432130676951  $\times 10^{-5}$

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