A CORNUCOPIA OF LABELED DIAGRAMS AND THEIR GENERATING POLYNOMIALS

George Wang

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Supervisor of Dissertation

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Jim Haglund, Professor of Mathematics

Graduate Group Chairperson

____

Julia Hartmann, Professor of Mathematics

Dissertation Committee:
Jim Haglund, Professor of Mathematics
Robin Pemantle, Merriam Term Professor of Mathematics
Mark Skandera, Visiting Faculty

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ABSTRACT

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AND THEIR GENERATING POLYNOMIALS

George Wang

Jim Haglund, Advisor

Combinatorics on tableaux-like objects and understanding the relationships of various polynomial bases with each other are classical explorations in algebraic combinatorics. This type of exploration is the focus of this dissertation. In the world of symmetric polynomials and their corresponding objects, we prove some partial results for the Schur expansion of Jack polynomials in certain binomial coefficient bases. As a result, we conjecture a bijection between tableaux and rook boards, which spurs some further exploration of quasi-Yamanouchi tableaux as combinatorial objects of their own merit.

We then move to the general polynomial ring and two of its bases, key and lock polynomials. These are each generating polynomials of certain kinds of Kohnert diagrams, and we use this connection to say something about their relationship. Each of the objects that they are generating polynomials of have a nice crystal structure. We prove that the crystal structure corresponding to lock polynomials is connected and can be embedded into the crystal structure corresponding to key polynomials.

Contents

Chapter 1

Introduction

The focus of this dissertation is on understanding families of polynomials through the tableaux that they are generating polynomials of. Perhaps the most well known example is the Schur polynomials viewed as generating polynomials of semistandard Young tableaux. Schur polynomials are central to the theory of symmetric polynomials, which in turn plays an important role in many areas of mathematics, including combinatorics, representation theory, and algebraic geometry. Young tableaux themselves have been studied extensively, and yet it seems like there are constantly new things to discover about them.

Schur polynomials have various generalizations as well. In one direction, they may be generalized by adding additional parameters; Jack polynomials are a generalization of Schur polynomials that add one parameter, while Macdonald polynomials are a further generalization of Schur polynomials that add two parameters.

In another direction, we may move from the ring of symmetric polynomials to the ring of quasisymmetric polynomials or to the full ring of polynomials. An important polynomial generalization of Schurs in this direction are key polynomials, also known as Demazure characters.

It is a common theme in algebraic combinatorics to show that some symmetric polynomial is Schur-positive, meaning it expands positively in the Schur basis. One nice consequence of this is that the Schur polynomials encode irreducible repreprresentations of the symmetric group, and so a Schur positive decomposition of a polynomial corresponds to a decomposition into irreducible components with multiplicity of the algebraic structure that the polynomial corresponds to. Therefore, considering the Schur basis expansion of a polynomial as a generating function of combinatorial objects within that basis means that computing multiplicities of irreducible representations is the same as enumeration of those combinatorial objects.

In Chapter 2, we introduce three closely related types of tableaux: semistandard Young tableaux, standard Young tableaux, and quasi-Yamanouchi tableaux. We then explore two conjectures on the Schur-positivity of Jack polynomials and find that for a special case, Jack polynomials are a generating function for quasi-Yamanouchi tableaux in the Schur basis. Our work on these conjectures on Jack polynomials then leads us to an exploration of the relationship between quasi-Yamanouchi tableaux and some other well-known types of combinatorial objects.

In Chapter 3, we move to the full polynomial ring. Lock polynomials and lock

tableaux are natural analogues to key polynomials and Kohnert tableaux, respectively. We compare lock polynomials to the much-studied key polynomials and give an explicit description of a crystal structure on lock tableaux. We then construct an injective, weight-preserving map from lock tableaux to Kohnert tableaux that intertwines with their crystal operators to show that the crystal structure on lock tableaux has a natural embedding into the Demazure crystal.

Chapter 2

The Symmetric World

2.1 Schur polynomials

A *partition* $\lambda = (\lambda_1 \geq \cdots \geq \lambda_k)$ is a weakly decreasing sequence of positive integers. The *size* of λ is denoted $|\lambda|$ and is the sum of the integers of the sequence. The *length* of λ, denoted $\ell(\lambda)$, is the number of integers in the partition. We say that a partition λ dominates a partition μ if for all $i \geq 1$, we have $\lambda_1 + \cdots + \lambda_i \geq \mu_1 + \cdots + \mu_i$. We also write $n(\lambda) = \sum_i^k (i-1)\lambda_i)$.

We identify a partition with its *diagram*, which we visualize in French notation. That is, rows are counted from bottom to top, the number of boxes in the ith row equals λ_i, and boxes are left justified. The conjugate of a partition λ is written λ' and is obtained by reflecting the diagram across the diagonal. The Cartesian coordinate $u = (i, j)$ is identified with the box in the ith column and jth row. The *content* of a square is $c(u) = i - j$. The *arm* of a box u, which we write as $\mathrm{arm}(u)$, is the number of boxes $v = (a, j)$ in the diagram such that $a > i$. Similarly, the *leg* of a box u, written $\mathrm{leg}(u)$, is the number of boxes $v = (i, a)$ such that $a > j$. The *hook-length* of a box is $h(u) = \mathrm{arm}(u) + \mathrm{leg}(u) + 1$.

Permutations $\pi \in S_n$ are written in one line notation, $\pi = \pi_1 \cdots \pi_n$, where $\pi_i = \pi(i)$. The *descent set* of π is $\mathrm{Des}(\pi) = \{i \in [n-1] \mid \pi_i > \pi_{i+1}\}$, and its size is $|\mathrm{Des}(\pi)| = \mathrm{des}(\pi)$. The *major index* of a permutation is $\mathrm{maj}(\pi) = \sum_{i \in \mathrm{Des}(\pi)} i$. Permutations act on polynomials in multiple variables $x_1, \ldots, x_n$ by permuting the indices of the variables, so that $x_i \mapsto x_{\pi(i)}$. Within the polynomial ring in n variables, there is the ring of *symmetric polynomials* in n variables, which are those that are invariant under the action of S_n.

Bases of the ring of symmetric polynomials are indexed by partitions. In particular, we have the *elementary* symmetric, *homogeneous* symmetric, *power-sum*, *monomial* symmetric, and *Schur* polynomials, denoted e_λ, h_λ, m_λ, p_λ, and s_λ respectively. The first three are defined as follows for $\lambda = (\lambda_1, \ldots, \lambda_k)$.

$$e_j(x_1, \ldots, x_n) = \sum_{1 \le i_1 < \cdots < i_j \le n} x_{i_1} \cdots x_{i_j}, \qquad e_\lambda = e_{\lambda_1} \cdots e_{\lambda_k}.$$

$$h_j(x_1, \ldots, x_n) = \sum_{1 \le i_1 \le \cdots \le i_j \le n} x_{i_1} \cdots x_{i_j}, \qquad h_\lambda = h_{\lambda_1} \cdots h_{\lambda_k}.$$

$$p_j(x_1, \ldots, x_n) = \sum_{i=1}^{n} x_i^j, \qquad p_\lambda = p_{\lambda_1} \cdots p_{\lambda_k}.$$

A monomial term in the polynomial ring in n variables can be written as $x_1^{a_1} \cdots x_n^{a_n}$. If we write $\mathbf{a} = (a_1, \ldots, a_n)$, we can abbreviate this monomial to $x^{\mathbf{a}}$. Then the monomial symmetric polynomial $m_{\mathbf{a}}(x_1, \ldots, x_n)$ is the sum of all monomials $x^{\mathbf{a}'}$ where $\mathbf{a}'$ is a distinct permutation of $\mathbf{a}$. Since $m_{\mathbf{a}} = m_{\mathbf{b}}$ for any rearrangement $\mathbf{b}$ of $\mathbf{a}$, we can consider only indices $\mathbf{a}$ where $a_1 \geq \cdots \geq a_n$. In particular, such an $\mathbf{a}$ is some partition λ that possibly also has trailing zeroes, and so we write $m_{\mathbf{a}}$ as m_λ.

We will think of Schur polynomials as the generating polynomials of certain combinatorial objects called *semistandard Young tableaux* (SSYT). An SSYT is a filling of a partition λ using positive integers that weakly increase to the right and strictly increase upwards. The set of such fillings of λ is denoted $\mathrm{SSYT}(\lambda)$, and if we restrict the maximum value of an entry to m, then the set of such fillings is denoted $\mathrm{SSYT}_m(\lambda)$. We can enumerate $\mathrm{SSYT}_m(\lambda)$ by Stanley's hook-content formula, which we reproduce below.

Theorem 2.1.1 (Hook-content formula [33])**.** *Given a partition λ,*

$$\mathrm{SSYT}_m(\lambda) = \prod_{u \in \lambda} \frac{m + c(u)}{h(u)}.$$

$$\begin{array}{|c|c|}\hline 2&2\\\hline 1&1\\\hline\end{array}\quad\begin{array}{|c|c|}\hline 2&3\\\hline 1&1\\\hline\end{array}\quad\begin{array}{|c|c|}\hline 2&3\\\hline 1&2\\\hline\end{array}\quad\begin{array}{|c|c|}\hline 3&3\\\hline 1&1\\\hline\end{array}\quad\begin{array}{|c|c|}\hline 3&3\\\hline 1&2\\\hline\end{array}\quad\begin{array}{|c|c|}\hline 3&3\\\hline 2&2\\\hline\end{array}$$

Figure 2.1: All 6 elements of $\mathrm{SSYT}_3(2,2)$.

The weight of an SSYT T is $\mathbf{wt}(T) = (w_1, w_2, \ldots)$ where w_i is the number of times that i appears, and given partitions λ, μ, the *Kostka* numbers $K_{\lambda\mu}$ count the number of SSYT of shape λ and weight μ. A *standard Young tableau* (SYT) of shape λ with size n is a semistandard Young tableau of shape λ with weight (1^n), and the set of such fillings is denoted $\mathrm{SYT}(\lambda)$. Frame, Robinson, and Thrall counted standard fillings using the hook-length formula.

Theorem 2.1.2 (Hook-length formula [15])**.** *Given a partition λ,*

$$\mathrm{SYT}(\lambda) = \frac{n!}{\prod_{u\in\lambda} h(u)}.$$

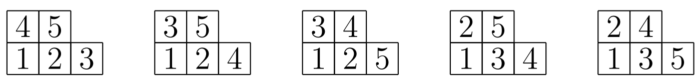


Figure 2.2: All 5 elements of $\mathrm{SYT}(3,2)$.

The descent set for $T \in \mathrm{SYT}(\lambda)$ is $\mathrm{Des}(T) = \{i \in [n-1] \mid i+1 \text{ is above } i\}$. If we write the descent set as $\{a_1, a_2, \ldots, a_{k-1}\}$ in increasing order, then the first *run* of the tableau is the set of boxes that contain all the entries from 1 to a_1. Then for $1 < i < k$, the ith run is the set of boxes containing entries from $d_{i-1}+1$ to d_i, and the kth run starts at d_{k+1} and ends at n. As with permutations, we define the *major index* of a tableau $T \in \mathrm{SYT}(\lambda)$ to be $\mathrm{maj}(T) = \sum_{i\in\mathrm{Des}(T)} i$. We also have the *charge* statistic for standard Young tableaux: each entry i in T has a charge defined recursively, where $ch(1) = 0$, $ch(i+1) = ch(i)$ if $i \notin \mathrm{Des}(T)$, $ch(i+1) = ch(i)+1$ if $i \in \mathrm{Des}(T)$, and $ch(T) = \sum_{i=1}^{|\lambda|} ch(i)$.

9	**10**	12		
4	5	7	**11**	
1	2	3	6	8

Figure 2.3: This tableau has descent set $\{3, 6, 8, 11\}$ and has the fourth run bolded.

An SSYT is a *quasi-Yamanouchi tableau* (QYT) if when i appears in the tableau, some instance of i is in a higher row than some instance of $i-1$ for all i. We write $\mathrm{QYT}(\lambda)$ to denote the set of QYT of shape λ, $\mathrm{QYT}_{\le m}(\lambda)$ to denote those with largest entry at most m, and $\mathrm{QYT}_{=m}(\lambda)$ to denote those with largest entry exactly m.

4			
2	3		
1	2	2	4

4			
3	3		
1	2	2	5

Figure 2.4: The left is a quasi-Yamanouchi filling, while the right is not.

3	
2	2
1	1

3	
2	3
1	1

3	
2	3
1	2

4	
2	3
1	2

3	
2	4
1	3

Figure 2.5: QYT of shape $(2, 2, 1)$, showing that $\mathrm{QYT}_{=3}(2, 2, 1) = 3$ and $\mathrm{QYT}_{=4}(2, 2, 1) = 2$.

As a generating function of SSYT, we can define the Schur polynomial in m variables indexed by λ as a sum over semistandard Young tableaux.

Definition 2.1.3. The Schur polynomial $s_\lambda(x_1, \ldots, x_n)$ is given by

$$s_\lambda(x_1, \ldots, x_n) = \sum_{T \in \mathrm{SSYT}_n(\lambda)} x^{\mathbf{wt}(T)}, \tag{2.1.1}$$

where $x^{(a_1,\ldots,a_n)} = x_1^{a_1} \cdots x_n^{a_n}$.

This summation requires a number of semistandard Young tableaux that varies depending on which integers are allowed. In contrast, the number of *standard* Young tableaux of a particular shape λ is always fixed, and so perhaps it would be nice to define s_λ as a sum over $\mathrm{SYT}(\lambda)$ instead. We can accomplish this using the *fundamental quasisymmetric* polynomial basis F_α for quasisymmetric polynomials, defined by Gessel [16] in 1984. The ring of *quasisymmetric polynomials* lies between symmetric polynomials and the full polynomial ring. A polynomial in the variables $x_1, \ldots, x_n$ is quasisymmetric in those variables if the coefficients of any two monomials agree whenever their ordered sequence of nonzero exponents agree, including monomials with a coefficient of zero.

Bases of the ring of quasisymmetric functions are indexed by *strong compositions*. A strong composition is a sequence of positive integers $\alpha = (\alpha_1, \ldots, \alpha_k)$, and the *size* of a strong composition is the sum of those integers. To define the fundamental basis, it is convenient to first define the *monomial quasisymmetric* polynomials M_α. For $\alpha = (\alpha_1, \ldots, \alpha_k)$ where each α_i is some positive integer, we have

$$M_\alpha(x_1,\dots,x_n) = \sum_{1\le i_1,<\cdots<i_k\le n} x_{i_1}^{\alpha_1}\cdots x_{i_k}^{\alpha_k}. \tag{2.1.2}$$

Given two compositions α and β of the same size, we say that β *refines* α if there exist indices $i_1 < \cdots < i_\ell$ such that $\beta_{i_j+1} + \cdots + \beta_{i_{j+1}} = \alpha_{j+1}$. The fundamental quasisymmetric polynomial [16] is defined as

$$F_\alpha(x_1,\dots,x_n) = \sum_{\beta \text{ refines } \alpha} M_\beta(x_1,\dots,x_n). \tag{2.1.3}$$

Descent sets of standard Young tableaux can be mapped to strong compositions by taking the number of boxes in each run in increasing order, and we call such a composition the *descent composition* of a tableau. We will engage in some abuse of notation and write $F_{\mathrm{Des}(T)}$ to mean the fundamental quasisymmetric polynomial indexed by the descent composition of T.

Theorem 2.1.4 ([16]). *The Schur polynomial $s_\lambda(x_1,\dots,x_n)$ is given by*

$$s_\lambda(x_1,\dots,x_n) = \sum_{T\in\mathrm{SYT}(\lambda)} F_{\mathrm{Des}(T)}(x_1,\dots,x_n). \tag{2.1.4}$$

It may be the case that certain terms of this summation are equal to zero when the number of variables is small. In particular, this happens if a term is indexed by a standard Young tableau T that has $\mathrm{des}(T) > n-1$, in which case $F_{\mathrm{Des}(T)} = 0$. In order to tighten this expansion, we look to quasi-Yamanouchi tableaux. Standard Young tableaux and quasi-Yamanouchi tableaux of the same shape have a natural correspondence given by the *standardization* map (QYT to SYT) and *destandard-*

ization map (SYT to QYT). The destandardization map is defined in [3] by Assaf and Searles.

Definition 2.1.5 (Definition 2.5, [3])**.** Define the *destandardization* of a standard Young tableau T, denoted by $\operatorname{dst}(T)$, to be the tableau constructed as follows. If the leftmost i lies strictly right of the rightmost $i-1$, then decrement every i to $i-1$. Repeat until no i satisfies the condition.

An equivalent description is to change every label in the ith run of T to i, for all runs of T. The standardization map is the inverse: a quasi-Yamanouchi tableau Q maps to a standard Young tableau T whose ith run is exactly the boxes with label i in Q. This bijection between SYT and QYT is a special case of Theorem 4.9 of [3] and is re-proven by the author in Proposition 3.2 of [36] with a tighter bound.

Proposition 2.1.6 ([3, 36])**.** *Let* $\lambda = (\lambda_1, \ldots, \lambda_k)$ *be a partition of* n *and* $m \geq n - (\lambda_1 - 1)$*. Then*

$$\mathrm{QYT}_{\leq m}(\lambda) \cong \mathrm{SYT}(\lambda). \tag{2.1.5}$$

Through this bijection, we can define the major index and charge of a quasi-Yamanouchi tableau Q to be the respective statistic of the standard Young tableau that Q maps to. Note that by definition, $\mathrm{QYT}_{=m}(\lambda) = 0$ for any $|\lambda| = n$ and $m > n$, so $\mathrm{QYT}_{\leq n}(\lambda)$ contains all quasi-Yamanouchi tableaux of shape λ. Since we can partition $\mathrm{QYT}_{\leq n}(\lambda)$ into $\{\mathrm{QYT}_{=m}(\lambda) \mid 1 \leq m \leq n\}$, this bijection gives a refinement on standard Young tableaux based on the number of runs of each tableaux. We will also later use the following result, which is obtained through a bijection consisting of standardizing a quasi-Yamanouchi tableau, conjugating, and then destandardizing.

Lemma 2.1.7 ([36]). *Given a partition λ of n, its conjugate λ', and $1 \leq k \leq n$,* $\mathrm{QYT}_{=k}(\lambda) \cong \mathrm{QYT}_{=(n+1)-k}(\lambda')$.

Return to Gessel's expansion, recall that the zero terms occur when there are too many descents compared to the number of variables x_i. Then using quasi-Yamanouchi tableaux, we can make the following improvement.

Theorem 2.1.8 (Theorem 2.7, [3]). *The Schur polynomial $s_\lambda(x_1, \ldots, x_n)$ is given by*

$$s_\lambda(x_1, \ldots, x_n) = \sum_{T \in \mathrm{QYT}_{=n}(\lambda)} F_{\mathbf{wt}(T)}(x_1, \ldots, x_n), \tag{2.1.6}$$

where all terms on the right hand side are nonzero.

Another correspondence that we will take advantage of in later sections is the celebrated *Robinson-Schensted-Knuth* (RSK) correspondence. RSK is a bijective algorithm between two line arrays and pairs (P, Q) of semistandard Young tableaux. A two line array here is defined as

$$w = \begin{pmatrix} i_1 & i_2 & \cdots & i_m \\ \\ j_1 & j_2 & \cdots & j_m \end{pmatrix}$$

in which the columns are in lexicographic order, that is $i_1 \leq i_2 \leq \cdots \leq i_m$ and if $i_r = i_s$, then $j_r \leq j_s$.

We also define an insertion procedure as follows. Given a semistandard Young tableau T, insert the value x_1 by scanning for the first entry in the first row from the left which is larger than x_1. If none exists, then adjoin a new cell with x_1 to the end of this row and terminate the procedure. Else if such an $x_2 > x_1$ does exist, replace its entry with x_1 and scan the second row for the first entry from the left larger than x_2. If none exists, adjoin x_2 to the end of the second row and terminate the procedure. Else if such an $x_3 > x_2$ does exist, replace its entry with x_2 and repeat this process in the third row. As this continues upwards, the procedure must eventually terminate, and we are left with a new tableau T'.

The RSK correspondence takes a two line array w and successively inserts the second row $j_1, \ldots, j_m$ into an empty diagram to get the insertion tableau P. The recording tableau Q records the order in which cells of P are added by adjoining a cell containing i_r after the insertion of j_r into P such that the insertion tableau and recording tableau maintain the same shape at every step. When the two line array w contains the integers $1, 2, \ldots, n$ in order in the first row and some permutation $\pi_1, \pi_2, \ldots, \pi_n = \pi \in S_n$ in the second row, we can identify w with the permutation π, and when restricted to permutations, RSK is a bijection between S_n and pairs of standard Young tableau of the same shape and of size n.

2.2 Jack polynomials

The (integral form, type A) Jack polynomials $J_\lambda^{(\alpha)}(x_1, \ldots, x_n)$ are an important family of symmetric functions with applications to many areas, including statistics, mathematical physics, representation theory, and algebraic combinatorics. While the symmetric polynomials of the previous section depend on a set of variables $x_1, \ldots, x_n$, the Jack polynomials add a parameter α and specialize into several families of symmetric polynomials with no parameter: m_λ at $\alpha = \infty$, $e_{\lambda'}$ at $\alpha = 0$, s_λ at $\alpha = 1$, and zonal polynomials at $\alpha = 1/2, 2$.

Despite their relations to many well studied families of polynomials, Jack polynomials are comparatively poorly understood. One area that has seen some progress is their positivity in other symmetric polynomial bases. From the definition of Jacks, it is not obvious that the coefficients of the monomial basis expansion are in $\mathbb{Z}[\alpha]$, but this integrality conjecture was proven by Lapointe and Vinet [28]. A result of Knop and Sahi [26] obtained later gives an explicit combinatorial formula for the expansion in the monomial basis, implying the stronger result that the coefficients lie in $\mathbb{N}[\alpha]$.

There has not been much exploration of the Schur basis expansion; the integrality result of Lapointe and Vinet implies that the coefficients of the Schur expansion are in $\mathbb{Z}[\alpha]$, but computations show that they are not generally in $\mathbb{N}[\alpha]$, and so a positive combinatorial formula is impossible here. The author, working jointly with Alexandersson and Haglund [1], explored a different approach towards a positive combinatorial formula. We define

$$\tilde{J}_\lambda^{(\alpha)}(x_1, \ldots, x_n) = \alpha^n J_\lambda^{1/\alpha}(x_1, \ldots, x_n), \tag{2.2.1}$$

then take the coefficient of a given Schur function s_μ in $\tilde{J}_\lambda^{(\alpha)}(x_1, \ldots, x_n)$ and expand it either in the basis $\{\binom{\alpha+k}{n}\}$ or in $\{\binom{\alpha}{k}k!\}$. This exploration grew from a conjecture by Haglund about the Schur expansion of (integral form, type A) Macdonald polynomials that Yoo [38, 39] proved for some special cases. Since Jack polynomials are a particular limit of the Macdonald polynomials, investigating the Schur expansion of Jacks may shed some light on the Macdonald case.

Experimentally, the coefficients appear to be nonnegative integers. When seeing nonnegative integer coefficients, the obvious question to ask is whether there is an interesting combinatorial interpretation. We were unable to find such an interpretation for general λ, μ but were able to produce some promising partial results in [1], which we reproduce here. The tableaux of the previous section also make a reappearance among the combinatorics of the coefficients explored in this section. The conjectures considered in this work were tested using Stembridge's Maple package SF [35]. A table of some computed coefficients in these bases can be found in Appendix A.

Conjecture 2.2.1 ([1]). *Let λ, μ be partitions of n. Then setting*

$$\langle \tilde{J}_\lambda^{(\alpha)}(x_1, \ldots, x_n), s_\mu \rangle = \sum_{k=0}^{n-1} a_k(\lambda, \mu) \binom{\alpha + k}{n},$$

we have $a_k(\lambda, \mu) \in \mathbb{N}$. Furthermore, the polynomial $\sum_{k=0}^{n} a_k(\lambda, \mu) z^k$ has only real zeros.

Conjecture 2.2.2 ([1]). *Let λ, μ be partitions of n. Then setting*

$$\langle \tilde{J}_\lambda^{(\alpha)}(x_1, \ldots, x_n), s_\mu \rangle = \sum_{k=1}^{n} b_{n-k}(\lambda, \mu) \binom{\alpha}{k} k!,$$

we have $b_{n-k}(\lambda,\mu) \in \mathbb{N}$*. Furthermore, the polynomial* $\sum_{k=0}^{n} b_{n-k}(\lambda,\mu) z^k$ *has only real zeros.*

We note that Conjecture 2.2.1 almost implies Conjecture 2.2.2. The identity $\binom{\alpha+k}{n} = \sum_i \binom{\alpha}{i}\binom{k}{n-i}$ shows that if the $a_k(\lambda,\mu) \in \mathbb{N}$, then $k! b_{n-k}(\lambda,\mu) \in \mathbb{N}$, so if Conjecture 2.2.1 is true, the only issue is whether or not the $b_{n-k}(\lambda,\mu)$ are integers.

2.2.1 Eulerian and Stirling numbers

The *Eulerian number* $A(n,k)$ is the number of permutations in S_n with k descents, and the *Stirling number* (of the second kind) $S(n,k)$ is the number of ways to partition n labeled objects into k nonempty, unlabeled subsets. For $|\lambda| = n$, we define the set of λ-*restricted permutations* to be permutations where $1, 2, \ldots, \lambda_1$ must appear in order from left to right, $\lambda_1 + 1, \ldots, \lambda_1 + \lambda_2$ appear in order, etc., but these sequences may be shuffled among each other. We define the λ-*restricted Eulerian number* $A(\lambda,k)$ to be the number of λ-restricted permutations in S_n with k descents.

We first became interested in Conjecture 2.2.1 and 2.2.2 due to the following observation, which follows from the normalization property of Jacks and the way that α^n is written in these two bases.

Proposition 2.2.3 ([1])**.** *For a partition* λ*, the coefficient of* m_{1^n} *in* $\tilde{J}_\lambda^{(\alpha)}(x_1,\ldots,x_n)$ *is*

$$\langle \tilde{J}_\lambda^{(\alpha)}(x_1,\ldots,x_n), h_{1^n}\rangle = n!\alpha^n = \sum_{k=0}^{n-1} n! A(n,k)\binom{\alpha+k}{n} = \sum_{k=1}^{n} n! S(n,k)\binom{\alpha}{k} k!. \quad (2.2.2)$$

Combined with computer data confirming the two conjectures up to $n = 11$, we have a first hint that there may be some nice combinatorics here. Two immediate corollaries come from extracting the coefficient of m_{1^n} from s_λ in the Schur expansion of $\tilde{J}_\lambda^{(\alpha)}$.

Corollary 2.2.4 ([1]). *Given a partition λ, we have*

$$\sum_{|\mu|=n} \sum_{k=0}^{n-1} a_k(\lambda,\mu) K_{\mu(1^n)} \binom{\alpha+k}{n} = \sum_{k=0}^{n-1} n! A(n,k) \binom{\alpha+k}{n}. \tag{2.2.3}$$

Corollary 2.2.5 ([1]). *Given a partition λ, we have*

$$\sum_{|\mu|=n} \sum_{k=1}^{n-1} b_{n-k}(\lambda,\mu) K_{\mu(1^n)} \binom{\alpha}{k} k! = \sum_{k=1}^{n-1} n! S(n,k) \binom{\alpha}{k} k!. \tag{2.2.4}$$

Furthermore, if $a_k(\lambda,\mu) \in \mathbb{N}[\alpha]$ or $b_k(\lambda,\mu) \in \mathbb{N}[\alpha]$ in general, then the respective result above would indicate some refinement on Eulerian numbers or Stirling numbers of the second kind.

2.2.2 Quasi-Yamanouchi tableaux

In the case of $\lambda = (n)$ and $|\mu| = n$, we noticed that the equality

$$\sum_{k=0}^{n-1} a_k((n),\mu) = |\mathrm{SYT}(\mu)| \tag{2.2.5}$$

held for the computer generated data. Upon closer inspection, it appeared that in fact the following theorem was true.

Theorem 2.2.6 ([1]). *Let μ be a partition of n and μ' be its conjugate. Then for the coefficient of s_μ in $\tilde{J}_\lambda^{(\alpha)}(x_1,\ldots,x_n)$,*

$$\langle \tilde{J}_{(n)}^{(\alpha)}(x_1,\ldots,x_n), s_\mu \rangle = \sum_{k=0}^{n-1} a_k((n),\mu) \binom{\alpha+k}{n}, \tag{2.2.6}$$

we have $a_k((n),\mu) = n!\mathrm{QYT}_{=k+1}(\mu')$.

We split the proof into several parts, starting with the coefficient of m_μ in $J_\lambda^{(\alpha)}(X)$. By example 3 in chapter IV, section 10 of Macdonald [31], this is $\frac{n!}{\mu!}\prod_{s\in\mu}(arm(s)\alpha + 1)$ where $\mu! = \mu_1!\mu_2!\cdots$. Converting to $\tilde{J}_\lambda^{(\alpha)}(X)$, this becomes $\frac{n!}{\mu!}\prod_{s\in\mu}(\alpha + arm(s))$. The next step is to convert these coefficients to the new basis.

Lemma 2.2.7 ([1])**.** *Given a partition* μ *of* n*,*

$$\frac{n!}{\mu!}\prod_{s\in\mu}(\alpha + arm(s)) = n!\sum_{k=0}^{n-1} A(\mu,k)\binom{\alpha+n-1-k}{n}$$

where $A(\mu,k)$ *is the number of* μ*-restricted permutations with* k *descents.*

Proof. Cancel $n!$ and rewrite the left hand side to get

$$\prod_{i=1}^{\ell(\mu)}\binom{\alpha+\mu_i-1}{\mu_i} = \sum_{k=0}^{n-1} A(\mu,k)\binom{\alpha+n-1-k}{n}.$$

We show that this equality holds with a bijection. Assume $\alpha \in \mathbb{N}$ and $\alpha \geq n$. On the left hand side we count diagrams where we take a rectangle of cells with $\alpha - 1$ rows and $\ell(\mu)$ columns, then adjoin the conjugate shape of μ at the bottom. In the ith column of the diagram, we choose μ_i many cells and mark them with dots. On the right hand side, we count pairs where, for some k, the first element is a μ-restricted permutation with k descents and the second element is a column of cells of height $\alpha + n - 1 - k$ with n cells marked by dots. Given a diagram counted by the left hand side, we apply the following algorithm to get a pair counted by the right hand side.

1. Label the dots in the diagram so that the first column's dots read $1,\ldots,\mu_1$ from top to bottom, the second column's dots read $\mu_1+1,\ldots,\mu_2$ from top to

bottom, etc. Extend the diagram downwards by adjoining cells to the bottom (without moving any dots) so that it becomes a rectangle of height $\alpha + n - 1$ and width $\ell(\mu)$. Set $i = 1$ and start with a pair where the first element is the empty word and the second element is a column of height 0.

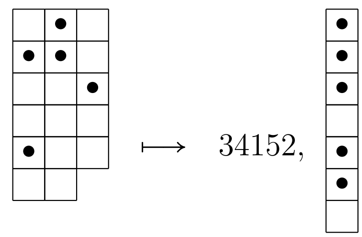
2. Read across row i, where rows are counted starting from the top. If there is no dot, go to 3a. If there is a dot in this row, go to 3b.

3. (a) Do nothing to the word in your pair and adjoin a blank cell to the bottom of your column. Go to step 4.

   (b) Adjoin the label of the first dot from the left in this row to the end of the word in your pair. If this is not a descent, add a new cell to the bottom of the column in your pair and mark it with a dot. If this is a descent, then do not add a new box to the bottom of the column, but do mark the bottom cell in the column (which will be blank if it is a descent) with a dot. Delete the dot that was hit and and push all dots that are not in the same column down by one row. Go to step 4.

4. If $i = \alpha + n - 1$, then terminate the algorithm, else increment i by one and go back to step 2.

Example 2.2.8. An example of the algorithm for $\mu = (2, 2, 1)$.



The algorithm must terminate, because the loop always goes through step 4. To show that it is well-defined, we need to check that no dot can be pushed below row $\alpha + n - 1$. If we consider a dot in column i at the lowest possible position, row $\alpha + \mu_i - 1$, then it needs to be pushed down $n - \mu_i + 1$ times to leave the diagram. However, there are only $n - \mu_i$ dots outside of this column that can contribute to pushing this dot down, so no dot can be pushed outside of the diagram.

We can obtain an inverse by reversing the steps of the algorithm. In this direction, the row that a dot comes from is encoded by both the marked column and the μ-restricted permutation, and the μ-restricted permutation also encodes which column a dot comes from. By similar reasoning as above, we also have to end with all dots in column i at or above row $\alpha + \mu_i - 1$. This algorithm gives a bijection that holds for any $\alpha \geq n$. Since both sides of the equality we are trying to prove are finite degree polynomials in α, this is sufficient to prove equality. □

We now wish to relate these μ-restricted Eulerian numbers to quasi-Yamanouchi tableaux. We can achieve this using RSK.

Lemma 2.2.9 ([1]). *Given a partition μ of n, it holds that*

$$\sum_{k=0}^{n-1} A(\mu,k)\binom{\alpha+n-1-k}{n} = \sum_{\substack{|\nu|=n \\ \nu\geq\mu}} K_{\nu\mu}\sum_{k=0}^{n-1} \mathrm{QYT}_{=k+1}(\nu)\binom{\alpha+n-1-k}{n}.$$

Proof. By comparing coefficients of $\binom{\alpha+n-1-k}{n}$, it is sufficient to show that for a fixed k,

$$A(\mu,k) = \sum_{\substack{|\nu|=n \\ \nu\geq\mu}} K_{\nu\mu}\mathrm{QYT}_{=k+1}(\nu).$$

We prove this through a bijection between μ-restricted permutations with k descents and pairs of tableaux (P,Q) of the same shape, where P is a standard Young tableau with $k+1$ runs and Q is a semistandard Young tableaux with weight μ.

Given a μ-restricted permutation π with k descents, obtain π' by decrementing all integers $1+\sum_{i=0}^{j}\mu_i,\ldots,\sum_{i=0}^{j+1}\mu_i$ to $j+1$ where $\mu_0=0$ for all $0\leq j<\ell(\mu)$. Create a two line array with π' in the top row and the integers $1,\ldots,n$ in order on the bottom, then reorder this array so that the columns have pairs in lexicographic order. Map this array via RSK to a pair (P,Q), where P is a standard Young tableau and Q is a semistandard Young tableau with weight μ. We want to show that P has $k+1$ runs.

If $\pi_i<\pi_{i+1}$, then $i+1$ is inserted after i in P, and RSK will keep i and $i+1$ in the same run of P. If $\pi_i>\pi_{i+1}$, then $i+1$ must be inserted before i. In this case, RSK will force $i+1$ to stay weakly left of i. Thus, descents in π correspond to descents in P, and P has $des(P)+1=k+1$ runs. This shows that RSK maps the two line arrays defined by μ-restricted permutations to the desired set of pairs (P,Q). It remains to show that the inverse map has image contained in the μ-restricted permutations.

Take some pair of tableaux (P, Q) where P is standard with $k+1$ runs and Q is semistandard with weight μ. The inverse map will give a two line array that, when rearranged to give $1, \ldots, n$ on the bottom row, will give a descent in the top row between columns i and $i+1$ exactly when $i+1$ starts a new run in P. The top row will also have weight μ, since Q has weight μ. The decrementing process described above on μ-restricted permutations has a natural inverse, so we reverse that process and end up with a μ-restricted permutation with k descents in the top row of the array as desired. Since RSK is a bijection, we know that both directions are injective, so the proof is complete.

□

Corollary 2.2.10 ([1])**.** *It holds that*

$$\sum_{|\mu|=n}\sum_{k=0}^{n-1} A(\mu,k)\binom{\alpha+n-1-k}{n} m_\mu = \sum_{|\nu|=n}\sum_{k=0}^{n-1} \mathrm{QYT}_{=k+1}(\nu)\binom{\alpha+n-1-k}{n} s_\nu.$$

Proof. We proceed by induction on the poset of partitions induced by the dominance order. For a given partition μ, our inductive hypothesis is that the coefficient of s_ν matches the claim for all $\nu > \mu$. From this, we show that the coefficient of s_μ is correct as well.

First we need the base case. $A((n),k) = 1$ when $k = 0$ and is zero otherwise. Therefore, the coefficient of m_n on the left hand side is $\binom{\alpha+n-1}{n}$. On the right hand side, we can only look to the expansion of s_n to get an m_n term, so it is clear that the coefficient of s_n on this side must also be $\binom{\alpha+n-1}{n}$. By definition of quasi-Yamanouchi tableaux, $\mathrm{QYT}_{=1}((n)) = 1$, which confirms the base case.

Now let μ be an arbitrary partition of size n and assume the inductive hypothesis. The expansion of Schur functions into monomial symmetric functions forces the coefficient of m_μ on either side to be

$$\sum_{k=0}^{n-1} A(\mu,k)\binom{\alpha+n-1-k}{n} = \sum_{k=0}^{n-1} C_{\mu,k+1}\binom{\alpha+n-1-k}{n} + \sum_{\substack{|\nu|=n\\ \nu>\mu}} K_{\nu\mu}\sum_{k=0}^{n-1}\mathrm{QYT}_{=k+1}(\nu)\binom{\alpha+n-1-k}{n}, \tag{2.2.7}$$

where $\sum_{k=0}^{n-1} C_{\mu,k+1}\binom{\alpha+n-1-k}{n}$ is the coefficient of s_μ and the other sum comes from each s_ν for $\nu > \mu$. Applying Lemma 2.2.9 immediately proves that $C_{\mu,k+1} = \mathrm{QYT}_{=k+1}(\mu)$, completing the inductive argument. □

Linking these together and applying Lemma 2.1.7 completes the proof of Theorem 2.2.6, which shows that the Schur expansion of $\tilde{J}^{(\alpha)}_{(n)}(x_1,\ldots,x_n)$ is in fact a generating function for quasi-Yamanouchi tableaux up to a constant of $n!$, thus proving Conjecture 1 for the case of $\lambda = (n)$. Chen, Yang, and Zhang [11] adapted a result by Brenti [9] to show that the polynomial $\sum_{T\in\mathrm{SYT}(\mu)} t^{des(T)}$ has only real zeroes. Using this and the definition of quasi-Yamanouchi tableaux, Theorem 2.2.6 also proves the $\lambda = (n)$ case of the second part of Conjecture 1.

2.2.3 Fundamental quasisymmetric expansion

One more look at Theorem 2.2.6 takes us into a brief digression towards the fundamental quasisymmetric expansion. First, we will need Assaf's dual equivalence graphs [4], although not in their full generality. Define the *elementary dual equiv-*

alence involution d_i on $\pi \in S_n$ for $1 < i < n$ by $d_i(\pi) = \pi$ if i occurs between $i-1$ and $i+1$ in π and by $d_i(\pi) = \pi'$ where π' is π with the positions of i and whichever of $i \pm 1$ is further from i interchanged when they do not appear in order. Two permutations π and τ are *dual equivalent* when $d_{i_1} \cdots d_{i_k}(\pi) = \tau$ for some $i_1, \ldots, i_k$. The reading word of a tableau is obtained by reading the entries from left to right, top to bottom, which for standard Young tableaux produces a permutation, and two standard Young tableaux of the same shape are dual equivalent if their reading words are. We also use Assaf's [4] characterization of Gessel's expansion of the Schur function into the fundamental quasisymmetric basis,

$$s_\lambda = \sum_{T \in [T_\lambda]} F_{Des(T)}(x), \tag{2.2.8}$$

where $[T_\lambda]$ is the dual equivalence class of all standard Young tableaux of shape λ, which is in fact all standard Young tableaux of shape λ.

In the $\binom{\alpha+k}{n}$ basis, we can obtain the fundamental quasisymmetric expansion of $\tilde{J}^{(\alpha)}_{(n)}(X)$ as a corollary of the following result.

Theorem 2.2.11 ([1])**.** *It holds that*

$$\sum_{\pi \in S_n} t^{des(\pi)} F_{Des(P(\pi))}(x) = \sum_{|\mu|=n} \sum_{k=0}^{n-1} \mathrm{QYT}_{=k+1}(\mu) t^k s_\mu, \tag{2.2.9}$$

where $P(\pi)$ is the insertion tableau of π given by RSK.

Proof. Connect all $\pi \in S_n$ with colored edges corresponding to elementary dual equivalence involutions to get a graph G. By looking at properties of the bump paths in RSK, we can see that RSK respects dual equivalence relations in the P

insertion tableaux, so applying RSK to every vertex to get G' maintains the edge relations between the P tableaux. For μ a partition, let G_μ be the dual equivalence graph on $\mathrm{SYT}(\mu)$. Each connected component of G' will be isomorphic to G_μ for some $|\mu| = n$, and furthermore, there will be exactly $|\mathrm{SYT}(\mu)|$ copies of G_μ contained in G' for each μ.

Dual equivalence relations do not change the descent set of a permutation, and the number of descents of a permutation is equal to the number of runs of its Q recording tableau minus one. Therefore, since the descent set is constant on the vertices of a connected component of G, the number of runs of each corresponding Q tableau is also constant. Among pairs (P, Q) of shape μ, Q ranges over all $\mathrm{SYT}(\mu)$ with $|\mathrm{SYT}(\mu)|$ of each appearing, so a counting argument tells us there must be exactly $\mathrm{QYT}_{=k}(\mu)$ many connected components isomorphic to G_μ which have k runs in each of the Q tableaux of its vertices. Then taking the sum

$$\sum_{\pi\in S_n} t^{des(\pi)} F_{Des(P(\pi))}(x) = \sum_{(P,Q)\in G'} F_{Des(P)}(x) t^{des(Q)} \tag{2.2.10}$$

and applying the expansion of Schur functions into the fundamental quasisymmetric basis to the right hand side completes the proof. □

Corollary 2.2.12 ([1])**.** *It holds that*

$$\tilde{J}^{(\alpha)}_{(n)}(X) = n! \sum_{\pi\in S_n} \binom{\alpha + n - 1 - des(\pi)}{n} F_{Des(P(\pi))}, \tag{2.2.11}$$

where $P(\pi)$ is the insertion tableau of π given by RSK.

Proof. Apply Theorems 2.2.6 and 2.2.11 and Lemma 2.1.7. □

This result prompted the following conjecture on the quasisymmetric expansion for general partitions λ.

Conjecture 2.2.13 ([1]). *For a partition λ of size n, it holds that*

$$\tilde{J}_\lambda^{(\alpha)}(X) = \sum_{\pi,\tau \in S_n} \binom{\alpha + n - 1 - des(\pi)}{n} F_{\sigma(\pi,\tau,\lambda)}(x) \tag{2.2.12}$$

for some set-valued function σ depending on π, τ, and λ and with image in $\{1, \ldots, n-1\}$.

Corollary 2.2.12 proves this in the case of $\lambda = (n)$, where $\sigma(\pi, \tau, (n)) = Des(P(\pi))$, and Proposition 2.2.3 proves it in the case of $\lambda = (1^n)$, where $\sigma(\pi, \tau, (1^n)) = \{1, \ldots, n-1\}$ for all $\pi, \tau \in S_n$. Furthermore, if we momentarily assume that the Jack polynomials are Schur positive in this basis, Corollary 2.2.4 along with the expansion of Schurs into fundamental quasisymmetrics shows that this conjecture is true in general for some σ, although it does not tell us what σ should be. Finally, while the fundamental quasisymmetric expansion would be interesting in its own right, it may also lead to a proof of Schur positivity by a generalization of the method used in Theorem 2.2.11. Corollary 2.2.12 and Conjecture 5 have the following analogous conjectures in the $\binom{\alpha}{k}k!$ basis, where B_n is the set of set partitions of $\{1, \ldots, n\}$.

Conjecture 2.2.14 ([1]). *For a partition λ of size n, it holds that*

$$\tilde{J}_\lambda^{(\alpha)}(X) = \sum_{\substack{\pi \in S_n \\ \beta \in B_n}} \binom{\alpha}{|\beta|} |\beta|! F_{\rho(\pi,\beta,\lambda)}(x) \tag{2.2.13}$$

for some set-valued function ρ depending on π, β, and λ with image in $\{1, \ldots, n-1\}$.

For the $\lambda = (n)$ case, we first define a function. Given $\pi \in S_n$ and $\beta \in B_n$, define $f_\beta(\pi)$ to be a rearrangement of π so that if $\{b_1, \ldots, b_k\} \in \beta$, then $b_1, \ldots, b_k$ appear in increasing order in $f_\beta(\pi)$ without changing the position of the subsequence. For example, given $\beta = \{\{1,4\},\{2,3,5\}\}$ and $\pi = 24531$, $f_\beta(\pi) = 21354$.

Conjecture 2.2.15 ([1]). *It holds that*

$$\tilde{J}_{(n)}^{(\alpha)}(X) = \sum_{\substack{\pi \in S_n \\ \beta \in B_n}} \binom{\alpha}{|\beta|} |\beta|! F_{Des(P(f_\beta(\pi)))}(x), \tag{2.2.14}$$

where $P(f_\beta(\pi))$ is the insertion tableau of $f_\beta(\pi)$ given by RSK.

Proposition 2.2.3 also proves the $\lambda = (1^n)$ case here, and we can make similar remarks as above. That is, if we assume Schur positivity, that Conjecture 2.2.14 is true for some ρ and that the fundamental quasisymmetric expansion could help prove Schur positivity in this basis.

2.2.4 Rook Boards

Returning to the problem of Schur positivity, we also had some success approaching the problem with rook boards. Given an $n \times n$ grid, we can choose a subset B, which we call a *board*. The kth *rook number* of B, denoted $r_k(B)$, is the number of ways to place k nonattacking rooks on B, and the kth *hit number* of B, denoted $h_k(B)$, is the number of ways to place n nonattacking rooks on the grid with exactly k on B. A *Ferrers board* is one where if (x,y) is in B, then every (i,j) weakly southeast is also in B. We will use the following result from Goldman, Joichi, and White [17] which translates certain products of factors into each of our bases.

Proposition 2.2.16 ([17]). *Let $0 \leq c_1 \leq c_2 \leq \cdots \leq c_n \leq n$ with $c_i \in \mathbb{N}$, and let $B = B(c_1, \ldots, c_n)$ be the Ferrers board whose ith column has height c_i. Then*

$$\prod_{i=1}^{n}(\alpha + c_i - i + 1) = \sum_{k=0}^{n} h_k(B)\binom{\alpha + k}{n} = \sum_{k=0}^{n} r_{n-k}(B)\binom{\alpha}{k}k!$$

We first use Proposition 2.2.16 to obtain a combinatorial interpretation of the coefficient of s_μ in our binomial bases when $\lambda = \mu$ for a hook shape. In general for $\lambda = \mu$, the coefficient of s_μ is the same as the coefficient of m_μ in the monomial expansion, so we can obtain from the combinatorial formula for the monomial expansion [26] that

$$\langle \tilde{J}_\lambda^{(\alpha)}(x_1, \ldots, x_n), s_\lambda \rangle = \prod_{s \in \lambda}(arm(s) + \alpha(leg(s) + 1)). \tag{2.2.15}$$

When $\lambda = (n - \ell, 1^\ell)$ is a hook shape, this product becomes

$$\begin{aligned} \langle \tilde{J}_\lambda^{(\alpha)}(X),\, s_\lambda(X) \rangle &= \ell! \alpha^\ell ((\ell + 1)\alpha + (n-1))(\alpha + (n-2)) \cdots (\alpha + 1)\alpha \\ &= \ell \cdot \ell! (\alpha + (n - \ell - 2)) \cdots (\alpha + 1)\alpha^{\ell+2} + \ell!(\alpha + (n - \ell - 1)) \cdots (\alpha + 1)\alpha^{\ell+1}, \end{aligned} \tag{2.2.16}$$

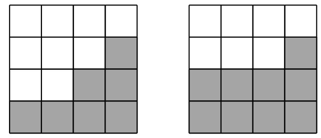
then applying Proposition 2.2.16 gives the following result.

Proposition 2.2.17 ([1]). *For $\lambda = \mu = (n - \ell, 1^\ell)$, we have*

$$\begin{aligned} \langle \tilde{J}_\lambda^{(\alpha)}(X), s_\lambda \rangle &= \sum_{k=0}^{n} (\ell \cdot \ell! h_k(B(c_1, \ldots, c_n)) + \ell! h_k(B(d_1, \ldots, d_n)))\binom{\alpha + k}{n} \\ &= \sum_{k=0}^{n} (\ell \cdot \ell! r_k(B(c_1, \ldots, c_n)) + \ell! r_k(B(d_1, \ldots, d_n)))\binom{\alpha}{k}k! \end{aligned} \tag{2.2.17}$$

where $c_1 = c_2 = \cdots = c_{n-\ell-1} = n - \ell - 2$ and $c_{n-\ell+i} = n - \ell - 1 + i$ for $0 \geq i \geq \ell$ and $d_1 = d_2 = \cdots = d_{n-\ell} = n - \ell - 1$ and $d_{n-\ell+i} = n - \ell - 1 + i$ for $1 \geq i \geq \ell$.

Example 2.2.18. $B(c_1,\dots,c_4)$ and $B(d_1,\dots,d_4)$ for $\lambda = \mu = (3,1)$.



This approach also yields a combinatorial interpretation for both bases in the case of $\lambda = (n)$. We can obtain $J_\lambda^{(1/\alpha)}(x_1,\dots,x_n)$ via the specialization of Macdonald polynomials $J_\lambda(x_1,\dots,x_n;q,t)$:

$$\begin{aligned} J_\lambda^{(1/\alpha)}(x_1,\dots,x_n) &= \lim_{t\to 1} \frac{J_\lambda(x_1,\dots,x_n;t^{1/\alpha},t)}{(1-t)^n} = \lim_{q\to 1} \frac{J_\lambda(x_1,\dots,x_n;q,q^\alpha)}{(1-q^\alpha)^n} \\ &= \lim_{q\to 1} \frac{J_\lambda(x_1,\dots,x_n;q,q^\alpha)}{(1-q)^n} \frac{(1-q)^n}{(1-q^\alpha)^n} = \frac{1}{\alpha^n} \lim_{q\to 1} \frac{J_\lambda(x_1,\dots,x_n;q,q^\alpha)}{(1-q)^n} \end{aligned} \tag{2.2.18}$$

so that

$$\tilde{J}_\lambda^{(\alpha)}(x_1,\dots,x_n) = \lim_{q\to 1} \frac{J_\lambda(x_1,\dots,x_n;q,q^\alpha)}{(1-q)^n}. \tag{2.2.19}$$

Then when $\lambda = (n)$, we can apply the limit as $q \to 1$ to a result of Yoo [38, Theorem 3.2] to obtain

$$\lim_{q\to 1} \frac{J_{(n)}(x_1,\dots,x_n;q,q^\alpha)}{(1-q)^n} = \sum_{|\mu|=n} s_\mu K_{\mu,1^n} \prod_{(i,j)\in\mu} (\alpha + i - j), \tag{2.2.20}$$

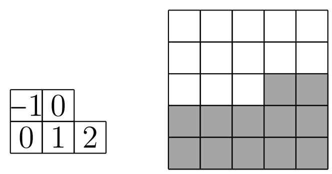
where $(i,j) \in \mu$ refers to cells of the diagram of μ identified with their Cartesian coordinates. Arrange the values of $i - j$ in non-increasing order and rewrite to get the desired $\prod_{i=1}^n (\alpha + c_i - i + 1)$ form. It is clear that for any partition μ, this produces a sequence $0 \le c_1 \le \cdots \le c_n \le n$, so we can apply Proposition 2.2.16 again to obtain the following.

Theorem 2.2.19 ([1])**.** *It holds that*

$$\langle \tilde{J}^{(\alpha)}_{(n)}(X), s_\mu \rangle = \sum_{k=0}^{n} K_{\mu,1^n} h_k(B(c_1,\ldots,c_n)) \binom{\alpha+k}{n} = \sum_{k=0}^{n} K_{\mu,1^n} r_{n-k}(B(c_1,\ldots,c_n)) \binom{\alpha}{k} k!, \tag{2.2.21}$$

with $c_1,\ldots,c_n$ *given above.*

Example 2.2.20. The values $i-j$ and $B(c_1,\ldots,c_5)$ for $\mu=(3,2)$.



In [19] it is shown that the rook and hit polynomials of Ferrers boards have only real zeros, so the two results of this section also prove the second part of Conjecture 2.2.1 for these special cases. We note that Theorem 2.2.19 provides a very different looking combinatorial interpretation to the one seen in Theorem 2.2.6 for the $\binom{\alpha+k}{n}$ basis. It would be interesting to find a relationship between the rook board interpretation of Theorem 2.2.19 and the tableau interpretation of Theorem 2.2.6. We explore this briefly among other combinatorics related to quasi-Yamanouchi tableaux in the next section.

2.3 Quasi-Yamanouchi tableaux

Tableaux play a substantial role in the previous sections as a way of interpreting certain families of polynomials as generating functions over tableaux. Therefore, we may gain further understanding of these polynomials by proxy by exploring the combinatorics of the tableaux themselves.

To begin with, we can ask whether there is a nice enumeration of these objects, in the same way that SYT and SSYT have the hook-length and hook-content formulas respectively. We explored the possibility of a product formula in [36] and found one for the special case of Durfee size two. However, computations revealed that large primes begin appearing very quickly after generalizing from this case, so a general product formula is unlikely. Similar work was done by Keith on enumeration formulas for descents and major index of standard Young tableaux [24, 25], which translates through the correspondence of QYT with SYT.

Instead of searching for a product formula, we then began focusing our investigation on the relation of quasi-Yamanouchi tableaux to other combinatorial objects. One such example is noted in Section 2.2 by comparing the quasi-Yamanouchi tableau interpretation of the coefficients of Jack polynomials with the rook board interpretation.

2.3.1 Rook Boards

If λ is a partition of size n, then we can construct a Ferrers board B_λ as follows. Take the contents $c_1, \ldots, c_n$ of λ arranged in weakly decreasing order, then let the heights of the columns of B_λ be $(c_i + i - 1)$. Let $B_\lambda \times 1$ be the board obtained by incrementing the height of every column by one. We note that it is always possible to do this once for any B_λ, as the construction never creates a column with height n. From the definitions, we get a relation between B_λ and $B_{\lambda'}$.

Proposition 2.3.1. *Given a partition λ, the complement of $B_\lambda \times 1$ is $B_{\lambda'}$ up to rotation.*

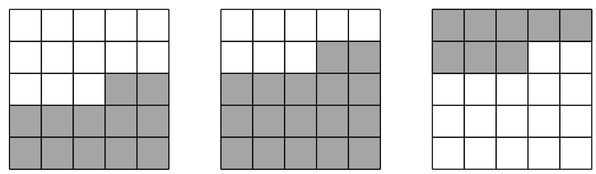


Figure 2.6: $B_{(3,2)}$, $B_{(3,2)} \times 1$, and $B_{(2,2,1)}$ rotated.

By comparing the two interpretations of Jack polynomial coefficients, we can then obtain the following hit number formula for quasi-Yamanouchi tableaux.

Theorem 2.3.2. *Given a partition λ of n and $0 \le k \le n-1$,*

$$\mathrm{QYT}_{=k+1}(\lambda) = \frac{h_k(B_{\lambda'})}{\prod_{u\in\lambda} h(u)}. \tag{2.3.1}$$

In this section, we prove two q-analogues of this theorem. The first q-analogue weights each tableaux by its major index and the second by its charge, and we relate these to *q-hit numbers* of rook boards. Dworkin [14] gave a combinatorial interpretation of a q-analogue of hit numbers for Ferrers boards, which we use as the definition. For $\pi \in S_n$, place a cross at each square in $\Gamma(\pi)$, and for any square to the right of a cross, put a bullet. Then from each cross, draw circles going up and wrapping around the top edge of the $[n] \times [n]$ array, skipping over bullets, and stopping after hitting the top border of the Ferrers board. The q weight of π is the number of circles at the end of this process. The kth q-hit number $T_k(B)$ is the sum of q weights over all permutations that hit the board exactly k times.

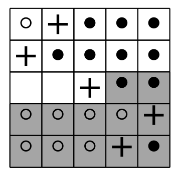


Figure 2.7: The q weight of 45312 on $B_{3,2}$ is 8.

In order to relate these q-analogues of tableau statistics and rook board statistics, we use the theory of posets and (P, ω)-partitions, which were introduced by Stanley [34].

For a partition $\lambda = (\lambda_1, \ldots, \lambda_k)$, let P_λ be the subposet of $\mathbb{N} \times \mathbb{N}$ such that $(i, j) \in P_\lambda$ if $1 \leq j \leq k$, $1 \leq i \leq \lambda_j$. Given a poset P with n elements, a *labeling* ω is a map $\omega : P \to [n]$. It is called a *natural* labeling if it is order preserving and *strict* if it is order reversing. For P_λ, there are also *column-strict* labelings, which are strict on columns and natural on rows.

For a fixed ω, a (P, ω)*-partition* of size p is a map $\sigma : P \to \mathbb{N}_{\geq 0}$ satisfying

1) $x \leq y \in P \implies \sigma(x) \geq \sigma(y)$, meaning σ is order reversing.

2) $x < y \in P$ and $\omega(x) > \omega(y) \implies \sigma(x) > \sigma(y)$.

3) $|\sigma| = \sum_{x \in P} \sigma(x) = p$.

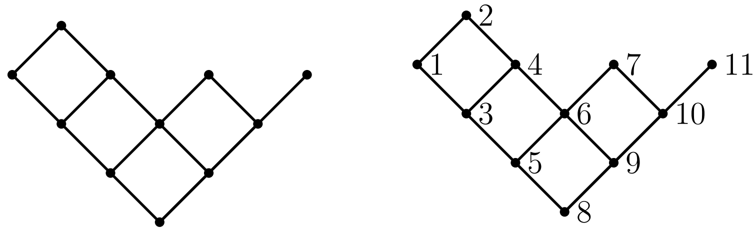


Figure 2.8: $P_{4,3,2,2}$ and a column-strict labeling of $P_{4,3,2,2}$.

The values $\sigma(x)$ are called the parts of σ, and a $(P,\omega;m)$-partition is a (P,ω)-partition with largest part at most m. $\mathcal{A}(P,\omega)$ denotes the set of (P,ω)-partitions, and $\mathcal{A}(P,\omega;m)$ denotes the set of $(P,\omega;m)$-partitions, which have generating function

$$U_m(P,\omega;m) = \sum_{\sigma\in\mathcal{A}(P,\omega;m)} q^{|\sigma|}.$$

The ω-*separator* $\mathcal{L}(P,\omega)$ is the set of permutations in S_n of the form $\omega(x_{i_1}),\ldots,\omega(x_{i_n})$ where $x_{i_1} < \ldots < x_{i_n}$ forms a linear extension of P. For each $0 \le k \le n-1$, define

$$W_k(P,\omega) = W_k(P,\omega;q) = \sum_{\pi} q^{\mathrm{maj}(\pi)},$$

where the sum is over all $\pi \in \mathcal{L}(P,\omega)$ with $des(\pi) = k$.

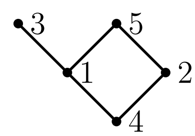


Figure 2.9: $\mathcal{L}(P,\omega) = \{42153, 42135, 41235, 41253, 41325\}$

and $W_2(P,\omega;q) = q^3 + q^4 + q^5$.

Major index formula

Fix a partition λ, and let ω be a column-strict labeling on P_λ. By Proposition 21.3 of [34],

$$U_m(P_\lambda, \omega; m) = q^{n(\lambda)} \prod_{u \in \lambda} \frac{[m + c(u) + 1]}{[h(u)]}.$$

By the definition of $W_k(P_\lambda, \omega)$, when ω is column-strict,

$$W_k(P_\lambda, \omega) = \sum_{T \in \mathrm{QYT}_{=k+1}(\lambda)} q^{\mathrm{maj}(T)}.$$

Proposition 8.2 of [34] gives

$$U_m(P_\lambda, \omega) = \sum_{k=0}^{n-1} \begin{bmatrix} m+n-k \\ n \end{bmatrix} W_k(P_\lambda, \omega).$$

Then since there is no restriction on m, it follows that

$$\sum_{k=0}^{n-1} \begin{bmatrix} x+n-k \\ n \end{bmatrix} W_k(P_\lambda, \omega) = q^{n(\lambda)} \prod_{u \in \lambda} \frac{[x + c(u) + 1]}{[h(u)]}.$$

We then apply to the right hand side the following q-analogue [18] of the Goldman, Joichi, Write identity [17]

$$\prod_{i=1}^{n} [x + b_i - i + 1] = \sum_{k=0}^{n} \begin{bmatrix} x+k \\ n \end{bmatrix} T_k(B),$$

where B is a Ferrers board with column heights b_i. Comparing coefficients of $\left[\begin{smallmatrix} x+k \\ n \end{smallmatrix}\right]$ gives the following theorem.

Theorem 2.3.3. *Given a partition λ and $0 \le k \le n-1$,*

$$\sum_{T \in QYT_{=k+1}(\lambda)} q^{maj(T)} = \frac{q^{n(\lambda)}}{\prod_{u \in \lambda} [h(u)]} T_{n-k}(B_\lambda \times 1).$$

Setting $q = 1$ and applying Proposition 2.3.1 recovers Theorem 2.3.2. We note that since $T_k(B)$ is Mahonian [14] for a Ferrers board, summing over k gives a nice (known) q-analogue of the hook-length formula,

$$\sum_{T \in \mathrm{SYT}(\lambda)} q^{\mathrm{maj}(T)} = \frac{q^{n(\lambda)}[n]!}{\prod_{u \in \lambda}[h(u)]}.$$

We briefly attempted to prove Theorem 2.3.3 bijectively but were unsuccessful. It would be nice to know what such a bijective algorithm might look like, and such an algorithm could be an interesting project to revisit in the future.

Charge formula

Fix a permutation λ of size n, and let ω be a column-strict labeling on P_λ. We write P_λ^* for the dual of P_λ and write ω^* for the labeling defined by $\omega^*(x_i) = n + 1 - \omega(x_i)$ for all $x_i \in P_\lambda$. Proposition 12.1 of [34] details what this dualization on P_λ and ω does to W_k, which is that

$$W_k(P_\lambda^*, \omega^*; q) = q^{nk} W_k(P_\lambda, \omega; \frac{1}{q}).$$

We note that since there are k descents,

$$q^{nk} W_k(P_\lambda, \omega; \frac{1}{q}) = \sum_{T \in \mathrm{QYT}_{=k+1}(\lambda)} q^{nk - \mathrm{maj}(T)} = \sum_{T \in \mathrm{QYT}_{=k+1}(\lambda)} \sum_{i \in \mathrm{Des}(T)} q^{n-i}.$$

Then since a descent at position i increments the charge value of the $n - i$ remaining entries by one, we get

$$W_k(P_\lambda^*, \omega^*; q) = \sum_{T \in \mathrm{QYT}_{=k+1}(\lambda)} q^{ch(T)}.$$

Using the facts that $[k] \mapsto [k]_{\frac{1}{q^{k-1}}}$ when substituting $1/q$ and that $\sum_{u\in\lambda} h(u) = n + n(\lambda) + n(\lambda')$, we get

$$W_k(P_\lambda, \omega; \frac{1}{q}) = \frac{q^{n(\lambda')}}{\prod_{u\in\lambda}[h(u)]} T^*_{n-k}(B_\lambda \times 1),$$

where $T^*_k(B_\lambda \times 1)$ gives a weight of $1/q$ to each circle instead of q. Multiplying this by $q^{\binom{n}{2}}$ changes the circles to a q^0 weight and empty squares to a q^1 weight, which is identical to drawing circles downwards instead of upwards from crosses and giving circles a q^1 weight. Then by Proposition 2.3.1, taking the complement of the board and reflecting vertically gives $B_{\lambda'}$ up to column permutation. By Theorem 7.13 of [14], $T_k(B)$ is invariant on column permutations for Ferrers boards, so it follows that

$$T^*_{n-k}(B_\lambda \times 1) = \frac{T_k(B_{\lambda'})}{q^{\binom{n}{2}}}.$$

This gives the following result, which clearly reduces to Theorem 2.3.2 when $q = 1$.

Theorem 2.3.4. *Given a partition λ and $0 \le k \le n-1$,*

$$\sum_{T\in QYT_{=k+1}(\lambda)} q^{ch(T)} = \frac{q^{nk+n(\lambda')-\binom{n}{2}}}{\prod_{u\in\lambda}[h(u)]} T_k(B_{\lambda'}).$$

Summing over k in this case also gives some sort of q-analogue of the hook-length formula, although it does not appear to immediately give a nice form.

2.3.2 A summation formula

Although it has been noted that a product formula is too much to hope for, we were able to prove the following summation formula, which we prove in two ways. First we use a q-hit number identity and then use (P,ω)-partitions. This gives a relatively clean enumeration for quasi-Yamanouchi tableaux compared to the fairly messy product formula of [36], the downside being that it is not a positive summation.

Theorem 2.3.5. *Given a partition λ and $0 \le k \le n-1$*

$$\mathrm{QYT}_{=k+1}(\lambda) = \sum_{m=0}^{k} \binom{n+1}{k-m} (-1)^{k-m} \mathrm{SSYT}_{m+1}(\lambda).$$

First proof. Setting $t = n$ in equation (24) in [18] gives

$$\frac{T_{n-k}(B)}{\prod_{i=1}^{n}[d_i]!} = \sum_{m=0}^{k} \begin{bmatrix} n+1 \\ k-m \end{bmatrix} (-1)^{k-m} q^{\binom{k-m}{2}} \prod_{i=1}^{n} \begin{bmatrix} m+H_i-D_i+d_i \\ d_i \end{bmatrix},$$

where $d_i = 1$ for all i, $D_i = i$, and H_i is the height of the ith column of B. By the way B_λ is constructed, the sequence $H_i - D_i$ for $B_\lambda \times 1$ becomes exactly the cell contents of λ, so setting $q = 1$ gives

$$h_{n-k}(B_\lambda \times 1) = \sum_{m=0}^{k} \binom{n+1}{k-m} (-1)^{k-m} \prod_{i=1}^{n} (m + c_i + 1).$$

Substituting this into Theorem 2.3.2 after applying Proposition 2.3.1 and comparing with the hook-content formula proves Theorem 2.3.5. □

Second proof. When ω is a column-strict labeling on P_λ, $\mathcal{A}(P_\lambda,\omega;m)$ is the set of $\mathrm{SSYT}_{m+1}(\lambda)$ with each entry decremented by one. Therefore, setting $q = 1$ in $U_m(P_\lambda,\omega)$ gives $|\mathrm{SSYT}_{m+1}(\lambda)|$. Proposition 8.4 in [34] says that

$$W_k(P,\omega) = \sum_{m=0}^{k} (-1)^m q^{\binom{n}{2}} \begin{bmatrix} n+1 \\ m \end{bmatrix} U_{k-m}(P,\omega).$$

Then setting $q = 1$, and reversing the order of summation proves Theorem 2.3.5. □

2.3.3 Weighted lattice paths and the polynomials $P_{n,k}$

For any partition λ with $|\lambda| = n$, we can express $\mathrm{QYT}_{=k+1}(\lambda)$ in terms of certain symmetric functions $P_{n,k}$. We begin with Lemma 4 of [18], where we set $q = 1$, $t = n$, $d_i = 1$, $e_i \in \{0,1\}$, E_i the partial sums of the e_i, and $D_i = i$. We also recall as before that for $B = B_\lambda \times 1$, we have $H_i - D_i = c_i$, the cell contents of λ in some order. After all of that, we get

$$h_{n-k}(B_\lambda \times 1) = \sum_{e_1+\cdots+e_n=k} \prod_{i=1}^{n} \binom{c_i + E_i + d_i - e_i}{d_i - e_i} \binom{i - 1 - c_i - E_i + e_i}{e_i}.$$

Since exactly one of $d_i - e_i$ or e_i are 1 and the other is 0, we get

$$h_{n-k}(B_\lambda \times 1) = \sum_{e_1+\cdots+e_n=k} \prod_{i=1}^{n} (c_i + E_i + 1)^{d_i - e_i} (i - c_i - E_i)^{e_i}.$$

This is the same as summing over weighted lattice paths with n steps from $(0,0)$ to $(k, n-k)$. Let E_i count the cumulative east steps and $N_i = i - E_i$ count the cumulative north steps. Then for each path, weight the ith step by $x_i + E_i + 1$ if it is a north step and $N_i - x_i$ if it is an east step, and let the weight of a path be the product of the weights of its steps.

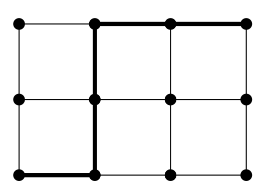


Figure 2.10: A path with weight $(-x_1)(x_2+2)(x_3+2)(2-x_4)(2-x_5)$.

Let $P_{n,k}(x_1, \ldots, x_n)$ denote the sum of the weights of all such paths. This gives the following weighted lattice path interpretation for QYT enumeration.

Theorem 2.3.6. *Given a partition λ of n with contents $c_1, \ldots, c_n$ and $0 \le k \le n$,*

$$\mathrm{QYT}_{=k+1}(\lambda) = \frac{P_{n,k}(c_1, \ldots, c_n)}{\prod_{u \in \lambda} h(u)}.$$

Such paths can be split recursively into ones that end on an east step and ones that end on a north step.

Proposition 2.3.7. *The polynomials $P_{n,k}$ satisfy the relation*

$$P_{n,k}(x_1, \ldots, x_n) = (x_n + k + 1)P_{n-1,k}(x_1, \ldots, x_{n-1}) + (n - k - x_n)P_{n-1,k-1}(x_1, \ldots, x_{n-1}).$$

We can use this to get a more concrete idea of what these polynomials look like. By their construction, it is not obvious that these polynomials are symmetric, but computing small cases seems to indicate they are.

$$\begin{aligned}
P_{1,0}(x_1) =& e_1(x_1) + 1 \\
P_{1,1}(x_1) =& e_1(x_1) \\
P_{2,0}(x_1, x_2) =& e_2(x_1, x_2) + e_1(x_1, x_2) + 1 \\
P_{2,1}(x_1, x_2) =& -2e_2(x_1, x_2) - e_1(x_1, x_2) + 1 \\
P_{2,2}(x_1, x_2) =& e_2(x_1, x_2) \\
P_{3,0}(x_1, x_2, x_3) =& e_3(x_1, x_2, x_3) + e_2(x_1, x_2, x_3) + e_1(x_1, x_2, x_3) + 1 \\
P_{3,1}(x_1, x_2, x_3) =& -3e_3(x_1, x_2, x_3) - 2e_2(x_1, x_2, x_3) + 4 \\
P_{3,2}(x_1, x_2, x_3) =& 3e_3(x_1, x_2, x_3) + e_2(x_1, x_2, x_3) - e_1(x_1, x_2, x_3) + 1 \\
P_{3,3}(x_1, x_2, x_3) =& e_3(x_1, x_2, x_3)
\end{aligned}$$

Let $a(n,k,m)$ denote the coefficient of e_m in $P_{n,k}$, and assume that $P_{i,k}$ is symmetric for $i < n$. Using the recursion, it is clear that the coefficient of the degree m monomials containing x_n in $P_{n,k}$ is $a(n-1,k,m-1) - a(n-1,k-1,m-1)$ and that the coefficient of the degree m monomials not containing x_n is $(k+1)a(n-1,k,m) + (n-k)a(n-1,k-1,m)$. Then to show that $P_{n,k}$ is symmetric, it is sufficient to show that $a(n-1,k,m-1) - a(n-1,k-1,m-1) = (k+1)a(n-1,k,m) + (n-k)a(n-1,k-1,m)$, which can be done with a straightforward induction argument.

Theorem 2.3.8. *Given a partition λ of n with contents $c_1, \ldots, c_n$ and $0 \leq k \leq n$,*

$$\mathrm{QYT}_{=k+1}(\lambda) = \frac{\sum_{m=0}^{n} a(n,k,m) e_m(c_1, \ldots, c_n)}{\prod_{u \in \lambda} h(u)}.$$

By the recursion and initial conditions, we have that $a(n,k,0)$ is the Eulerian number $A(n,k)$ and that for $1 < m \leq n$, it is easy to generate these coefficients recursively using the relation $a(n,k,m) = a(n-1,k,m-1) - a(n-1,k-1,m-1)$. We also note that for a fixed value of $n - m$ with varying n and k, this gives something close to a Pascal's triangle for the coefficients. Each term contributes its positive absolute value and its negative absolute value to the next line of the triangle, so summing over a line gives 0 except when $m = 0$. Therefore, summing over $P_{n,k}$ for all $0 \leq k \leq n$ leaves only the constant terms, and the hook-length formula is easily recovered.

$n = 3$			1	4	1	
$n = 4$		1	3	-3	-1	
$n = 5$	1	2	-6	2	1	
$n = 6$	1	1	-8	8	-1	-1

Figure 2.11: $a(n,k,m)$ for fixed $n - m = 3$ and $0 \le k \le n-1$ increasing along rows.

2.3.4 q-analogues of generating functions

The Schur basis generating function for quasi-Yamanouchi tableaux is

$$\sum_{|\lambda|=n} \sum_{k=1}^{n} \mathrm{QYT}_{=k}(\lambda) t^{k-1} s_\lambda, \tag{2.3.2}$$

which has a natural q-analogue

$$\sum_{|\lambda|=n} \sum_{T \in \mathrm{QYT}(\lambda)} q^{\mathrm{maj}(T)} t^{\mathrm{des}(T)} s_\lambda. \tag{2.3.3}$$

In this section, we present the fundamental quasisymmetric and monomial expansions of this q-analogue of the generating function. We note that the fundamental quasisymmetric expansion is an extension of Theorem 2.2.11. [*define dual knuth relations?*].

Theorem 2.3.9. *For $n \in \mathbb{N}$,*

$$\sum_{\pi \in S_n} q^{\mathrm{maj}(\pi)} t^{\mathrm{des}(\pi)} F_{\mathrm{Des}(\pi^{-1})}(x) = \sum_{|\lambda|=n} \sum_{T \in \mathrm{QYT}(\lambda)} q^{\mathrm{maj}(T)} t^{\mathrm{des}(T)} s_\lambda.$$

Proof. Connect all $\pi \in S_n$ by colored edges corresponding to dual Knuth relations to get a graph G and identify each permutation π with its image $(P(\pi), Q(\pi))$ through

RSK. Dual Knuth relations do not change the descent set of a permutation, and the descent set of a permutation corresponds to the descent set of its recording tableau $Q(\pi)$. Therefore, all permutations in a connected component of G have the same descent and major index statistics and map to the same recording tableau.

On the other hand, RSK respects dual Knuth relations between permutations and their insertion tableaux, so the equivalence classes formed by dual Knuth relations guarantee that the insertion tableaux on a connected component range over exactly all $T \in \mathrm{SYT}(\lambda)$ for some λ. The descent set of an insertion tableau $P(\pi)$ is the same as the descent set of π^{-1}. Then give each vertex of a connected component the weight $q^{\mathrm{maj}(\pi)}t^{\mathrm{des}(\pi)}F_{\mathrm{Des}(\pi^{-1})}(x)$ and apply Gessel's fundamental quasisymmetric expansion to show that each connected component has summed weight $q^{\mathrm{maj}(Q)}t^{\mathrm{des}(Q)}s_{sh(Q)}$, where Q is the recording tableau shared by the connected component. RSK forms a bijection between $\pi \in S_n$ and pairs of SYT (P,Q) of the same shape, so summing over all connected components of G, applying a counting argument, and using the correspondence between SYT and QYT completes the proof. $\square$

For the monomial symmetric function expansion, we use multiset permutations. We can define descents and major index for multiset permutations in the same way as for permutations in S_n, and we write S_λ for the set of multiset permutations of $\{1^{\lambda_1}, 2^{\lambda_2}, \ldots\}$.

Lemma 2.3.10. *Given a partition λ of n,*

$$\sum_{\pi \in S_\lambda} q^{\mathrm{maj}(\pi)} t^{\mathrm{des}(\pi)} = \sum_{\nu \geq \lambda} K_{\nu\lambda} \sum_{T \in \mathrm{QYT}(\nu)} q^{\mathrm{maj}(T)} t^{\mathrm{des}(T)}.$$

Proof. RSK gives a bijection between multiset permutations $\pi \in S_\lambda$ and pairs of tableaux (P, Q) of the same shape $\nu \geq \lambda$. In particular, P is an SYT with descents in the same positions as π and Q has weight λ. Then since the descent set and major index are preserved, using the correspondence between SYT and QYT proves the claim. □

Theorem 2.3.11. *For $n \in \mathbb{N}$,*

$$\sum_{|\lambda|=n} \sum_{\pi \in S_\lambda} q^{\mathrm{maj}(\pi)} t^{\mathrm{des}(\pi)} m_\lambda = \sum_{|\nu|=n} \sum_{T \in \mathrm{QYT}(\nu)} q^{\mathrm{maj}(T)} t^{\mathrm{des}(T)} s_\lambda.$$

Proof. We proceed by induction on the poset of partitions of n under dominance order. The inductive claim is that the coefficient of s_λ on the right hand side is the desired coefficient, and the inductive assumption is that the claim is true for all $\nu > \lambda$. As a base case, this clearly holds for $\lambda = (n)$ by computation. By the triangularity of the expansion of Schur functions into monomials, the coefficients of m_λ on each side forces

$$\sum_{\pi \in S_\lambda} q^{\mathrm{maj}(\pi)} t^{\mathrm{des}(\pi)} = C_\lambda + \sum_{\nu > \lambda} K_{\nu\lambda} \sum_{T \in \mathrm{QYT}(\nu)} q^{\mathrm{maj}(T)} t^{\mathrm{des}(T)},$$

where C_λ is the coefficient of s_λ on the right hand side, and the second term comes from the expansion of each s_ν, $\nu > \lambda$. Applying Lemma 2.3.10 immediately shows that $C_\lambda = \sum_{T \in \mathrm{QYT}(\lambda)} q^{\mathrm{maj}(T)} t^{\mathrm{des}(T)}$. Continuing this induction downwards on the poset eventually proves the claim for all partitions of n. □

Chapter 3

The Not Necessarily Symmetric World

In this chapter, some notation will be reused or redefined. In cases where notation is overloaded, the new definition of this chapter overrides the definition of the previous chapter. The work in this chapter is reproduced from [37]

3.1 Key Polynomials

There are many bases for the polynomial ring that have deep geometric and representation theoretic significance. We begin with one such basis by defining it combinatorially using certain diagrams indexed by weak compositions.

A *diagram* is an array of finitely many cells in $\mathbb{N} \times \mathbb{N}$, and a *labeled diagram* is a diagram for which each cell contains a natural number, possibly with repetition. We draw all diagrams throughout this paper in French notation, that is, row indices will increase from bottom to top. The location of a cell in a diagram will be denoted using Cartesian coordinates. A *weak composition* is an ordered sequence of nonnegative integers written $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ for some $n \in \mathbb{N}$, and we call a_i, $i \in \mathbb{N}$, a *part* of $\mathbf{a}$. The length of a weak composition is the number of parts it has. We write $\text{rev}(\mathbf{a})$ to denote $(a_n, a_{n-1}, \ldots, a_1)$ and $\text{sort}(\mathbf{a})$ to denote the weak composition obtained from $\mathbf{a}$ by rearranging its parts in weakly decreasing order. We also write $\max(\mathbf{a})$ to denote the value of the (possibly not unique) largest part of $\mathbf{a}$. A *composition* is a weak composition where all parts are positive and $\text{flat}(\mathbf{a})$ denotes the composition with only the nonzero parts of $\mathbf{a}$ in order. The *weight* of a diagram D, denoted $\mathbf{wt}(D)$, is the weak composition whose ith part is the number of cells in row i. A diagram is a *key diagram* if the rows are left justified. For each weak composition $\mathbf{a}$, there is a unique key diagram of weight $\mathbf{a}$, which we simply call the *key diagram of* $\mathbf{a}$.

Starting from a particular diagram D, one can generate new diagrams using *Kohnert moves*. A Kohnert move on a diagram takes the rightmost cell of a given row and moves the cell to the first open position below, jumping over other cells if necessary. In the case of key diagrams, we call the set of diagrams generated by Kohnert moves on the key diagram of $\mathbf{a}$ the set of *Kohnert diagrams of* $\mathbf{a}$. Kohnert [27] showed that the key polynomial (also known as a Demazure character, introduced by Demazure [13]) parameterized by the weak composition $\mathbf{a}$ is the generating polynomial of the set of Kohnert diagrams of $\mathbf{a}$.

Assaf and Searles [7] defined *Kohnert tableaux*, which are unique labelings for Kohnert diagrams that track the original position of each cell in the key diagram that the Kohnert diagram is generated from, before any Kohnert moves are applied. We note that these are a reformulation of Mason's fillings of key diagrams [32].

Definition 3.1.1 ([8])**.** Given a weak composition $\mathbf{a}$ of length n, a *Kohnert tableau of content* $\mathbf{a}$ is a diagram filled with entries $1^{a_1}, 2^{a_2}, \ldots, n^{a_n}$, one per cell, satisfying the following conditions:

1. there is exactly one i in each column from 1 through a_i;
2. each entry in row i is at least i;
3. the cells with entry i weakly descend from left to right;
4. if $i < j$ appear in a column with i above j, then there is an i in the column immediately to the right of and strictly above j

The set of Kohnert tableaux of content $\mathbf{a}$ is denoted $\mathrm{KT}(\mathbf{a})$. We call condition (2) the *flagged* condition and say that a labeled diagram (not just a Kohnert tableaux) satisfying this condition is flagged. An occurrence of (4) in any labeled diagram is called an *inversion* and we say that i and j are inverted. We also use the notation $\mathbb{D}(T)$ to denote the underlying diagram for a given labeled diagram T.

Since each Kohnert diagram has a unique such labeling, we may define key polynomials as generating polynomials over Kohnert tableaux instead.

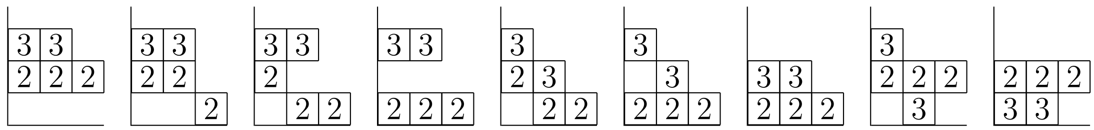


Figure 3.1: The set $\mathrm{KT}(0,3,2)$.

Definition 3.1.2. The *key polynomial* indexed by the weak composition $\mathbf{a}$ is

$$\kappa_{\mathbf{a}} = \sum_{T \in \mathrm{KT}(\mathbf{a})} x^{\mathbf{wt}(T)}. \tag{3.1.1}$$

For example, we have from Figure 3.1 that

$$\kappa_{(0,3,2)} = x_2^3x_3^2 + x_1x_2^2x_3^2 + x_1^2x_2x_3^2 + x_1^3x_3^2 + x_1^2x_2^2x_3 + x_1^3x_2x_3 + x_1^3x_2^2 + x_1x_2^3x_3 + x_1^2x_2^3.$$

Key polynomials are a polynomial generalization of the Schur polynomials, and Lascoux and Schützenberger [29] showed that if $\mathbf{a}$ is weakly increasing, then the corresponding key polynomial is a Schur polynomial and therefore symmetric.

Theorem 3.1.3 ([29])**.** *For a weak composition $\mathbf{a}$ of length n, the key polynomial $\kappa_{\mathbf{a}}$ is symmetric in $x_1, \ldots, x_n$ if and only if $\mathbf{a}$ is weakly increasing. Moreover, in this case, $\kappa_{\mathbf{a}} = s_{\mathrm{rev}(\mathbf{a})}(x_1, \ldots, x_n)$.*

We can also characterize directly when a key polynomial is quasisymmetric.

Proposition 3.1.4 ([37])**.** *For a weak composition $\mathbf{a}$ of length n, the key polynomial $\kappa_{\mathbf{a}}$ is quasisymmetric in $x_1, x_2, \ldots, x_n$ if and only if $\mathbf{a}$ has no zero parts or the parts are weakly increasing.*

Proof. We first consider when $\mathbf{a}$ is weakly increasing. By Theorem 3.1.3, $\kappa_{\mathbf{a}}$ is symmetric and so it is also quasisymmetric.

Next suppose that $\mathbf{a}$ has no zero parts. The diagram of $\mathbf{a}$ has a box in every row from 1 to n in the leftmost column, and any sequence of Kohnert moves preserves this property. Then $x^{\mathbf{wt}(T)}$ for any Kohnert tableau T of $\mathbf{a}$ has positive exponent for $x_1, \ldots, x_n$ and is as a result quasisymmetric in $x_1, \ldots, x_n$. Therefore, $\kappa_{\mathbf{a}}$ is a sum of monomials, which are each individually quasisymmetric polynomials, so $\kappa_{\mathbf{a}}$ is quasisymmetric.

Finally, suppose that $\mathbf{a}$ is not weakly increasing and has at least one part equal to zero. We consider two cases: either there exists some index i for which $a_i > a_{i+1} = 0$, or there does not.

Suppose first that such an index exists. Observe that for a given diagram D, $\mathbf{wt}(D)$ comes later in lexicographic order than the weights of any diagrams resulting from a sequence of Kohnert moves on D. Then since $\kappa_{\mathbf{a}}$ contains the term

$$x_1^{a_1} \cdots x_i^{a_i} x_{i+1}^{a_{i+1}} \cdots x_n^{a_n} = x_1^{a_1} \cdots x_i^{a_i} x_{i+1}^{0} \cdots x_n^{a_n}$$

but not the term $x_1^{a_1} \cdots x_i^{0} x_{i+1}^{a_i} \cdots x_n^{a_n}$, $\kappa_{\mathbf{a}}$ is not quasisymmetric.

Now suppose that no such index i exists, so that $\mathbf{a}$ has some positive number of leading zeroes followed by exclusively nonzero parts. Choose j such that $a_j > a_{j+1} > 0$. We can apply Kohnert moves to the diagram of $\mathbf{a}$ to push all nonempty rows below row j down by exactly one space, then apply a_{j+1} Kohnert moves to row $j+1$ to move the boxes in row $j+1$ to row $j-1$. Now we have a Kohnert diagram with associated monomial

$$x_1^{a_2} \cdots x_{j-2}^{a_{j-1}} x_{j_1}^{a_{j+1}} x_j^{a_j} x_{j+1}^0 x_{j+2}^{a_{j+2}} \cdots x_n^{a_n}.$$

If $\kappa_{\mathbf{a}}$ were quasisymmetric, then we would also need the monomial

$$x_1^{a_1} \cdots x_{j-1}^{a_{j-1}} x_j^{a_{j+1}} x_{j+1}^{a_j} x_{j+2}^{a_{j+2}} \cdots x_n^{a_n}.$$

However, the weight of the Kohnert diagram that this monomial would be associated with would come later in lexicographic order than **a**, which contradicts our observation above that Kohnert moves on a diagram must produce weights that come earlier in lexicographic order. Therefore, $\kappa_{\mathbf{a}}$ is not quasisymmetric. □

Notably, the only key polynomials that are quasisymmetric but not symmetric are those with nonzero parts that are not weakly increasing and also have no zero parts.

3.2 Lock polynomials

Assaf and Searles [8] introduced *lock polynomials* as a natural analogue to the combinatorial definition of key polynomials. The *lock diagrams* of **a** are all diagrams that can be obtained from applying a sequence of Kohnert moves to the unique right justified diagram with weight **a** and nonempty first column. As with Kohnert diagrams, lock diagrams of **a** have unique labelings, which we call *lock tableaux of content* **a**. We denote the set of lock tableaux of content **a** by $\mathrm{LT}(\mathbf{a})$.

Definition 3.2.1 ([8]). Given a weak composition of length n, a *lock tableau of content* $\mathbf{a}$ is a diagram filled with entries $1^{a_1}, 2^{a_2}, \ldots, n^{a_n}$, one per cell, satisfying the following conditions:

1. there is exactly one i in each column from $\max(\mathbf{a}) - a_i + 1$ through $\max(\mathbf{a})$;

2. each entry in row i is at least i;

3. the cells with entry i weakly descend from left to right;

4. the labeling strictly decreases down columns.

We can see that there is a unique such labeling for any lock diagram because condition (1) fixes the set of labels in each column and condition (4) fixes their order within each column. We reproduce the first part of [8, Theorem 6.9] in order to reference this fact later. Here, $L_{\mathbf{a}}$ refers to the explicit labeling algorithm for lock tableaux, which we will not need for this paper.

Theorem 3.2.2 ([8, Theorem 6.9]). *The labeling map $L_{\mathbf{a}}$ is a weight-preserving bijection between lock diagrams of $\mathbf{a}$ and lock tableaux of $\mathbf{a}$.*

We reiterate from earlier that we use the notation $\mathbb{D}(T)$ to denote the underlying diagram for a given labeled diagram T. We also use $T_{\mathbf{a}}$ to denote the unique lock tableau with weight $\text{flat}(\mathbf{a})$ and content $\mathbf{a}$.

The second half of [8, Theorem 6.9] allows us to define lock polynomials as the generating polynomials of lock tableaux.

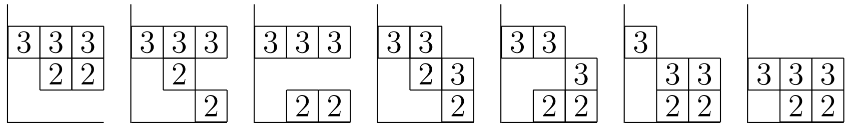


Figure 3.2: The set $\mathrm{LT}(0,2,3)$, where the rightmost tableau is $T_{(0,2,3)}$.

Definition 3.2.3 ([8, Theorem 6.9])**.** The *lock polynomial* indexed by the weak composition $\mathbf{a}$ is

$$\mathfrak{L}_{\mathbf{a}} = \sum_{T \in \mathrm{LT}(\mathbf{a})} x^{\mathbf{wt}(T)}. \tag{3.2.1}$$

For example, we have from Figure 3.2 that

$$\mathfrak{L}_{(0,2,3)} = x_2^2 x_3^3 + x_1 x_2 x_3^3 + x_1^2 x_3^3 + x_1 x_2^2 x_3^2 + x_1^2 x_2 x_3^2 + x_1^2 x_2^2 x_3 + x_1^2 x_2^3.$$

Lock polynomials also form a basis for the full polynomial ring, and they coincide with key polynomials if the nonzero parts of $\mathbf{a}$ are weakly decreasing.

Proposition 3.2.4 ([8, Corollary 6.2])**.** *The lock polynomials form a basis for the polynomial ring.*

Theorem 3.2.5 ([8, Theorem 6.12])**.** *Given a weak composition* $\mathbf{a}$ *of length* n *such that its nonzero parts are weakly decreasing, we have*

$$\mathfrak{L}_{\mathbf{a}} = \kappa_{\mathbf{a}}. \tag{3.2.2}$$

As with key polynomials, lock polynomials are not always symmetric or quasisymmetric, however we can characterize exactly when each happens. For the quasisymmetric case, the condition is the same as for key polynomials.

Proposition 3.2.6 ([37]). *For* $\mathbf{a}$ *a weak composition of length* n*,* $\mathfrak{L}_{\mathbf{a}}$ *is quasisymmetric in* $x_1, x_2, \ldots, x_n$ *if and only if* $\mathbf{a}$ *has no zero parts or the parts are weakly increasing.*

Proof. If there are no zero parts, then no Kohnert moves can be done on the lock diagram of $\mathbf{a}$. Then since lock tableaux of $\mathbf{a}$ are in bijection with lock diagrams of $\mathbf{a}$ by Theorem 3.2.2, $\mathfrak{L}_{\mathbf{a}}$ by definition consists of a single monomial with positive exponent for all variables $x_1, \ldots, x_n$, and therefore $\mathfrak{L}_{\mathbf{a}}$ is quasisymmetric in those variables.

Now suppose that $\mathbf{a}$ is weakly increasing with leading zeroes. Define maps p_i and d_i for $1 \leq i < n$ as follows. If row i (row $i+1$) has at least one box in it and row $i+1$ (row i) is empty, p_i (d_i) moves all boxes from row i (row $i+1$) to row $i+1$ (row i), preserving their columns and labels, otherwise p_i (d_i) does nothing. We can think of these as colored edges connecting different labeled diagrams, where a connected component has generating polynomial equal to a monomial quasisymmetric polynomial in n variables. Therefore, it is sufficient to show that when at least one labeled diagram in a connected component is a lock tableau with content $\mathbf{a}$, every labeled diagram in that connected component is a lock tableau with content $\mathbf{a}$, since then summing over the connected components with lock tableau gives the lock polynomial as a sum of monomial quasisymmetric polynomials.

When d_i is applied to a lock tableau of content $\mathbf{a}$, it is easy to check that all four properties in Definition 3.2.1 are preserved. For p_i, properties (1), (3), and (4) are also clear by construction. For property (2), suppose that some box in column j with label i is pushed to row $i+1$ by p_i. By properties (1) and (2) and the fact that $\mathbf{a}$ is weakly increasing, there must be boxes with labels $i+1, i+2, \dots, n$ strictly above row $i+1$ in column j. However, since there cannot be boxes above row n, we must have $n-i$ boxes fitting into $n-i-1$ rows, which is impossible. Therefore, property (2) must also hold, and any labeled diagram connected to a lock tableau of content $\mathbf{a}$ by a sequence of p_i, d_i is also a lock tableau of content $\mathbf{a}$.

Finally, consider the case where the parts of $\mathbf{a}$ are not weakly increasing and at least one part is equal to zero. The proof in this case is essentially identical to that of the same case in the proof of Proposition 3.1.4 and the analagous conclusion follows, that the lock polynomial of $\mathbf{a}$ is not quasisymmetric in this case. □

Symmetry for lock polynomials is less common than for key polynomials, as seen by comparing Theorem 3.1.3 with the following.

Proposition 3.2.7 ([37]). *For* $\mathbf{a}$ *a weak composition of length* n*, the lock polynomial* $\mathfrak{L}_{\mathbf{a}}$ *is symmetric in* $x_1, x_2, \dots, x_n$ *if and only if* $\mathbf{a} = 0^{n-k} \times m^k$ *for some integers* $m, k > 0$ *and* $k \leq n$*. Moreover, in this case, we have* $\mathfrak{L}_{\mathbf{a}} = s_{m^k}(x_1, \dots, x_n)$.

Proof. By Proposition 3.2.6, $\mathbf{a}$ must be weakly increasing or else $\mathfrak{L}_{\mathbf{a}}$ is not quasisymmetric, and so not symmetric.

Suppose then that $\mathbf{a}$ is weakly increasing and that there exists some index i such that $a_{i+1} > a_i > 0$, and let $s_i\mathbf{a}$ be $\mathbf{a}$ with the parts a_i and a_{i+1} swapped. The lock polynomial of $\mathbf{a}$ must contain a monomial $x^{\mathbf{a}}$, so if it is symmetric, it must also contain the monomial $x^{s_i \mathbf{a}}$. Consider a lock tableau that would be associated with this monomial.

By condition (2) in Definition 3.2.1, every box in rows $i+2$ to n must have a label between $i+2$ and n, and since there are $a_{i+2} + \cdots + a_n$ many such boxes and labels, every such box must have such a label, and there are no remaining labels between $i+2$ and n to place in lower rows.

Using condition (2) again, every one of the a_i boxes in row $i+1$ must have an $i+1$ label, since no smaller labels can exist in row $i+1$, and from above, no larger labels can either. Since $a_{i+1} > a_i$, this leaves $a_{i+1} - a_i$ many $i+1$ labels that must go in lower rows. Since columns strictly decrease, these excess $i+1$ labels must be to the left of column $\max(\mathbf{a}) - a_i + 1$. However, this would imply the existence of $i+1$ labels strictly lower and to the left of the $i+1$ labels in row $i+1$, which contradicts condition (3). Therefore, no such lock tableau can exist, and $\mathfrak{L}_{\mathbf{a}}$ is not symmetric.

The only remaining cases are those for which $\mathbf{a} = 0^{n-k} \times m^k$. By Theorem 3.2.5, we have $\mathfrak{L}_{\mathbf{a}} = \kappa_{\mathbf{a}}$, then by Theorem 3.1.3, we have $\kappa_{\mathbf{a}} = s_{\mathrm{rev}(\mathbf{a})}(x_1, \ldots, x_n)$, so $\mathfrak{L}_{\mathbf{a}}$ is always symmetric in these cases. □

3.3 Crystals

Kashiwara [21] introduced the notion of crystal bases in his study of the representation theory of quantized universal enveloping algebras at $q = 0$. Combinatorially for the general linear group (type A), a *crystal* is a set $\mathcal{B}$ not containing 0, a *weight map* $\mathbf{wt} : \mathcal{B} \to \mathbb{Z}^n$, and *raising and lowering operators* $e_i, f_i : \mathcal{B} \to \mathcal{B} \cup \{0\}$, for $i = 1, 2, \ldots, n-1$ that satisfy certain axioms including $e_i(b) = b'$ if and only if $f_i(b') = b$. In particular, we can deduce the lowering operators from the raising operators. For a more in depth introduction to crystals, see [20, 10]. We can also visualize a crystal by identifying it with an edge weighted directed graph where $b \xrightarrow{i} b'$ if and only if $b' = f_i b$, and we call this graph the *crystal graph*.

Demazure [13] introduced Demazure modules that arose in connection with Schubert calculus [12] and gave a character formula for them. The proof of this character formula turned out to have a gap, but it was later proven by Andersen [2]. Littelmann [30] conjectured and Kashiwara [22] proved that Demazure modules have crystal bases, which are now called Demazure crystals. These Demazure crystals are certain truncations of crystals on semistandard Young tableaux that were constructed explicitly by Kashiwara and Nakashima [23] and Littelmann [30]. Assaf and Schilling [6, Definition 3.7] gave an explicit combinatorial construction of Demazure crystals with raising and lowering operators that act on semistandard key tableaux. These tableaux were reformulations of the fillings of key diagrams defined by Mason [32] and can be translated into the language of Kohnert diagrams and tableaux, as presented by Assaf and González [5]. In this paper, we focus specifically on these crystal operators on Kohnert diagrams and tableaux.

Definition 3.3.1 ([5]). Given any diagram D with $n \geq 1$ rows and $1 \leq i < n$, define the *vertical i-pairing* of D as follows: i-pair any boxes in rows i and $i+1$ that are located in the same column and then iteratively vertically i-pair any unpaired boxes in row $i+1$ with the rightmost unpaired box in row i located in a column to its left whenever all the boxes in rows i and $i+1$ in the columns between them are already vertically i-paired.

Definition 3.3.2 ([5])**.** Given any integer $n \geq 0$ and any diagram D with at most n rows, for any integer $1 \leq i < n$, define the *raising operator* e_i on the space of diagrams as the operator that pushes the rightmost vertically unpaired box in row $i+1$ of D down to row i. If D has no vertically unpaired boxes in row $i+1$, then $e_i(D) = 0$.

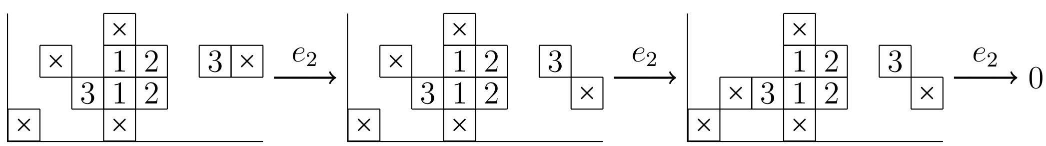


Figure 3.3: Boxes that share a number label in these diagrams are vertically 2-paired.

Assaf and González show in [5, Proposition 5.23] that these raising operators on Kohnert diagrams coincide with their raising operators on Kohnert tableaux. Therefore, we simply define raising operators on Kohnert tableaux through identification with Kohnert diagrams.

We can do the same for raising operators on lock tableaux. That is, given T a lock tableau of content $\mathbf{a}$ with underlying diagram D, the raising operator on T

produces the unique lock tableau of content $\mathbf{a}$ with underlying diagram $e_i(D)$ if it exists, otherwise $e_i(T) = 0$. Note that in this case, we also specify that the resulting diagram must have a valid lock tableau labeling. This is because while the raising operator e_i on a Kohnert tableau always produces another Kohnert tableau of the same content, the same may not be true for a given lock tableau. Put another way, the minimal k such that $e_i^{k+1}(T) = 0$ is the number of unpaired boxes in row $i+1$ for a Kohnert tableau but may be smaller for a lock tableau.

We also provide the following equivalent formulation for raising operators on lock tableaux for completeness, where boxes are vertically paired based on the underlying diagram.

Definition 3.3.3. Given a weak composition $\mathbf{a}$, $T \in \mathrm{LT}(\mathbf{a})$, and $1 \leq i < n$, the *raising operator* e_i acts on T by $e_i(T) = 0$ if T has no vertically unpaired boxes in row $i+1$ or if the rightmost unpaired box in row $i+1$ has the same label as a box to its right in the same row. Otherwise, e_i pushes the rightmost vertically unpaired box in row $i+1$ of T down to row i.

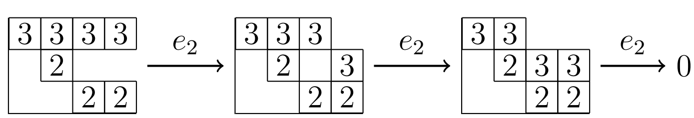


Figure 3.4: Raising operators acting on a lock tableau of content $(0, 3, 4)$. Notice that the third tableau is sent to zero despite there being an unpaired box in row 3.

It is straightforward to see that this coincides with the previous definition on the underlying diagram. To avoid excessive notation, we will use e_i for any raising operator on lock or Kohnert diagrams or tableaux and f_i for any lowering operator, where the type of object being acted on will either be clear from context or specified if not.

Since there is a natural bijection from Kohnert tableaux to semistandard key tableaux, the crystal graph on Kohnert tableaux of content $\mathbf{a}$ is the Demazure crystal parametrized by $\mathbf{a}$. We will refer to the crystal graph on lock tableaux of content $\mathbf{a}$ as the *lock crystal of* $\mathbf{a}$. See Figure 3.5 for an example. It is well-known that Demazure crystals are connected, and it turns out that the same is true for lock crystals.

Theorem 3.3.4 ([37]). *For* $\mathbf{a}$ *a weak composition, the raising and lowering operators on semistandard lock tableaux generate a connected, edge weighted directed graph on* $\mathrm{LT}(\mathbf{a})$.

Proof. See Figure 3.6 for an explicit example of the argument below. Recall that $T_{\mathbf{a}}$ denotes the LT of content $\mathbf{a}$ with weight $\mathrm{flat}(\mathbf{a})$. We can check that this is unique by the definition of lock tableaux. It is sufficient to show that for $T \in \mathrm{LT}(\mathbf{a})$ with highest box in row m, T is connected to $T_{\mathbf{a}}$ using only the crystal operators $e_1, f_1, \ldots, e_{m-1}, f_{m-1}$. We prove this by inducting on the size of $\mathbf{a}$. The base case consists of weak compositions of size 1, where the single box in row m is always connected to the single box in row 1 by the sequence of crystal operators $e_1 \circ e_2 \circ \cdots \circ e_{m-1}$.

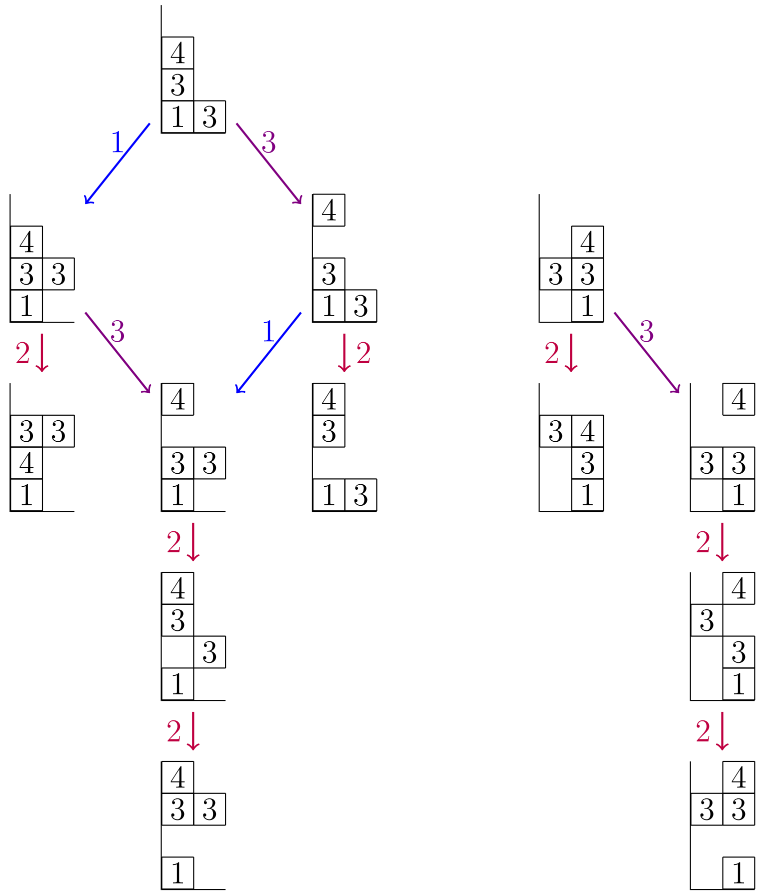


Figure 3.5: On the left is the Demazure crystal of $\mathbf{a} = (1,0,2,1)$ and on the right is the lock crystal of $\mathbf{a}$. Here, an arrow labeled with i denotes a lowering operator f_i.

Suppose that $\mathbf{b}$ is a weak composition of size at most $n-1$. If $S \in \mathrm{LT}(\mathbf{b})$ with highest box in row i, we can connect S to $T_{\mathbf{b}}$ using only the operators $e_1, f_1, \ldots, e_{i-1}, f_{i-1}$. Fix $\mathbf{a}$ to be a weak composition of size n with nonzero parts $\{a_{j_1}, \ldots, a_{j_k}\}$, and let $T \in \mathrm{LT}(\mathbf{a})$ with highest box in row m.

Let T' be the LT obtained by removing all boxes of T in row m, and let T' have shape $\mathbf{a}'$ and highest box in row $m' < m$. By the inductive assumption, there is some

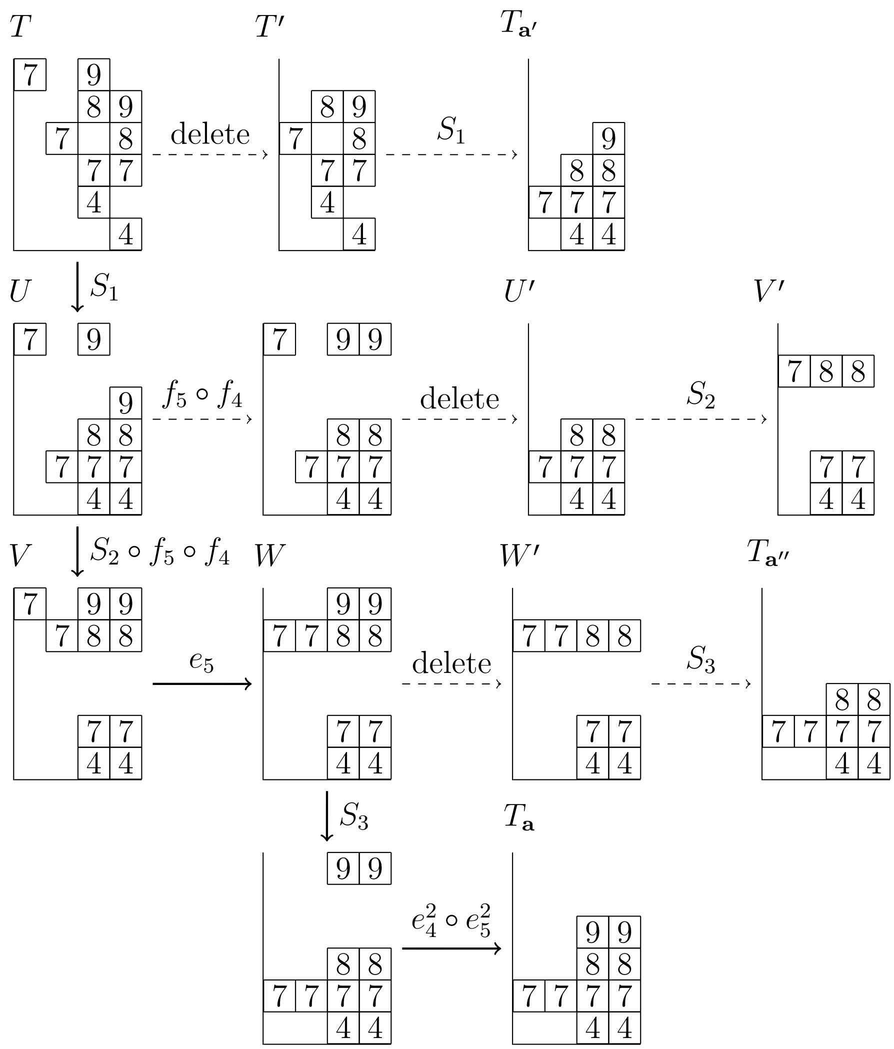


Figure 3.6: An explicit example of the inductive argument in the proof of Theorem 3.3.4 with diagrams labeled. Each S_i is a sequence of operators given by the inductive assumption, and the full sequence applied to T to get to $T_{\mathbf{a}}$ is given by following the southwest border.

sequence of crystal operators $e_1, f_1, \ldots, e_{m'-1}, f_{m'-1}$ that sends T' to $T_{\mathbf{a}'}$. Since these crystal operators only check the positions of boxes in row m' and below, applying the same sequence of crystal operators to T gives a tableau U which boxes in row m everywhere that T does and which has boxes below row m everywhere that $T_{\mathbf{a}'}$ does. Suppose that U has some number t of boxes with label j_k in row k. There are no boxes in the rows strictly between k and m and every box with label j_k in row k must be strictly right of every box in row m, so applying $f_{m-1}^t \circ \cdots \circ f_k^t$ to U brings all t of the boxes that were in row k with label j_k to row m. Therefore, we can assume every box of U with label j_k must be in row m.

If all of the boxes in row m have label j_k, then set $W = U$ and advance to the step in the last paragraph of this proof. Otherwise, U has some boxes in row m with label smaller than j_k, so let c be the rightmost column containing such a box and let that box have label ℓ. Obtain U' from U by removing all boxes in row m, then obtain V' from U' by pushing the highest box of each column to the right of c up to row $m-1$ while preserving their label. This clearly still satisfies the column strict condition on LT, and the sets of labels in each column are unchanged so condition (1) holds as well.

Suppose that condition (3) of Definition 3.2.1 is not satisfied in V' because of some pair of boxes x left of y with label $p \neq \ell$, where y is pushed above x. By construction, x must be weakly left of column c and y must be strictly right. In U, column c contains a box in row m with label ℓ and no boxes above row m. Then the column strict condition implies that there cannot be any labels larger than ℓ in column c, and then condition (1) implies there cannot be labels to the right of c with label larger than ℓ either. It also implies that the highest box in each column to the right of c in U' must have label at least ℓ. Therefore, if $p > \ell$, then x cannot exist, and if $p < \ell$, then y is not the highest box in its column is therefore not pushed upwards.

Since U is an LT with a label ℓ in row m, we have $\ell \geq m$. Then using the observation that the highest box in each column to the right of c in U' has label at least ℓ, every box that is pushed up to row $m-1$ in V' has label at least $m-1$. Since all conditions are satisfied, V' is an LT by definition. Then by the inductive assumption, some sequence of the operators $e_1, f_1, \ldots, e_{m-2}, f_{m-2}$ sends U' to V', and therefore the same sequence of operators on U gives a tableau V which has boxes in row m everywhere that U does and which has boxes below row m everywhere that V' does.

By construction, all boxes in row m with label j_k of V must be paired and all other boxes of row m, which have label smaller than j_k, are unpaired. We can then apply e_{m-1} operators until all the unpaired boxes of row m are in row $m-1$ and call the new tableau W.

In either case, the tableau W has every box with label j_k in row m and every box with label smaller than j_k below row m. Obtain W' a LT of content $\mathbf{a}''$ from W by removing all boxes in row m. By the inductive assumption, some sequence of crystal operators $e_1, f_1, \ldots, e_{m-2}, f_{m-2}$ sends W' to $T_{\mathbf{a}''}$. Then applying the same sequence of operators to W followed by applying $e_k^{a_{j_k}} \circ \cdots \circ e_{m-1}^{a_{j_k}}$ sends W to $T_{\mathbf{a}}$ and we are done. □

3.4 Rectification and Unlock

In this section, we define the map $U_{\mathrm{flat}(\mathbf{a})}$ and prove the following theorem about it.

Theorem 3.4.1 ([37])**.** *Let* $\mathbf{a}$ *be a weak composition. Then* $U_{\mathrm{flat}(\mathbf{a})} : \mathrm{LT}(\mathbf{a}) \mapsto \mathrm{KT}(\mathbf{a})$ *embeds the lock crystal of* $\mathbf{a}$ *into the Demazure crystal of* $\mathbf{a}$.

We will show this by comparing the rectification operators of Assaf and González [5] that act on diagrams and operators that we call *unlock operators* that act on labeled diagrams. We will see that, for the cases we consider, unlock operators on labeled diagrams act on the underlying diagram in the same way as rectification operators with the added benefit that unlock operators can track the movement of labels through each step. We begin by defining rectification operators.

Definition 3.4.2 ([5])**.** Given any diagram D with $n \geq 1$ columns and integer $1 \leq i < n$, define the *horizontal i-pairing* of D as follows: i-pair any boxes in columns i and $i+1$ that are located in the same row and then interatively i-pair any unpaired box in column $i+1$ with the lowest unpaired box in column i located in a row above it whenever all the boxes in columns i and $i+1$ in the rows between them are already horizontally i-paired.

Definition 3.4.3 ([5])**.** Given any integer $n \geq 0$ and any diagram D with at most n columns, for any integer $1 \leq i < n$, define the *rectification operator* ϑ_i on the space of diagrams as the operator which pushes the bottom-most horizontally unpaired box in column $i+1$ of D left to column i. If D has no unpaired boxes in column $i+1$, then $\vartheta_i(D) = 0$.

As Assaf and González note, these operators can be viewed as a rotation of raising operators on diagrams. We also have the following equivalent formulation from [7, Lemma 2.2], which we find more convenient to work with in the proofs to follow. Given a diagram D and an integer $i \geq 1$, define

$$M^i(D, r) = \#\{(i+1, s) \in D | s \geq r\} - \#\{(i, s) \in D | s \geq r\}, \tag{3.4.1}$$

$$M^i(D) = \max_r(M^i(D, r)). \tag{3.4.2}$$

Proposition 3.4.4 ([5, Lemma 2.2])**.** *Let $i \geq 1$ and D be a diagram. If $M^i(D) \leq 0$, then $\vartheta_i(D) = 0$; otherwise, letting r be the largest row index such that $M^i(D, r) = M^i(D)$, $\vartheta_i(D)$ is the result of pushing the cell in position $(r, i+1)$ left to position (r, i).*

We can see that this is equivalent because the largest row index on which $M^i(D, r)$ achieves its maximum is the same row as the lowest row containing a horizontally unpaired box in column $i+1$. Now for $\mathbf{a}$ a weak composition, $m = \max(a_i)$, and $\alpha = \text{flat}(\mathbf{a})$, let $R_{\alpha,i}$ denote the composition of rectification moves

$$R_{\alpha,i} = (\vartheta_{\alpha_i} \circ \cdots \circ \vartheta_{m-1}) \circ \cdots \circ (\vartheta_2 \circ \cdots \circ \vartheta_{m-\alpha_i+1}) \circ (\vartheta_1 \circ \cdots \circ \vartheta_{m-\alpha_i}). \tag{3.4.3}$$

Here, the indices within a pair of parentheses are incremented by 1 moving from left to right, and there are always $m - \alpha_i$ many $\reflectbox{e}$ within a pair of parentheses. When $m - \alpha_i = 0$, $R_{\alpha,i}$ is considered to be the identity. Let R_α denote the composition of rectification moves

$$R_\alpha = R_{\alpha,\ell(\alpha)} \circ \cdots \circ R_{\alpha,2} \circ R_{\alpha,1}. \tag{3.4.4}$$

We will sometimes refer to R_α as the *rectification algorithm* (for $\mathbf{a}$) and to each individual rectification operator that R_α is composed of as the *steps* of the algorithm. We note that the order of rectification operators applied here is different in general from the order used by Assaf and González.

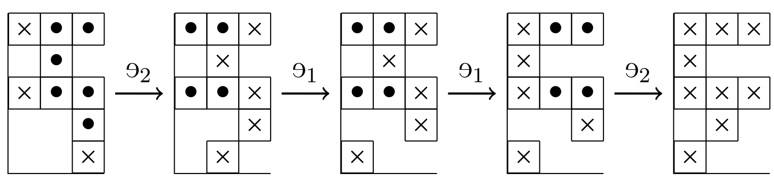


Figure 3.7: For $\mathbf{a} = (1,0,3,0,3,2)$, we have $\alpha = (1,3,3,2)$ and $R_\alpha = \reflectbox{e}_2\reflectbox{e}_1\reflectbox{e}_1\reflectbox{e}_2$. On the left is a lock diagram of $\mathbf{a}$ and each step of the rectification algorithm for $\mathbf{a}$ on that diagram with relevant horizontally paired boxes represented by bullets.

We have the following crucial properties of rectification operators which we will leverage in our proof of Theorem 3.4.1.

Proposition 3.4.5 ([37])**.** *Given a weak composition* $\mathbf{a}$ *and a lock diagram* D*, if* $R_\alpha(D) \neq 0$*, then* R_α *is weight preserving and injective.*

Proof. A rectification operator only pushes boxes to the left, so at every step, the number of boxes in each row remains unchanged and thus the weight is unchanged. For an arbitrary diagram D, if $ɘ_i(D) \neq 0$, then we claim that we can obtain D from $ɘ_i(D)$ by pushing the top-most horizontally unpaired box in column i of $ɘ_i(D)$ right to column $i+1$. We just need to check that this box exists and is the same box that $ɘ_i$ pushed to the left in D, and to prove this, it is sufficient to show that the horizontal pairings are unchanged by $ɘ_i$. We can view rectification operators as rotated raising operators and raising operators do not change the vertical pairing of boxes, so it follows that $ɘ_i(D)$ can be inverted and R_α is injective. □

Theorem 3.4.6 ([5, Theorem 5.33])**.** *The rectification operators and the raising operators on diagrams commute.*

Corollary 3.4.7. *Given a weak composition* $\mathbf{a}$*,* $\alpha = \mathrm{flat}(\mathbf{a})$*, and* $T \in \mathrm{LT}(\mathbf{a})$*, we have* $R_\alpha(T) \neq 0$*.*

Proof. It is straightforward to check that $R_\alpha(T_{\mathbf{a}})$ is the unique Kohnert tableau of content $\mathbf{a}$ and weight α, and therefore $R_\alpha(T_{\mathbf{a}}) \neq 0$. Then since rectification operators intertwine with crystal operators on diagrams by Theorem 3.4.6 and the lock crystal is connected by Theorem 3.3.4, we must have $R_\alpha(T) \neq 0$ as well. □

We now define the unlock algorithm. Again, we will see that the rectification algorithm and the unlock algorithm act in the same way, except that the unlock algorithm is translated through the natural correspondence between lock and Kohnert diagrams and tableaux.

Definition 3.4.8 ([37]). Given a positive integer $i \geq 1$, define the *unlock operators* u_i on labeled diagrams as follows. The *string* ℓ, for ℓ a label of a labeled diagram T, is the set of boxes of T with label ℓ. A box x in string ℓ is *left justified* if every column to the left of x contains a box with label ℓ. We say that a box x in a string ℓ in column $i+1$ *crosses* a string $\ell' \neq \ell$ when string ℓ' contains a box in column i weakly above x and a box in column $i+1$ strictly below x. Let x be the box in column $i+1$ that has minimal label ℓ among those in column $i+1$ that are not left justified. If no such x exists, then u_i returns 0. Otherwise, u_i returns the diagram resulting from iterating the following steps until x is pushed into column i.

1. If x does not cross any strings, then push x one space to the left and terminate the algorithm. Otherwise, go to step 2.

2. Fix string ℓ' to be the string with highest row index in column i among those strings that x crosses. Let y be the box in column $i+1$ of string ℓ' and swap the row indices of x and y so that x with label ℓ is below y with label ℓ' in column $i+1$. Return to step 1.

For $\mathbf{a}$ a weak composition with $m = \max(a_i)$ and $\alpha = \text{flat}(\mathbf{a})$, let $U_{\alpha,i}$ denote the composition of unlock operators

$$U_{\alpha,i} = (u_{\alpha_i} \circ \cdots \circ u_{m-1}) \circ \cdots \circ (u_2 \circ \cdots \circ u_{m-\alpha_i+1}) \circ (u_1 \circ \cdots \circ u_{m-\alpha_i}), \tag{3.4.5}$$

and let U_α denote the composition of unlock operators

$$U_\alpha = U_{\alpha,\ell(\alpha)} \circ \cdots \circ U_{\alpha,2} \circ U_{\alpha,1}. \tag{3.4.6}$$

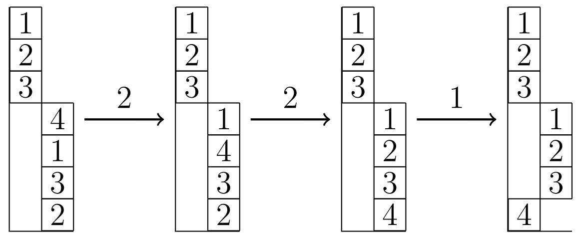


Figure 3.8: The steps of the unlock operator u_1 on the given labeled diagram, where arrows are labeled by the relevant step.

As with R_α, we will sometimes refer to U_α as the *unlock algorithm* (for $\mathbf{a}$) and to each individual unlock operator that it is composed of as the *steps* of the algorithm. We note that the unlock operators are not well defined for all labeled diagrams, for example if the box x that the unlock operator would like to push left is not crossing any strings but already has a box directly to its left. It turns out that for any lock tableau T of shape $\mathbf{a}$, $U_\alpha(T)$ is well defined, and this is formalized later in Lemma 3.4.13. For now, we will assume that the unlock algorithm is well defined on lock tableaux.

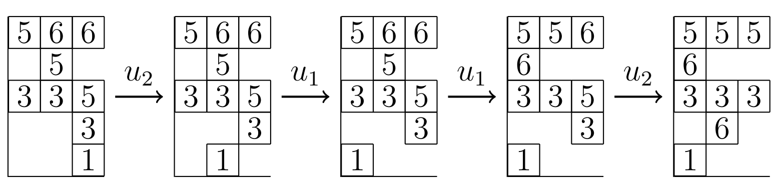


Figure 3.9: For $\mathbf{a} = (1,0,3,0,3,2)$, we have $\alpha = (1,3,3,2)$ and $U_\alpha = u_2u_1u_1u_2$. On the left is a lock tableau of content $\mathbf{a}$ and each step of the unlock algorithm for $\mathbf{a}$ on that tableau. Compare with Figure 3.7.

We also note that the order of the unlock operators in U_α is very intentionally chosen so that the boxes of $T \in \mathrm{LT}(\mathbf{a})$ are left justified in a particular order. See Figure 3.9 for a small example.

Proposition 3.4.9 ([37]). *Let* $\mathbf{a}$ *be a weak composition with* $\alpha = \mathrm{flat}(\mathbf{a})$ *and with nonzero parts* $\{a_{\ell_1}, a_{\ell_2}, \ldots, a_{\ell_k}\}$. *If* U_α *is well defined on* $T \in \mathrm{LT}(\mathbf{a})$, *then in order from* $i = 1, \ldots, k$, *the operator* $U_{\alpha,i}$ *left justifies the boxes of string* ℓ_i *in order from left to right. Furthermore, at each step of the unlock algorithm, a box* x *with label* ℓ *can only cross strings with labels strictly smaller than* ℓ.

Proof. The first claim on the order of boxes moved by U_α can be seen by construction from a straightforward examination of the definition of unlock operators and lock tableaux.

For the second claim, if a box x in T has label ℓ, then any box y with label $\ell_j > \ell$ must lie strictly north, strictly east, or both. The unlock operators move boxes weakly south and strictly west, and by the first claim, any box y with label $\ell_j > \ell$ will be left justified at a later step than x. Therefore, as x moves southwest, it can never cross a string with label $\ell_j > \ell$. □

It would be nice if each lock crystal had a unique lowest weight element. In this case, we would only need to show that this unique element maps to a Kohnert tableau via rectification, and then we could use the connectivity of the lock crystal and the commutativity of rectification operators and raising operators to prove Theorem 3.4.1. This is unfortunately not the case, and so instead we organize the proof of Theorem 3.4.1 as follows. It is easier to first assume that step by step for a given lock tableau T, $R_\alpha(\mathbb{D}(T))$ and $U_\alpha(T)$ *agree on the level of diagrams.* That is, if we let $R_\alpha = \backepsilon_{j_t} \circ \cdots \circ \backepsilon_{j_1}$ and $U_\alpha = u_{j_t} \circ \cdots \circ u_{j_1}$, then for all $1 \le s \le t$, we have

$$\backepsilon_{j_s} \circ \cdots \circ \backepsilon_{j_1}(\mathbb{D}(T)) = \mathbb{D}(u_{j_s} \circ \cdots \circ u_{j_1}(T)).$$

Given this assumption, we show that the resulting diagram $U_\alpha(T)$ is a Kohnert tableau of content $\mathbf{a}$ (a consequence of Lemma 3.4.12). We then show that the assumption always holds that R_α and U_α agree on the level of diagrams for lock tableaux (a claim of Lemma 3.4.13). We begin with the following technical results (Lemma 3.4.10 and Corollary 3.4.11).

In all the lemmas below, T is a lock tableau of content $\mathbf{a} = (a_1, \ldots, a_m)$ that contains the labels $\ell_1 < \cdots < \ell_k$, and $\alpha = \text{flat}(\mathbf{a})$. We also define a *truncation* of T, denoted $T^{<\ell}$, by deleting all boxes of T with label ℓ or larger. From the definition of lock tableaux, $T^{<\ell}$ is clearly still a lock tableau.

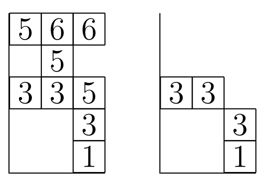


Figure 3.10: On the left is a lock tableau T and on the right is $T^{<5}$.

Lemma 3.4.10 ([37]). *Fix $1 \le p < k$ and $\ell > \ell_p$, and let t be given by writing $R_{\alpha,p} \circ \cdots \circ R_{\alpha,1} = \backepsilon_{p_t} \circ \cdots \circ \backepsilon_{p_1}$. Then for all $1 \le s < t$, $\backepsilon_{p_s}$ pushes a box from position*

$(c+1,r)$ to (c,r) in $\mathbb{D}(\ni_{p_{s-1}} \circ \cdots \circ \ni_{p_1}(T^{<\ell}))$ if and only if $\ni_{p_s}$ pushes a box from position $(c+1,r)$ to (c,r) in $\mathbb{D}(\ni_{p_{s-1}} \circ \cdots \circ \ni_{p_1}(T))$.

Proof. Let D be an arbitrary diagram and let columns $c, c+1$ be such that column $c+1$ is nonempty. Furthermore, let h_c, h_{c+1} be the highest row indices occupied by boxes in columns $c, c+1$ respectively, where $h_c = 0$ if column c is empty. Suppose that $M^c(D) > 0$ and r_0 is the highest row index for which $M^c(D,r)$ achieves its maximum. Then the following hold from the definition of M^c by examining $M^c(D_i,r)$ compared to $M^c(D,r)$ in each case over all rows.

1. Let $r_{c+1} > h_{c+1}$ and let D_1 be obtained from D by adding a box to position $(c+1, r_{c+1})$. Then r_0 is the highest row index for which $M^c(D_1,r)$ achieves its maximum.

2. Let $r_c \geq r_{c+1}$ with $r_c > h_c$ and $r_{c+1} > h_{c+1}$. Obtain D_2 from D by adding boxes to positions (c, r_c) and $(c+1, r_{c+1})$. Then r_0 is the highest row index for which $M^c(D_2,r)$ acheives its maximum.

3. Suppose that $h_{c+1} > r_0$ and obtain D_3 from D by removing the box in position $(c+1, h_{c+1})$. If $M^c(D_3) > 0$, then r_0 is the highest index for which $M^c(D_3,r)$ achieves its maximum.

4. Suppose $h_c \geq h_{c+1} > r_0$, and obtain D_4 from D by removing the boxes in positions (c, h_c) and $(c+1, h_{c+1})$. If $M^c(D_4) > 0$, then r_0 is the highest index for which $M^c(D_4,r)$ achieves its maximum.

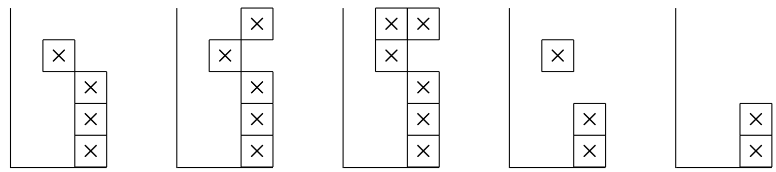


Figure 3.11: In order from left to right, we have an example of a possible diagram D and diagrams D_1 through D_4. In all cases, $\ni^2$ pushes the box in position $(3,1)$ to $(2,1)$.

See Figure 3.11 for an example of each case. By the definition of lock tableaux, going from $T^{<\ell}$ to T by adding back strings one at a time either has no effect on a pair of columns $c, c+1$ or it has the effect of one of the cases (1) or (2) above, which proves one direction of the claim.

Similarly, removing strings one at a time from T to obtain $T^{<\ell}$ either has no effect on a pair of columns $c, c+1$ or it has the effect of one of the cases (3) or (4) above. We do need to check that it is still true that the $M^c(D_3) > 0$ and $M^c(D_4) > 0$ conditions hold in cases (3) and (4) respectively. Using Corollary 3.4.7, we see that $R_{\alpha',p} \circ \cdots \circ R_{\alpha',1}$ is nonzero on $T^{<\ell}$, which must mean that $M^c(D_i) > 0$ does hold for cases (3) and (4). □

Using the same notation as above, we obtain the following corollary.

Corollary 3.4.11 ([37]). *Suppose that for every $1 \le s \le t$, $u_{p_s} \circ \cdots \circ u_{p_1}(T)$ is well defined and we have*

$$\ni_{p_s} \circ \cdots \circ \ni_{p_1}(\mathbb{D}(T)) = \mathbb{D}(u_{p_s} \circ \cdots \circ u_{p_1}(T)).$$

Then for all $1 \leq s \leq t$, $u_{p_s} \circ \cdots \circ u_{p_1}(T^{<\ell_q})$ *is well-defined and we have*

$$\backepsilon_{p_s} \circ \cdots \circ \backepsilon_{p_1}(\mathbb{D}(T^{<\ell_q})) = \mathbb{D}(u_{p_s} \circ \cdots \circ u_{p_1}(T^{<\ell_q})).$$

Proof. Proposition 1.5 tells us that the operators $U_{\alpha,p} \circ \cdots \circ U_{\alpha,1}$ left justify the boxes in strings ℓ_1 through ℓ_p. By the weakly decreasing row conditions and strictly decreasing column conditions on lock tableaux as well as the fact that unlock operators only push boxes southwest, the left justification of the strings $\ell_1, \ldots, \ell_p$ can only depend on the positions of boxes in those strings. Therefore, removing any string $\ell_m > \ell_p$ from T has no effect on the steps of the unlock algorithm up through the left justification of string ℓ_p. It follows that $u_{p_s} \circ \cdots \circ u_{p_1}(T^{<\ell_q})$ and $u_{p_s} \circ \cdots \circ u_{p_1}(T)$ have identical strings $\ell_1, \ldots, \ell_p$ for all $1 \leq s \leq t$, then combining with Lemma 3.4.10 proves the claim. □

We will use this corollary in proving the following lemma that if U_α and R_α agree on the level of diagrams on lock tableaux, then the Unlock algorithm preserves the properties necessary for the resulting tableau to be a Kohnert tableau of the same content as the inputted lock tableau. In particular, compare claims (2), (3), and (4) to Definition 3.1.1.

Lemma 3.4.12 ([37]). *Write*

$$R_\alpha = R_{\alpha,k} \circ \cdots \circ R_{\alpha,1} = \backepsilon_{k_t} \circ \cdots \circ \backepsilon_{k_1}$$

$$U_\alpha = U_{\alpha,k} \circ \cdots \circ U_{\alpha,1} = u_{k_t} \circ \cdots \circ u_{k_1}$$

and suppose that for $1 \le s \le t$, $u_{k_s} \circ \cdots \circ u_{k_1}(T)$ *is well defined and we have*

$$\ni_{k_s} \circ \cdots \circ \ni_{k_1}(\mathbb{D}(T)) = \mathbb{D}(u_{k_s} \circ \cdots \circ u_{k_1}(T))$$

Then the following hold:

1. *An operator* u_{k_i}, $1 \le i \le s$, *never tries to push a box* x *from* $(c+1, r)$ *to* (c, r) *where column* c *contains the rightmost box of a different string in some row weakly above* r.

2. *After all steps of* $U_{\alpha,i}$ *have been completed, the string* ℓ_i *is left justified and weakly descending in row index from left to right and remains so through every subsequent step of* U_α. *Furthermore, while the steps of* $U_{\alpha,i}$ *are in progress, all other strings than* ℓ_i *maintain their weakly decreasing property.*

3. *(inversions) For each intermediate labeled diagram* $u_{k_i} \circ \cdots \circ u_{k_1}(T)$ *with* $1 \le i \le t$, *if a column* c *has boxes* x, y *where* x *is both below* y *and has a larger label, then in column* $c+1$, *there is a box* z *strictly above the row index of* x *with the same label as* y.

4. *(flagged) For each intermediate diagram* $u_{k_i} \circ \cdots \circ u_{k_1}(T)$ *with* $1 \le i \le t$, *every box with label* ℓ *is no higher than row* ℓ.

Proof. We proceed by induction on the strings of T in increasing value of label. For the base case, the claims of Proposition 3.4.9 make it straightforward to check that while applying $U_{\alpha,1}$, the claims hold at every step. Now suppose that for $1 < p \le k$, the claims hold through all steps of $U_{\alpha,1}, U_{\alpha,2}, \ldots, U_{\alpha,p-1}$.

Proof of claim (1). Suppose that all conditions hold up to some u_{k_i}, and let $u_{k_{i-1}} \circ \cdots \circ u_{k_1}(T) = T'$. Suppose that u_{k_i} chooses a box x with label ℓ. Conduct all swaps of x that occur in step 2 while applying u_{k_i} to T', but stop just before u_{k_i} tries to push x left after all swaps have occurred. Let y be the rightmost box of some other string that is in column k_i and weakly above x. At this point, the underlying diagram is unchanged, so if we delete all boxes with labels larger than x to get $T'^{<\ell}$, then by Lemma 3.4.10, $R_\alpha(\mathbb{D}(T)) \neq 0$ means that $\ni_{k_i}(\mathbb{D}(T'^{<\ell})) \neq 0$, so $M^{k_i}(\mathbb{D}(T'^{<\ell})) > 0$.

Since x is weakly below y, deleting both x and y from $T'^{<\ell}$ to get $T''^{<\ell}$ preserves $M^{k_i}(\mathbb{D}(T''^{<\ell})) > 0$. Since all smaller labeled strings are left justified, columns $k_i, k_i{+}1$ of $T''^{<\ell}$ can either contain the rightmost box of a string or one box in each column from a string. Therefore, since smaller labeled strings are also weakly decreasing from left ot right, each string must contribute either 0 or -1 to a given row, and so it must hold that $M^{k_i}(\mathbb{D}(T''^{<\ell})) \leq 0$, which is a contradiction. Therefore, condition (1) must hold.

Proof of claim (2). We observe that if all strings with labels smaller than ℓ_p are weakly decreasing, then by definition, any swaps that occur during step 2 of an unlock operator between a box of string ℓ_p and a string $\ell_s < \ell_p$ will preserve the weakly descending property of string ℓ_s. Proposition 3.4.9 tells us that an unlock operator trying to push a box with label ℓ_p left cannot change the position of any boxes in a string $\ell_t > \ell_p$. Therefore, strings $\ell_s > \ell_p$ remain weakly descending because they are in the original tableau T.

It remains to check that string ℓ_p is weakly descending from left to right after the steps of $U_{\alpha,p}$ are completed. Index the boxes of string ℓ_p from left to right as $x_1, \ldots, x_t$. Suppose that for all x_j for $1 < j \leq i < t$ it holds that x_j is weakly lower than x_{j-1} after they have been left justified, where the base case for x_1 is vacuously true. Suppose also that over the course of being left justified, x_i was swapped m times from positions $(c_1, r_0), \ldots, (c_m, r_{m-1})$ to $(c_1, r_1), \ldots, (c_m, r_m)$ respectively, with $r_0 > r_1 > \cdots > r_m$ and $c_1 \geq c_2 \geq \cdots \geq c_m$, and index the respective strings that x_i swaps with as $\ell_{i_1}, \ldots, \ell_{i_m}$.

Since string ℓ_p is weakly decreasing to begin with, x_{i+1} must start in some row $r'_0 \leq r_0$. We know that string ℓ_{i_1} has a box in position (c_1, r_0), since x_i swapped from position (c_1, r_0) to (c_1, r_1). By condition (1), x_{i+1} cannot end up in the same column and strictly lower than a box in string ℓ_{i_1} unless there is some column $c'_1 > c_1$ in which x_{i+1} either swaps with string ℓ_{i_1} or swaps with some other string such that it ends up below some box of string ℓ_{i_1} in column c'_1. In either case, since string ℓ_{i_1} was already weakly decreasing before x_i swapped with it in column c_1, its box in column $c_1 + 1$ must have a row index weakly less than r_1, and so by the time x_{i+1} is pushed into column c_1, it must have a row index $r'_1 \leq r_1$. If $r'_1 \leq r_k$, then we are done. If we suppose instead that $r_j \geq r'_1 > r_{j+1}$ for some $1 \leq j < k$, then we can repeat the above argument with string $\ell_{i_{j+1}}$ to show that x_{i+1} must end up in some row $r'_2 \leq r_{j+1}$ before it reaches column c_{j+1}. Iterating this eventually forces x_{i+1} to end up weakly below row r_k, and therefore weakly below x_i. Since i was arbitrary, the entire string ℓ_p must be weakly decreasing left to right.

Proof of claim (3). By Proposition 3.4.9, no strings $\ell_s > \ell_p$ have inversions at any step of $U_{\alpha,1}, \ldots, U_{\alpha,p}$.

If an unlock operator swaps a box x of ℓ_p so that it is below the string $\ell_t < \ell_p$ in the same column, the operator terminates with a left push, so combined with the weakly decreasing property of string ℓ_t, x satisfies the inversion condition with the boxes of string ℓ_t directly after that operator is applied. Each successive unlock operator that left justifies x moves it left or down, so condition (2) ensures that x continues to satisfy the inversion condition with string ℓ_t. Otherwise, x stays above string ℓ_t, and the inversion condition is also satisfied.

It remains to show that, given an intermediate diagram in which inversion conditions are satisfied everywhere at all previous steps, any subsequent swaps that occur in $U_{\alpha,p}$ do not violate inversion conditions between pairs of strings $\ell_s, \ell_t < \ell_p$. To do this, we consider the following two diagrams (with other boxes suppressed).

We claim that if x in row r_1 swaps with a box y in row r_2, then any labels that appear between x and y have a smaller label than y. The diagram on the left gives an example of how there might be a larger label between x and y. However, if $j > i$, then the inversion condition is violated between the boxes in positions $(2,4), (2,5), (3,1)$, which contradicts that our given diagram satisfies inversion conditions. The right diagram shows the only way a swap might cause a trio of boxes that violates the inversion condition, with $j > i$, where a box from string j remains below the box of string i in the same column, but is moved weakly above a box of string i in the next column to the right.

The crux is how x made it to that position. If it was pushed left into that position, then it failed to swap with string i, so that cannot be possible. It could also have swapped with string i into that position, but then prior to that swap, the i in position $(3,3)$, the j in position $(2,4)$, and the i in column 2 above the j would violate the inversion condition. The last option is if x swapped with some box z with label k. However, by our previous claim, $k > i$, and then prior to x and z swapping, the inversion condition is not satisfied with z in the position of x, which is again a contradiction.

Proof of claim (4). Proposition 3.4.9 shows that no string $\ell_s > \ell_p$ is changed while any string $\ell_1, \dots, \ell_p$ is left justified, so boxes of string ℓ_s continue to satisfy the flagged condition because they did to begin with in T. Boxes of string ℓ_p can only move south or west while $U_{\alpha,p}$ is applied, so they must also continue to satisfy the flagged condition. Finally, the leftmost box of any string $\ell_t < \ell_p$ satisfies the flagged condition before $U_{\alpha,p}$ is applied by the inductive assumption. Unlock operators cannot change the position of the leftmost boxes of left justified strings, and such strings remain weakly decreasing from left to right by condition (2), so all boxes of strings $\ell_t < \ell_p$ must also satisfy the flagged condition through all steps of $U_{\alpha,p}$. $\square$

Up to this point, we have been examining the consequences of the assumption that the unlock algorithm is well defined on lock tableaux and that it agrees with rectification on the level of diagrams. We now show that this assumption indeed holds in general on lock tableaux.

Lemma 3.4.13 ([37]). *Write* $U_\alpha = u_{k_t} \circ \cdots \circ u_{k_1}$ *and* $R_\alpha = \ni_{k_t} \circ \cdots \circ \ni_{k_1}$. *The function* U_α *is well defined and*

$$\ni_{k_s} \circ \cdots \circ \ni_{k_1}(\mathbb{D}(T)) = \mathbb{D}(u_{k_s} \circ \cdots \circ u_{k_1}(T))$$

holds for all $1 \leq s \leq t$.

Proof. We proceed by induction, noting that the following argument proves both the base case at $m = 1$ and the inductive steps for $m > 1$. Suppose that for some m, we have

$$\ni_{k_s} \circ \cdots \circ \ni_{k_1}(\mathbb{D}(T)) = \mathbb{D}(u_{k_s} \circ \cdots \circ u_{k_1}(T))$$

for all $1 \leq s \leq m-1$, where $\ni_{k_{m-1}} \circ \cdots \circ \ni_{k_1}$ and $u_{k_{m-1}} \circ \cdots \circ u_{k_1}$ are the identity at $m = 1$. We first show that u_{k_m} is well defined on $u_{k_{m-1}} \circ \cdots \circ u_{k_1}(T) = T'$.

Proposition 3.4.9 shows that by construction, u_{k_m} must have a box that it tries to push left, so the only way that it can not be well defined is if the box it attempts to push left is in a position where it is directly to the right of and in the same row as the rightmost box of a different string. In this case, there is nothing to swap with, but it still cannot be pushed left into an open space. The proof of condition (1) of Lemma 3.4.12 can be repeated here to show that this cannot happen (noting that the proof of condition (1) does not require the assumption that u_{k_m} and $\ni_{k_m}$ agree on the level of diagrams), and therefore u_{k_m} must be well defined on T'.

Now we check that

$$\ni_{k_m} \circ \cdots \circ \ni_{k_1}(\mathbb{D}(T)) = \mathbb{D}(u_{k_m} \circ \cdots \circ u_{k_m}(T)).$$

Suppose that u_{k_m} chooses a box x to push left, with label ℓ. Due to the weakly descending arrangement of labels in columns $k_m, k_m + 1$ of $T'^{<\ell+1}$ as discussed above in the proof of condition (1) of Lemma 3.4.12, $\mathbb{D}(T'^{<\ell+1})$ has at most one horizontally unpaired box in column k_{m+1}, and it follows that we can at most have $M^{k_m}(T'^{<\ell+1}) = 1$, and if that maximum is acheived, it must be in the row containing the horizontally unpaired box.

Using Lemma 3.4.10 and $\ni_{k_m}(T') \neq 0$, we know this maximum must be achieved somewhere. Let r_0 be the row of x in $T'^{<\ell+1}$, and suppose u_{k_m} swaps it to rows $r_1, r_2, \ldots, r_t$ before being pushed left. The descending arrangement of labels in columns $k_m, k_m + 1$ means that a first upper bound for $r_{\max}$, the maximal row index such that $M^{k_m}(T'^{<\ell+1}, r_{\max}) = 1$, is r_0. However, since x swaps into row r_1, it must cross some string ℓ_{i_1} that has boxes at (k_m, r_1') and (k_m, r_1) with $r_1 < r_0 \leq r_1'$. Again using the descending arrangement of other labels, the string of x is the only string that can cumulatively contribute +1 to $M^{k_m}(T'^{<\ell+1}, r)$, so since string ℓ_{i_1} cumulatively contributes -1 to $M^{k_m}(T'^{<\ell+1}, r)$ for all $r_1 < r \leq r_1'$, we must have

$M^{k_m}(T'^{<\ell+1}, r_0) \leq 0$. Therefore $r_1 < r_0$ gives a new upper bound on $r_{\max}$. Iterating this argument eventually gives an upper bound of r_t.

Now x is in row r_t and is not crossing any strings. Once again following the proof of condition (1) of Lemma 3.4.12, we get that all labels in columns $k_m, k_m + 1$ that are above x must have a box in both columns. Therefore, $M^{k_m}(T'^{<\ell+1}, r_t) = 1$ so the upper bound is achieved and $r_{\max} = r_t$. Then we have

$$\text{϶}_{k_s} \circ \cdots \circ \text{϶}_{k_1}(\mathbb{D}(T)) = \mathbb{D}(u_{k_s} \circ \cdots \circ u_{k_1}(T)),$$

which completes the proof of the inductive step. □

Combining Lemmas 3.4.12 and 3.4.13 shows that the final diagram after applying the unlock algorithm to a lock tableau is a Kohnert tableau of the same content and that the underlying diagram is the same as the one resulting from rectification. The rectification operators are weight-preserving, injective, and intertwine with crystal operators on diagrams, so Theorem 3.4.1 follows.

Appendix A

Jack Polynomial Data

Some experimental data follows for the Schur expansion coefficients of various Jack polynomials. The first column gives the indexing partition of the Jack polynomial, and the third and fourth columns give the coefficient of the Schur polynomial s_μ in the Schur expansion of $\tilde{J}_\lambda^{(\alpha)}$. A shorthand is used for the respective bases used in the third and fourth columns that is given by the column header.

λ	μ	$r^k = \binom{\alpha+k}{\lvert\lambda\rvert}$	$r^k = \binom{\alpha}{k}k!$
1	1	1	1
20	20	$2r$	$2r + 1$
20	11	2	1
11	11	$2r + 2$	$2r + 2$
300	300	$6r^2$	$6r^2 + 6r + 1$
300	210	$12r$	$6r + 2$

300	111	6	1
210	210	$3r^2 + 8r + 1$	$3r^2 + 7r + 2$
210	111	$8r + 4$	$4r + 2$
111	111	$6r^2 + 24r + 6$	$6r^2 + 18r + 6$
4000	4000	$24r^3$	$24r^3 + 36r^2 + 12r + 1$
4000	3100	$72r^2$	$36r^2 + 24r + 3$
4000	2200	$24r^2 + 24r$	$12r^2 + 12r + 2$
4000	2110	$72r$	$12r + 3$
4000	1111	24	1
3100	3100	$8r^3 + 32r^2 + 8r$	$8r^3 + 28r^2 + 16r + 2$
3100	2200	$24r^2 + 24r$	$12r^2 + 12r + 2$
3100	2110	$40r^2 + 52r + 4$	$20r^2 + 22r + 4$
3100	1111	$36r + 12$	$6r + 2$
2200	2200	$12r^3 + 60r^2 + 24r$	$12r^3 + 48r^2 + 30r + 4$
2200	2110	$40r^2 + 52r + 4$	$20r^2 + 22r + 4$
2200	1111	$24r^2 + 60r + 12$	$12r^2 + 18r + 4$
2110	2110	$8r^3 + 72r^2 + 60r + 4$	$8r^3 + 48r^2 + 38r + 6$
2110	1111	$48r^2 + 84r + 12$	$24r^2 + 30r + 6$
1111	1111	$24r^3 + 264r^2 + 264r + 24$	$24r^3 + 168r^2 + 144r + 24$

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