GENERALIZATIONS OF MACMAHON'S THEOREM

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1 Preliminaries

1.1 Permutations

Let $[n] = \{1, 2, 3, \ldots, n\}$ be the set of the first n positive integers. A *permutation* π is a bijection from $[n]$ to $[n]$. We will express permutations in *one-line notation*, that is, $\pi = \pi_1\pi_2\cdots\pi_n$, where $\pi_i = \pi(i)$, as opposed to cycle notation or two-line notation. For example, the permutation $\pi = (32)(145)$ in cycle notation will be written as $\pi = 43251$. We use S_n to denote the set of permutations of length n. We let $|\cdot|$ denote the size of a set. For example, $|S_n| = n!$.

A *multiset permutation* π is an ordering of the multiset $\{1^{m_1}, 2^{m_2}, \ldots, n^{m_n}\}$, where m_i is the number of times i appears in the set. Let $\mathcal{R}(m_1, m_2, \ldots, m_n)$ denote the set of multiset permutations of $\{1^{m_1}, 2^{m_2}, \ldots, n^{m_n}\}$. Also, let $A_k = m_1 + m_2 + \ldots + m_k$. For example, $312232141 \in \mathcal{R}(3, 3, 2, 1)$ and $A_4 = 9$.

1.2 Permutation statistics

A *permutation statistic* is a function from S_n to the non-negative integers. Before defining some of the important statistics used in this paper, a few other definitions are required. Note that while the following statistics are defined on permutations, they can naturally be extended to multiset permutations without needing to change the definition at all.

A *descent* in a permutation is an entry followed by a smaller entry. The *descent* *set* of a permutation π, denoted $\mathrm{Des}(\pi)$, is defined by

$$\mathrm{Des}(\pi) = \{i : \pi_i > \pi_{i+1}\}.$$

The descent number of π, denoted $\mathrm{des}(\pi)$, is defined to be the number of descents:

$$\mathrm{des}(\pi) = |\,\mathrm{Des}(\pi)|.$$

Finally, a *descent top* is the larger entry in a descent, and a *descent bottom* is the smaller entry in a descent. For the example $\pi = 43251$, we have $\mathrm{Des}(\pi) = \{1, 2, 4\}$, and $\mathrm{des}(\pi) = 3$. The descent tops of π are $4, 3, 5$, and the descent bottoms are $3, 2, 1$.

An *inversion* in a permutation is a pair of entries, not necessarily adjacent, which appear in decreasing order. The *inversion number* of π, denoted $\mathrm{inv}(\pi)$, is the number of inversions in π. Continuing with the above example, the inversions of π are $\{43, 42, 41, 32, 31, 21, 51\}$, so $\mathrm{inv}(\pi) = 7$.

Another way of recording inversions which will be useful in later definitions is the *inversion sequence*, (a_i). This keeps track of the number of times entry π_i is the second entry in an inversion. Formally,

$$a_i = |\{j : j < i \text{ and } \pi_j > \pi_i \}|.$$

For example, the inversion sequence for 43251 is $(0, 1, 2, 0, 4)$.

One statistic we use in this paper is the *major index*, named after Major Percy MacMahon, who introduced it in [Mac]. The major index of a permutation π, denoted $\mathrm{maj}(\pi)$, is the sum of the indices of the descents of π. In other words,

$$\mathrm{maj}(\pi) = \sum_{i \in \mathrm{Des}(\pi)} i.$$

Continuing with the above example, $\mathrm{maj}(43251) = 1 + 2 + 4 = 7$.

1.3 Marked permutations

A *marked permutation* is a permutation with some subset of its descents marked. Let $\widehat{\mathrm{Des}}(\pi)$ denote the indices of the marked descents, so $\widehat{\mathrm{Des}}(\pi) \subseteq \mathrm{Des}(\pi)$. Let $\widehat{\mathrm{des}}(\pi) = |\widehat{\mathrm{Des}}(\pi)|$. We use a $>$ sign to denote the marked descents when writing a marked permutation. For example, 4>325>1 is a marked permutation with $\widehat{\mathrm{Des}}(\pi) = \{1, 4\}$ and $\widehat{\mathrm{des}}(\pi) = 2$. Let $\widehat{S_n}$ denote the set of marked permutations of length n.

The inversion number and major index statistics can be extended to $\widehat{S_n}$. The *marked inversion number*, denoted minv, is defined by

$$\mathrm{minv}(\pi) = \mathrm{inv}(\pi) - \sum_{i \in \widehat{\mathrm{Des}}(\pi)} (a_i + 1). \tag{1.1}$$

(note that $\mathrm{inv}(\pi)$ refers to the inversion number of the underlying unmarked permutation). Equivalently,

$$\mathrm{minv}(\pi) = \sum_{i \notin \widehat{\mathrm{Des}}(\pi)} a_i - \widehat{\mathrm{des}}(\pi). \tag{1.2}$$

To see this equivalence, note that inv is the sum of all the a_i's, and so subtracting $a_i + 1$ for every $i \in \widehat{\mathrm{Des}}(\pi)$ cancels out these a_i's and subtracts one for every element of $\widehat{\mathrm{Des}}(\pi)$.

The *marked major index*, denoted mmaj, is defined by

$$\mathrm{mmaj}(\pi) = \mathrm{maj}(\pi) - \sum_{i \in \widehat{\mathrm{Des}}(\pi)} |\{j : j \in \mathrm{Des}(\pi) \text{ and } j \leq i\}|. \tag{1.3}$$

In words, marked major index is defined as follows: compute the major index for the underlying permutation, and then for each marked descent, subtract the number of descents weakly preceding that descent (i.e. including the marked descent itself). For example,

$$\pi = 85{>}274{>}3{>}16$$

has $\mathrm{Des}(\pi) = \{1, 2, 4, 5, 6\}$, and $\widehat{\mathrm{Des}}(\pi) = \{2, 5, 6\}$ so

$$\mathrm{mmaj}(\pi) = (1 + 2 + 4 + 5 + 6) - (2 + 4 + 5) = 7.$$

Note that the notion of a descent extends naturally to multiset permutations (it is still just the index of an entry which is followed by a strictly smaller entry), so we can define a *marked multiset permutation* to be a multiset permutation with some subset of its descents marked. Let $\widehat{\mathcal{R}}(m_1, m_2, \ldots, m_n)$ denote the set of marked multiset permutations of $\{1^{m_1}, 2^{m_2}, \ldots, n^{m_n}\}$. For example,

$$\pi = 3{>}12232{>}141 \in \widehat{\mathcal{R}}(3, 3, 2, 1).$$

In this example, $\mathrm{Des}(\pi) = \{1, 5, 6, 8\}$ and $\widehat{\mathrm{Des}}(\pi) = \{1, 6\}$. Now, the definition of mmaj for marked multiset permutations is the same as in (1.3). So in this example,

$$\mathrm{mmaj}(\pi) = 1 + 5 + 6 + 8 - (1 + 3) = 16.$$

The definitions for inversions and inversion sequence also carry over to multiset permutations. Thus, minv for marked multiset permutations is defined as in (1.1) and (1.2). Continuing the above example, the inversion sequence for π is

$(0, 1, 1, 1, 0, 2, 5, 0, 6)$. It follows (using (1.2)) that

$$\mathrm{minv}(\pi) = (1 + 1 + 1 + 0 + 5 + 0 + 6) - 2 = 12.$$

2 MacMahon's Theorem

In this section, we state and give several proofs of MacMahon's Theorem, which says that inv and maj have the same distribution on S_n. More generally, any statistic which has this distribution is said to be *mahonian*. MacMahon first introduced the major index and showed it had the same distribution as the inversion number [Mac]. Here we present bijective constructions given by Carlitz [C], Foata [F], and Rawlings [R].

2.1 Carlitz's Proof

We state MacMahon's theorem and give a bijective proof due to Carlitz [C].

Theorem 2.1. *(MacMahon) The statistics* maj *and* inv *are equidistributed on the set of permutations of length n. That is,*

$$\sum_{\pi \in S_n} q^{\mathrm{inv}(\pi)} = \sum_{\pi \in S_n} q^{\mathrm{maj}(\pi)}. \tag{2.1}$$

We break up the proof into two lemmas which will be useful as we will use a similar method in the generalization that follows in the next section. In these lemmas, we think of a permutation as being built recursively by inserting n into a permutation of length $n-1$ and then considering the effect this has on inversion number and major index.

Lemma 2.2. *(Inversion Insertion) For $\pi \in S_{n-1}$ and $i \in \{0, 1, \ldots, n-1\}$, there is exactly one location where one can insert n that increases the inversion number by i.*

Proof. Note that n creates a new inversion with every element to its right, and creates no inversions with any element to its left, and does not affect any existing inversions. Therefore, to increase the number of inversions by i, insert n so that it has i elements to its right. □

Lemma 2.3. *(Major Index Insertion) For $\pi \in S_{n-1}$ and for each $i \in \{0, 1, \ldots, n-1\}$, there is exactly one location where one can insert n that increases the major index by i.*

Proof. Note that there are n possible spaces between existing entries of π (including the space before the first entry and the space after the last entry) where one can insert n. Classify these spaces as follows.

- *Ascent spaces* are spaces between a pair of entries which are in increasing order along with the space before the first entry.

- *Descent spaces* are spaces between a pair of entries which are in decreasing order along with the space after the last entry.

Let $\mathrm{des}(\pi) = k$ be the number of descents in π. Then there are $k+1$ descent spaces, and hence $n-k-1$ ascent spaces. Label the descent spaces from right to left $0, 1, 2, \ldots, k$. Then label the ascent spaces from left to right by $k+1, k+2, \ldots, n-1$. For example, if $\pi = 74236581$, then the spaces are numbered as follows.

$$\underset{5}{\sqcup}\,7\,\underset{4}{\sqcup}\,4\,\underset{3}{\sqcup}\,2\,\underset{6}{\sqcup}\,3\,\underset{7}{\sqcup}\,6\,\underset{2}{\sqcup}\,5\,\underset{8}{\sqcup}\,8\,\underset{1}{\sqcup}\,1\,\underset{0}{\sqcup}$$

We claim that the label for a space gives the increase in major index from inserting n into that space.

First, consider inserting n into a descent space. Note that this does not increase the number of descents because the descent formed by n and the entry following n is replacing the descent that already existed at that position. This increments (by one) the index of every descent at or to the right of the insertion point. So the effect on maj is an increase by the number of descent spaces to the right of the insertion space. Then by the way we have labeled the descent spaces, the increase in maj is the same as the label of the insertion space.

Second, consider inserting n into an ascent space. Here, n both creates a new descent and increases the index of each descent to its right. So the contribution to maj is the number of entries to the left of, and including, the insertion space plus the number of descents to the right of the insertion space. This is the same as the label of the ascent space, so we are done. □

Proof. (of Theorem 2.1) We give a recursive bijection φ from S_n to S_n such that $\text{maj}(\pi) = \text{inv}(\varphi(\pi))$. Given $\pi \in S_n$, let σ be the permutation which results from deleting n, and let $i = \text{maj}(\pi) - \text{maj}(\sigma)$ be the resulting change in major index. Recursively compute $\varphi(\sigma)$, and then insert n so that it increases the inversion number of $\varphi(\sigma)$ by i (i.e., so that n has i entries to its right). By the previous lemmas this is well-defined and bijective. □

2.2 Foata's bijection

In this section we present a bijection ϕ due to Foata [F] which takes major index to inversion number. This bijection is also recursive in nature, and although it is more complex than the bijection given in Section 2.1, it can be composed with other bijections to give an involution on permutations which interchanges maj and inv, as done by Foata and Schützenberger [FS]. This shows the joint equidistribution of major index and inversion number. That is,

$$\sum_{\pi \in S_n} q^{\mathrm{maj}(\pi)} t^{\mathrm{inv}(\pi)} = \sum_{\pi \in S_n} q^{\mathrm{inv}(\pi)} t^{\mathrm{maj}(\pi)}. \tag{2.2}$$

The bijection is defined as follows. Suppose $\pi = \pi_1 \pi_2 \cdots \pi_n$ is a permutation written in one-line notation. If $n \leq 2$, then $\phi(\pi) = \pi$. Let $\phi^{(k)} = \phi(\pi_1 \pi_2 \cdots \pi_k)$ be the result of applying the bijection to the first k entries of π.

Assume we have computed $\phi^{(n-1)}$ recursively. Then append π_n to the end. If $\pi_{n-1} < \pi_n$, then put a bar after every element of $\phi^{(n-1)}$ which is less than π_n. Otherwise, if $\pi_{n-1} > \pi_n$, then put a bar after every element of $\phi^{(n-1)}$ which is greater than π_n. Either way, this partitions the elements of $\phi^{(n-1)}$ into blocks. For each block, take the rightmost element of the block and move it to the beginning of the block. Then erase the bars.

As an example, let $\pi = 3417625$. Stepping through the process gives

$$\phi^{(2)} = 34$$

$$\phi^{(3)} = 3|4|1 \rightarrow 341$$

$$\phi^{(4)} = 3|4|1|7 \rightarrow 3417$$

$$\phi^{(5)} = 3417|6 \rightarrow 73416$$

$$\phi^{(6)} = 7|3|4|16|2 \rightarrow 734612$$

$$\phi^{(7)} = 73|4|61|2|5 \rightarrow 3741625.$$

Therefore, $\phi(3417625) = 3741625$. One can check that $\mathrm{maj}(3417625) = \mathrm{inv}(3741625)$, as desired.

2.3 Rawlings' proof

Rawlings [R] defined a permutation statistic on S_n called the r-major index, denoted r-maj, which interpolates between maj and inv as r varies from 1 to n. First, define the set of r-*descents*, denoted r-Des, and r-*inversions*, denoted r-Inv, as follows:

$$\begin{aligned} r\text{-Des}(\pi) &= \{i \,:\, \pi_i \geq \pi_{i+1} + r\}. \\ r\text{-Inv}(\pi) &= \{(i,j) \,:\, i < j \text{ and } r > \pi_i - \pi_j > 0\}. \end{aligned} \tag{2.3}$$

Now, r-maj is defined by

$$r\text{-maj}(\pi) = |r\text{-Inv}(\pi)| + \sum_{i \in r\text{-Des}(\pi)} i + . \tag{2.4}$$

Note that r-Des is the set of descents where the descent top and bottom differ by at least r. Conversely, r-Inv is the set of inversions where the larger element and smaller element differ by less than r. Notice that when $r = 1$, 1-Des is the set of (ordinary) descents and 1-Inv is empty, hence 1-maj = maj. And when $r = n$, n-Des is empty and n-Inv is the entire set of inversions, and so n-maj = inv on S_n.

As an example, consider the permutation $\pi = 6437521$. We have that $2\text{-Des}(\pi) = \{1, 4, 5\}$, since these are the indices of the descents whose top and bottom differ by at least 2. We find that $2\text{-Inv}(\pi) = \{(6, 5), (4, 3), (3, 2), (2, 1)\}$, since these are the inversions of π which differ by less than 2. Therefore, $2\text{-maj}(\pi) = 4+(1+4+5) = 14$.

Rawlings [R] showed bijectively that r-maj has the same distribution on S_n for every r.

Theorem 2.4. *For every* $1 \le r, s \le n$*,*

$$\sum_{\pi\in S_n} q^{r\text{-maj}(\pi)} = \sum_{\pi\in S_n} q^{s\text{-maj}(\pi)}. \tag{2.5}$$

Proof. We give Rawlings' construction, which is a generalization of Carlitz's construction given above, but we omit the proof that it is bijective (it is similar to the proofs of Lemmas 2.2 and 2.3). For a particular r and s, we describe the mapping $\Gamma_{rs} : S_n \to S_n$ such that $r\text{-maj}(\pi) = s\text{-maj}(\Gamma_{rs}(\pi))$.

Given $\pi \in S_n$, let $\sigma \in S_{n-1}$ be the result of deleting n from π. Let

$$i = r\text{-maj}(\pi) - r\text{-maj}(\sigma)$$

be the resulting change in r-maj. Then recursively compute $\Gamma_{rs}(\sigma)$, and reinsert n so that the resulting increase in s-maj is i.

To determine the increase in s-maj from inserting n in a given space, identify the spaces where inserting n would not create a new s-descent, and label these from right to left starting at 0. Continue labeling the remaining spaces (those which *do* create a new s-descent) from left to right. For example, we have the following possible increases in 4-maj from inserting 7 into the various spaces of 652341:

$$\underset{3}{\sqcup}\,6\,\underset{2}{\sqcup}\,5\,\underset{4}{\sqcup}\,2\,\underset{5}{\sqcup}\,3\,\underset{1}{\sqcup}\,4\,\underset{6}{\sqcup}\,1\,\underset{0}{\sqcup}\,.$$

Thus, after recursively computing $\Gamma_{rs}(\sigma)$, we can label the spaces as above to determine the (unique) correct location to reinsert n to increase s-maj by i. □

As an example of the mapping Γ_{rs}, consider again the permutation $\pi = 6437521$, and suppose $r = 2$ and $s = 4$. First, we keep deleting the largest element of the permutation, keeping track of the effect on 2-maj.

1. $6437521 \to 643521$; change to 2-maj is 5.
2. $643521 \to 43521$; change to 2-maj is 3.
3. $43521 \to 4321$; change to 2-maj is 3.
4. $4321 \to 321$; change to 2-maj is 1.
5. $321 \to 21$; change to 2-maj is 1.
6. $21 \to 1$; change to 2-maj is 1.

Now, we build $\Gamma_{24}(\pi)$ by inserting the next larger element into the position which increases 4-maj by the required amount.

1. $1 \to 21$ increases 4-maj by 1.
2. $21 \to 231$ increases 4-maj by 1.

3. $231 \to 2341$ increases 4-maj by 1.
4. $2341 \to 52341$ increases 4-maj by 3.
5. $52341 \to 652341$ increases 4-maj by 3.
6. $652341 \to 6527341$ increases 4-maj by 5.

Thus, $\Gamma_{24}(6437521) = 6527341$.

3 Generalizations of MacMahon's Theorem

In this section we give the generalization of MacMahon's theorem to marked permutations, and then to marked multiset permutations.

3.1 Marked Permutations

The origin of marked permutations and the marked inversion and major index statistics came about in the course of proving the following conjecture of Haglund:

$$\sum_{\pi \in S_n} q^{\mathrm{maj}(\pi)} \prod_{i=1}^{\mathrm{des}(\pi)} \left(1 + \frac{z}{q^i}\right) = \sum_{\pi \in S_n} q^{\mathrm{inv}(\pi)} \prod_{i=1}^{\mathrm{des}(\pi)} \left(1 + \frac{z}{q^{a_i+1}}\right). \tag{3.1}$$

Here, as before, (a_i) is the inversion sequence defined by

$$a_i = \#\{j : j < i \text{ and } \pi_j > \pi_i \}.$$

Note that by setting $z = 0$, we recover (2.1). Recall that $\widehat{\mathrm{des}}(\pi)$ denotes the number of marked descents in π. Then the left hand side of (3.1) can be written

$$\sum_{\pi \in \widehat{S_n}} q^{\mathrm{mmaj}(\pi)} z^{\widehat{\mathrm{des}}(\pi)}.$$

To see this, interpret the ith factor in the product on the left as a choice: either mark the ith descent of π, or leave the descent unmarked. A marked descent then contributes a factor of z and divides by a factor of q for every descent weakly before it, which agrees with how mmaj is defined.

Similarly, the right hand side of (3.1) can be written

$$\sum_{\pi \in \widehat{S_n}} q^{\mathrm{minv}(\pi)} z^{\widehat{\mathrm{des}}(\pi)}.$$

Again, think of the ith factor in the product as a choice to mark or leave unmarked the ith descent. Choosing to mark the ith descent contributes a factor of z and divides by $a_i + 1$ factors of q, which agrees with the definition of minv.

Thus, proving (3.1) is equivalent to finding a bijection $\phi : \widehat{S_n} \to \widehat{S_n}$ such that $\widehat{\mathrm{des}}(\phi(\pi)) = \widehat{\mathrm{des}}(\pi)$ and $\mathrm{minv}(\phi(\pi)) = \mathrm{mmaj}(\pi)$.

Define ϕ recursively as follows.

1. Given a marked permutation $\widehat{\pi} \in \widehat{S_n}$, let σ denote the (underlying) permutation which results from deleting n from π.

2. Let $k = \mathrm{maj}(\pi) - \mathrm{maj}(\sigma)$ (i.e. k is the difference in maj of the underlying permutations).

3. If n created a new descent in π, then take every marked descent weakly to the left of n and move it left by one. Otherwise (if n did not create a new descent in π), take every marked descent among the first k descents and move it left by one. If the leftmost descent is marked and gets bumped, then this mark disappears.

4. Now, delete n from $\widehat{\pi}$. If n was a marked descent top, then the mark remains in this space (which must still be a descent, or else the mark would have gotten shifted to the left in the previous step without another mark to replace it).

5. Record the change in mmaj, say i, which results and whether a mark disappeared in the previous step.

6. Find $\phi(\sigma)$ recursively and then insert n to increase minv by i and mark the newly created descent if a mark disappeared in the previous step.

This mapping can be defined more explicitly by repeatedly stripping out the largest element of π, making the appropriate shifts in marks, and keeping track of the change in mmaj and whether a mark was removed at each step. Then we build up $\phi(\pi)$ with $\text{minv}(\phi(\pi)) = \text{mmaj}(\pi)$ by repeatedly inserting the next larger element into the space which gives the prescribed increase in minv and marking the newly formed descent (if required). We demonstrate the mapping on the marked permutation

$$\pi = 85>274>3>16.$$

Removing 8 gives $\sigma = 5>274>3>16$. Note that

$$k = \text{maj}(\pi) - \text{maj}(\sigma) = 5,$$

but since 8 created a new descent, and all the marks are to the right of 8, nothing gets bumped. The decrease in mmaj is 2, and no mark was lost. Recording this first step and each of the subsequent steps gives:

1. 5>274>3>16, change: 2, no lost mark.
2. 524>3>16, change: 2, lost mark.
3. 524>3>1, change: 0, no lost mark.
4. 24>3>1, change: 1, no lost mark.
5. 23>1, change: 1, lost mark.
6. 21, change: 0, lost mark.
7. 1, change: 1, no lost mark.

Now, we build $\phi(\pi)$ according to the necessary increases in minv and marks:

1. $1 \to 21$, change: 1, no new mark.

2. 21 → 23>1, change: 0, new mark.
3. 23>1 → 4>23>1, change: 1, new mark.
4. 4>23>1 → 4>253>1, change: 1, no new mark.
5. 4>253>1 → 4>253>16, change: 0, no new mark.
6. 4>253>16 → 4>27>53>16, change: 2, new mark.
7. 4>27>53>16 → 4>27>583>16, change: 2, no new mark.

So ϕ(85>274>3>16) = 4>27>583>16.

Theorem 3.1. *ϕ is a bijection from $\widehat{S_n}$ to $\widehat{S_n}$ which fixes number of marks and takes* mmaj *to* minv*.*

Proof. The mapping ϕ is the same as the mapping defined in the next section when restricted to marked permutations. So the proof that it is correct follows from the proof of Theorem 3.2. □

3.2 Marked Multiset Permutations

We now turn to the generalization of MacMahon's theorem to marked multiset permutations. The statement can be expressed purely in terms of multiset permutations, as in (3.1):

$$\sum_{\pi \in \mathcal{R}(m_1,\ldots,m_n)} q^{\mathrm{maj}(\pi)} \prod_{i=1}^{\mathrm{des}(\pi)} \left(1 + \frac{z}{q^i}\right) = \sum_{\pi \in \mathcal{R}(m_1,\ldots,m_n)} q^{\mathrm{inv}(\pi)} \prod_{i=1}^{\mathrm{des}(\pi)} \left(1 + \frac{z}{q^{a_i+1}}\right). \tag{3.2}$$

As mentioned in Section 1.3, the definitions of marked inversion number and marked major index are the same for marked multiset permutations as for marked permutations. So (3.2) can be expressed equivalently as

$$\sum_{\pi \in \widehat{\mathcal{R}}(m_1,\ldots,m_n)} q^{\mathrm{mmaj}(\pi)} z^{\widehat{\mathrm{des}}(\pi)} = \sum_{\pi \in \widehat{\mathcal{R}}(m_1,\ldots,m_n)} q^{\mathrm{minv}(\pi)} z^{\widehat{\mathrm{des}}(\pi)}. \tag{3.3}$$

We require some notation and terminology before we can define the bijection which will prove (3.3). In a multiset permutation $\sigma \in \mathcal{R}(m_1, \ldots, m_{n-1})$, a *LR position* is a space where inserting an n would create a new descent. Recall that we let $A_{n-1} = m_1 + m_2 + \ldots + m_{n-1}$ be the length of σ. Note that there are $A_{n-1} - \mathrm{des}(\sigma)$ LR positions. In the following example, the LR positions (where inserting 5 would create a new descent) are indicated with a square bracket:

$$\sqcup 31 \sqcup 2 \sqcup 2 \sqcup 321 \sqcup 41.$$

A *RL position* is a space where inserting an n would not create a new descent. σ has $\mathrm{des}(\sigma) + 1$ RL positions (each of the existing descents of σ along with the space all the way to the right of σ). Note that once an n is inserted into a LR position, then the space to its left becomes a RL position. So, in the following example where some 5's have already been inserted, the following are RL positions (where inserting one or more 5's would not create a new descent):

$$3 \sqcup 1 \sqcup 52 \sqcup 523 \sqcup 2 \sqcup 1 \sqcup 54 \sqcup 1 \sqcup.$$

We recursively build a multiset permutation $\pi \in \mathcal{R}(m_1, \ldots, m_n)$ by inserting m_n n's into $\sigma \in \mathcal{R}(m_1, \ldots, m_{n-1})$. We break the insertion into two stages and record the insertion as follows. First, we insert some n's into some of the LR positions. Record this insertion with a 0-1 sequence where the jth entry is a 1 if an n was inserted into the jth LR space counting from the right (for technical reasons) and a 0 if not. Call this sequence τ_{LR}. Suppose i n's were inserted in this stage so that τ_{LR} consists of i 1's and $A_{n-1} - \text{des}(\sigma) - i$ 0's.

In the second stage, we insert the remaining $m_n - i$ n's into the $\text{des}(\sigma) + i + 1$ RL positions. Note that several n's may be inserted into the same RL position. This can be thought of as the classic balls-in-bins problem, where the RL positions are the bins (counted from right to left, for technical reasons) and the n's are the balls. We record this as a sequence, τ_{RL}, of 0's and 1's, where the 1's serve as the dividers between RL positions and the 0's represent the n's going into that RL position. Note that τ_{RL} consists of $m_n - i$ 0's and $\text{des}(\sigma) + i$ 1's (we require one less divider than the number of bins).

For example, inserting six 5's into the above permutation might give

$$315552523251541.$$

The first stage is inserting a 5 into the second, third, and fifth LR positions (and recording it with τ_{LR}, which records these from right to left):

$$\sqcup 31 \underset{5}{\sqcup} 2 \underset{5}{\sqcup} 2 \sqcup 321 \underset{5}{\sqcup} 41 \qquad \tau_{\text{LR}} = 10110.$$

The second stage is inserting 5's into the indicated RL positions (again, recording these with τ_{RL} from right to left):

$$3 \sqcup 1 \underset{55}{\sqcup} 52 \sqcup 523 \sqcup 2 \underset{5}{\sqcup} 1 \sqcup 54 \sqcup 1 \sqcup \qquad \tau_{\text{RL}} = 1110111001.$$

We will think of a marked multiset permutation $\widehat{\pi} \in \widehat{\mathcal{R}}(m_1, \ldots, m_n)$ to be a pair $\pi \in \mathcal{R}(m_1, \ldots, m_n)$ (the underlying multiset permutation) and a 1-2 sequence τ_D, the *descent mark sequence*, where the jth entry is a 2 if the jth descent of $\widehat{\pi}$ is marked, and 1 otherwise.

With these definitions in place, the mapping $\phi : \widehat{\mathcal{R}}(m_1, \ldots, m_n) \to \widehat{\mathcal{R}}(m_1, \ldots, m_n)$ is defined recursively as follows. It is a technical construction which resulted from parsing a purely algebraic proof by Remmel and Wilson [RW1].

1. Given $\widehat{\pi} \in \widehat{\mathcal{R}}(m_1, \ldots, m_n) = (\pi, \tau_D)$, as defined above. Let σ be the multiset permutation obtained by deleting the m_n n's from π, and τ_{LR} and τ_{RL} (as defined above) are the insertion sequences which are used to build π from σ.

2. Form a sequence τ_C of length $m_n + \mathrm{des}(\sigma)$ as follows. Begin with τ_{RL} and replace the 1's with the entries of τ_D. Then consider the subsequence consisting of 0's and 1's. Leaving the 2's in their current locations, perform a *reverse complement* on the subsequence of 0's and 1's (that is, reverse the order of them and replace 1's with 0's and 0's with 1's).

3. Form a sequence τ_A by concatenating τ_{LR} and τ_C.

4. Let β_1 be the subsequence of 0's and 1's in τ_A.

5. Let β_2 be τ_C where we have replaced any 0's with 1's.

6. Use the last $\mathrm{des}(\sigma)$ entries of β_2 as the descent mark sequence for $\widehat{\sigma}$, then discard these entries and call the result β_2^*.

7. Combine β_1 and β_2^* in a sequence γ by replacing the 1's in β_1 by the entries of the reverse of β_2^*.

8. Form the sequence γ_M from the subsequence of 0's and 2's in γ.

9. Form the sequence γ_U by replacing all the 2's with 0's in γ.

10. Recursively compute $\phi(\widehat{\sigma})$.

11. Insert n's with a mark using γ_M as an indicator sequence. That is, go from left to right through the possible insertion spaces and if the corresponding entry of γ_M is a 2, then insert an n with a mark. If the entry is a 0, then do not insert anything in that space.

12. Insert n's without a mark according to γ_U. Here, interpret the 0's as dividers and the 1's as the n's which go in a certain space, from left to right. That is, the number of 1's between the jth and $(j+1)$th 0 is the number of n's to insert in the $(j+1)$th possible insertion space.

As an example, we demonstrate each above step on the permutation

$$\widehat{\pi} = 31555{>}2523{>}25{>}15{>}4{>}1.$$

1. We have
$$\pi = 315552523251541 \quad \text{and} \quad \tau_D = 1212222.$$
As above,
$$\sigma = 312232141 \quad \tau_{\text{LR}} = 10110 \quad \tau_{\text{RL}} = 1110111001.$$
Note also that $\text{des}(\sigma) = 4$ and $m_5 = 6$.

2. Replacing the 1's in τ_{RL} with the entries of τ_D gives
$$1210222002.$$
The 0-1 subsequence is 11000, whose reverse complement is 11100. Replacing the 0-1 subsequence with its reverse complement gives
$$\tau_C = 1211222002.$$
3. $\tau_A = \tau_{\mathrm{LR}} \cdot \tau_C = 101101211222002.$
4. The 0-1 subsequence from τ_A gives $\beta_1 = 1011011100$.
5. Replacing the 0's with 1's in τ_C gives $\beta_2 = 1211222112$.
6. The final $\mathrm{des}(\sigma) = 4$ entries of β_2 are 2112, so using this as the descent mark sequence for $\widehat{\sigma}$, we find that
$$\widehat{\sigma} = 3{>}1223214{>}1.$$
Discarding these entries gives $\beta_2^* = 121122$.
7. Reversing β_2 gives 221121 and replacing the 1's in β_1 with this sequence gives $\gamma = 2021012100$.
8. The subsequence of 0's and 2's of γ gives $\gamma_M = 2020200$.
9. Replacing the 2's with 0's in γ gives $\gamma_U = 0001010100$.
10. Recursively computing $\phi(\widehat{\sigma})$ (these steps omitted) gives
$$\phi(\widehat{\sigma}) = 4{>}1321322{>}1.$$
11. Inserting n's with marks according to γ_M (that is, into the first, third, and fifth possible spaces) gives
$$\underset{5>}{\sqcup}\,4{>}1 \sqcup 3 \underset{5>}{\sqcup} 2 \sqcup 1 \underset{5>}{\sqcup} 3 \sqcup 2 \sqcup 2{>}1 \longrightarrow 5{>}4{>}135{>}215{>}322{>}1.$$
12. Inserting n's without marks according to γ_U gives
$$\sqcup\, 5{>}4{>}1 \sqcup 3 \sqcup 5{>}2 \underset{5}{\sqcup} 1 \underset{5}{\sqcup} 5{>}3 \underset{5}{\sqcup} 2 \sqcup 2{>}1 \sqcup \longrightarrow 5{>}4{>}135{>}25155{>}3522{>}1.$$

Theorem 3.2. *The above mapping ϕ is a bijection from $\widehat{\mathcal{R}}(m_1,\ldots,m_n)$ to itself which takes* mmaj *to* minv.

Proof. To see that ϕ is a bijection, we note that each of the above steps is reversible. In other words, given γ_U, γ_M, and $\widehat{\sigma}$, we can work our way back through each step of the construction to recover $\widehat{\pi}$.

Beginning with γ_U and γ_M, we can replace the subsequence of 0's in γ_U with γ_M to obtain γ. Replacing the 2's in γ with 1's gives β_1. Reversing the 1-2 subsequence in γ recovers the first m_n entries of β_2 (that is, β_2^*). The remaining entries of β_2 are formed by the descent mark sequence of $\widehat{\sigma}$.

The first $A_{n-1} - \mathrm{des}(\sigma)$ entries of β_1 form τ_{LR}. Taking the remaining portion of β_1 and replacing the subsequence of 1's in β_2 with these entries recovers τ_C.

Take the reverse complement of the 0-1 subsequence of τ_C. Then the 1-2 subsequence of this gives τ_D, and replacing the 1-2 subsequence with all 1's recovers τ_{RL}. Finally, τ_{LR} and τ_{RL} are used to insert n's into σ, which recovers π. And τ_D records the markings of $\widehat{\pi}$. Thus, ϕ is a bijection.

To see that ϕ takes mmaj to minv, we begin by relating $\mathrm{maj}(\pi)$ to $\mathrm{maj}(\sigma)$. This was studied in [HLR]. Let $i = \mathrm{des}(\pi) - \mathrm{des}(\sigma)$ (equivalently, i is the number of n's inserted into LR positions which is also the number of 1's in τ_{LR}). The possible increases from inserting n into an LR position are $\{\mathrm{des}(\sigma)+1, \mathrm{des}(\sigma)+2, \ldots, A_{n-1}\}$. At most one n can be inserted into any LR position. Note also that each n inserted into an LR position also increments the index of every subsequent n inserted into an LR position (hence adds one to maj for every n inserted into an LR position to its right). Thus, the minimum increase in maj, which results from inserting the i n's into the leftmost LR positions, is

$$\begin{aligned}
&(\operatorname{des}(\sigma)+1)+(\operatorname{des}(\sigma)+2)+\ldots+(\operatorname{des}(\sigma)+i)+(0+1+2+\ldots+i-1)\\
&=i\cdot\operatorname{des}(\sigma)+1+2+\ldots+i+\binom{i}{2}\\
&=i\cdot\operatorname{des}(\sigma)+\binom{i+1}{2}+\binom{i}{2}\\
&=i\cdot\operatorname{des}(\sigma)+i^2\\
&=i(\operatorname{des}(\sigma)+i)\\
&=i\operatorname{des}(\pi),
\end{aligned}$$

by our definition of i. When an n is moved to an LR position further right, its contribution to maj increases by one for every LR space to the right by which it moves. This means that the additional contribution to maj which results from spreading the n's further right is recorded by $\operatorname{inv}(\tau_{\mathrm{LR}})$, since an inversion in τ_{LR} comes from a 1 being moved left past a 0, which is equivalent to an n being inserted one space further to the right (since τ_{LR} records insertions into LR spaces from right to left).

Recall that inserting i n's into LR positions increases the number of RL positions to $\operatorname{des}(\sigma)+i+1$. So the possible contributions to maj are (from right to left) $\{0,1,\ldots,\operatorname{des}(\sigma)+i\}$. Recall that by the construction of τ_{RL} (which uses 1's to indicate dividers between RL positions and 0's to indicate n's going into an RL position), the number of 1's to the left of a 0 corresponds to which RL position (from right to left) that n is inserted. And the RL position into which an n is inserted corresponds to its contribution to maj. Thus, $\operatorname{inv}(\tau_{\mathrm{RL}})$ records the contribution to maj from inserting

$m_n - i$ n's into RL positions according to τ_{RL}. In total, we have

$$\mathrm{maj}(\pi) = \mathrm{maj}(\sigma) + i\,\mathrm{des}(\pi) + \mathrm{inv}(\tau_{\mathrm{LR}}) + \mathrm{inv}(\tau_{\mathrm{RL}}) \tag{3.4}$$

Now, we consider $\mathrm{mmaj}(\widehat{\pi})$, so we must subtract from $\mathrm{maj}(\pi)$ the number of descents weakly to the left of each marked descent. Let k be the number of marked descents of $\widehat{\pi}$. Then the decrease from marking descents comes from adding up k decreases from the set $\{1, 2, \ldots, \mathrm{des}(\pi)\}$. Then by a similar calculation to the above, the maximum decrease (from choosing to mark the k rightmost descents) is given by

$$\mathrm{des}(\pi) + (\mathrm{des}(\pi) - 1) + \ldots + (\mathrm{des}(\pi) - (k-1)) = k\,\mathrm{des}(\pi) - \binom{k}{2}.$$

Each time we move a marked descent to the left by one, it decreases the loss to mmaj by one. Recall that τ_D records marked descents as 2's and non-marked descents as 1's from left to right. So moving a marked descent to the left by one is equivalent to moving a 2 to the left of a 1 in τ_D. Therefore, $\mathrm{inv}(\tau_D)$ records how much we need to subtract from the total decrease to mmaj due to marked descents. In other words,

$$\mathrm{mmaj}(\widehat{\pi}) = \mathrm{maj}(\pi) - \left(k\,\mathrm{des}(\pi) - \binom{k}{2} - \mathrm{inv}(\tau_D)\right). \tag{3.5}$$

Therefore, combining (3.4) and (3.5) we find that

$$\mathrm{mmaj}(\widehat{\pi}) = \mathrm{maj}(\sigma) + i\,\mathrm{des}(\pi) + \mathrm{inv}(\tau_{\mathrm{LR}}) + \mathrm{inv}(\tau_{\mathrm{RL}}) - k\,\mathrm{des}(\pi) + \binom{k}{2} + \mathrm{inv}(\tau_D). \tag{3.6}$$

Next, we make a few observations about the various slicing and dicing operations on the various sequences defined in the course of the mapping, and how the inversion numbers relate to one another. Sometimes we will be interested in only the inversions between, say, the 2's in a sequence and the 0's in a sequence. We refer to these as the *2-0 inversions*, and similarly for other pairs of entries. We also briefly refer to the *coinversions* of a sequence, which are a pair of entries in increasing order.

First, we show that

$$\mathrm{inv}(\tau_C) = \mathrm{inv}(\tau_{\mathrm{RL}}) + \mathrm{inv}(\tau_D). \tag{3.7}$$

Recall that τ_C was formed by replacing the 1's in τ_{RL} with the entries of τ_D and then reverse complementing the 0-1 subsequence of the result. Note that reverse complementing the 0-1 subsequence does not change the inversion number, so we can consider τ_C before we apply the reverse complement. The inversions in τ_C between 2 or 1 and 0 come from the same positions as the inversions of τ_{RL} (since some of the 1's in τ_{RL} became 2's but otherwise nothing changed). The inversions between 2 and 1 are the inversions of τ_D. This accounts for all of the inversions of τ_C, hence establishes (3.7).

The next sequence in the construction is τ_A. This is just the concatenation of τ_{LR} and τ_C, and so inversions of τ_A are the inversions of τ_{LR}, the inversions of τ_C, and any new inversions formed between the two. Since τ_{LR} is a 0-1 sequence, we get new inversions between any 1 from τ_{LR} and any 0 from τ_C. The number of 1's in τ_{LR} was denoted i, and the number of 0's in τ_C is $\mathrm{des}(\pi) - k$. This shows

$$\mathrm{inv}(\tau_A) = \mathrm{inv}(\tau_{\mathrm{LR}}) + \mathrm{inv}(\tau_C) + i(\mathrm{des}(\pi) - k). \tag{3.8}$$

Now we look at β_1 and β_2. Recall that β_1 is the 0-1 subsequence of τ_A, which is the concatenation of τ_{LR} and τ_C. Therefore, the inversions of β_1 are the 1-0 inversions of τ_{LR} (namely, all the inversions of τ_{LR} since τ_{LR} is a 0-1 sequence), the 1-0 inversions of τ_C, and the 1-0 inversions between the two (which we counted above as part of (3.8)). This gives

$$\operatorname{inv}(\beta_1) = \operatorname{inv}(\tau_{\text{LR}}) + |\{\text{1-0 inversions in } \tau_C\}| + i(\operatorname{des}(\pi) - k). \tag{3.9}$$

Recalling that β_2 is the same as τ_C with all the 0's changed to 1's, we see that the inversions in β_2 are the same as the inversions of τ_C, except we lose the 1-0 inversions (the 2-0 inversions get converted into 2-1 inversions so we do not lose them). Thus

$$\operatorname{inv}(\beta_2) = \operatorname{inv}(\tau_C) - |\{\text{1-0 inversions in } \tau_C\}|. \tag{3.10}$$

Adding together (3.9) and (3.10) (cancelling the 1-0 inversions of τ_C) and using (3.7), we find that

$$\begin{aligned} \operatorname{inv}(\beta_1) + \operatorname{inv}(\beta_2) &= \operatorname{inv}(\tau_{\text{LR}}) + i(\operatorname{des}(\pi) - k) + \operatorname{inv}(\tau_C) \\ &= \operatorname{inv}(\tau_{\text{LR}}) + i(\operatorname{des}(\pi) - k) + \operatorname{inv}(\tau_{\text{RL}}) + \operatorname{inv}(\tau_D). \end{aligned} \tag{3.11}$$

Combining this with (3.6) gives

$$\operatorname{mmaj}(\widehat{\pi}) = \operatorname{maj}(\sigma) + \operatorname{inv}(\beta_1) + \operatorname{inv}(\beta_2) - k \operatorname{des}(\sigma) + \binom{k}{2}. \tag{3.12}$$

Now, let $\tau_D(\sigma)$ be the final $\operatorname{des}(\sigma)$ entries of β_2 (since these entries form the descent mark sequence for $\widehat{\sigma}$), and recall that β_2^* denotes the result of discarding these entries from β_2. Note that, similar to the calculation in (3.8), we have

$$\operatorname{inv}(\beta_2) = \operatorname{inv}(\beta_2^*) + \operatorname{inv}(\tau_D(\sigma)) + |\{\text{2's in } \beta_2^*\}| \cdot |\{\text{1's in } \tau_D(\sigma)\}|. \tag{3.13}$$

Let j be the number of 2's in β_2^*. Since β_2 had k 2's total, it follows that $\tau_D(\sigma)$ has $k-j$ 2's and hence $\operatorname{des}(\sigma)-(k-j)$ 1's. Thus (3.13) can be written

$$\operatorname{inv}(\beta_2) = \operatorname{inv}(\beta_2^*) + \operatorname{inv}(\tau_D(\sigma)) + j(\operatorname{des}(\sigma) - k + j) \tag{3.14}$$

Making this replacement in (3.12) gives

$$\operatorname{mmaj}(\widehat{\pi}) = \operatorname{maj}(\sigma)+\operatorname{inv}(\beta_1)+\operatorname{inv}(\beta_2^*)+\operatorname{inv}(\tau_D(\sigma))+j(\operatorname{des}(\sigma)-k+j)-k\operatorname{des}(\sigma)+\binom{k}{2}. \tag{3.15}$$

Notice that $k-j$ is the number of 2's in $\tau_D(\sigma)$, hence it is the number of marked descents in $\widehat{\sigma}$. This means we can rewrite (3.5) in terms of σ, which gives

$$\begin{aligned}\operatorname{mmaj}(\widehat{\sigma}) &= \operatorname{maj}(\sigma) - (k-j)\operatorname{des}(\sigma) + \binom{k-j}{2} + \operatorname{inv}(\tau_D(\sigma)).\\ &= \operatorname{maj}(\sigma) - k\operatorname{des}(\sigma) + j\operatorname{des}(\sigma) + \binom{k-j}{2} + \operatorname{inv}(\tau_D(\sigma)).\end{aligned} \tag{3.16}$$

Combining (3.15) and (3.16) gives

$$\operatorname{mmaj}(\widehat{\pi}) = \operatorname{mmaj}(\widehat{\sigma}) - \binom{k-j}{2} + \operatorname{inv}(\beta_1) + \operatorname{inv}(\beta_2^*) + \binom{k}{2} - jk + j^2. \tag{3.17}$$

Expanding and simplifying shows

$$\begin{aligned}\binom{k}{2} - \binom{k-j}{2} - jk + j^2 &= \frac{k(k-1) - (k-j)(k-j-1) - 2jk + 2j^2}{2}\\ &= \frac{k^2 - k - k^2 + 2jk - j^2 - j + k - 2jk + 2j^2}{2}\\ &= \frac{j^2 - j}{2}\\ &= \binom{j}{2}.\end{aligned}$$

So we find that

$$\operatorname{mmaj}(\widehat{\pi}) = \operatorname{mmaj}(\widehat{\sigma}) + \operatorname{inv}(\beta_1) + \operatorname{inv}(\beta_2^*) + \binom{j}{2}. \tag{3.18}$$

Now, we relate the inversions in β_1 and β_2^* with γ and then with γ_M and γ_U. Recall that γ_M is the 0-2 subseqence of γ and γ_U is γ where the 2's have been replaced by 0's. So the inversions of γ_M are the 2-0 inversions of γ. The inversions of γ_U are the inversions of γ excluding the 2-0 and 2-1 inversions, and including the 1-2 coinversions (since a 1-2 coinversion becomes a 1-0 inversion when the 2's become 0's in γ_U). Therefore,

$$\begin{aligned} \mathrm{inv}(\gamma_M) + \mathrm{inv}(\gamma_U) &= |\{\text{2-0 invs in } \gamma\}| + \mathrm{inv}(\gamma) - |\{\text{2-0 invs in } \gamma\}| \\ &\quad - |\{\text{2-1 invs in } \gamma\}| + |\{\text{1-2 coinvs in } \gamma\}| \\ &= \mathrm{inv}(\gamma) - |\{\text{2-1 invs in } \gamma\}| + |\{\text{1-2 coinvs in } \gamma\}|. \end{aligned} \tag{3.19}$$

Now, recall that γ is β_1 with its 1's replaced by the reverse of β_2^*. So the 1-2 coinversions in γ are precisely the 2-1 inversions in β_2^* (which are the only inversions in β_2^* since it is a 1-2 sequence). So

$$|\{\text{1-2 coinvs in } \gamma\}| = \mathrm{inv}(\beta_2^*). \tag{3.20}$$

Also, the inversions of β_1 become 2-0 and 1-0 inversions in γ, so

$$\begin{aligned} \mathrm{inv}(\gamma) - |\{\text{2-1 invs in } \gamma\}| &= |\{\text{2-0 invs in } \gamma\}| + |\{\text{1-0 invs in } \gamma\}| \\ &= \mathrm{inv}(\beta_1). \end{aligned} \tag{3.21}$$

Putting together (3.19), (3.20), and (3.21), we find that

$$\mathrm{inv}(\gamma_M) + \mathrm{inv}(\gamma_U) = \mathrm{inv}(\beta_1) + \mathrm{inv}(\beta_2^*). \tag{3.22}$$

Finally, making this replacement in (3.18), we find that

$$\begin{aligned} \text{mmaj}(\widehat{\pi}) &= \text{mmaj}(\widehat{\sigma}) + \text{inv}(\gamma_U) + \text{inv}(\gamma_M) + \binom{j}{2} \\ &= \text{minv}(\phi(\widehat{\sigma})) + \text{inv}(\gamma_U) + \text{inv}(\gamma_M) + \binom{j}{2}, \end{aligned} \tag{3.23}$$

by induction.

According to the construction, after recursively finding $\phi(\widehat{\sigma})$, we insert m_n n's into $\phi(\widehat{\sigma})$ first using γ_M to indicate where the marked n's should go, and then using γ_U to indicate where the unmarked n's should go. Inserting an n with a mark increases minv by 1 less than the number of elements to its right which are not marked descent tops. Note that in γ_M, every element of the sequence corresponds to a valid possible place to insert a marked n. But for every element of $\widehat{\sigma}$ that is not a marked descent top (other than the final element), there is a valid space to insert a marked n immediately following it. So the effect on minv is to add the number of elements to the right of each 2 in γ_M. This includes the $\text{inv}(\gamma_M)$ inversions of γ_M, as well as all the 2-2 occurences, which are counted by $\binom{j}{2}$.

The effect of inserting an n without a mark increases minv by the number of elements to its right which are not marked descent tops. Note that in γ_U, the 0's serve as dividers between possible spaces for unmarked n's. A possible space for an unmarked n comes right after an element which is not a marked descent top. Thus, a 0 to the right of a 1 in γ_U corresponds to an element which is not a marked descent top to the right of an n, and so it follows that the increase in minv due to the insertion of unmarked n's is $\text{inv}(\gamma_U)$.

Thus, the overall increase in minv from inserting n's according to γ_M and γ_U is

$$\operatorname{inv}(\gamma_U) + \operatorname{inv}(\gamma_M) + \binom{j}{2},$$

and so combining this with (3.23) shows that

$$\begin{aligned} \operatorname{mmaj}(\widehat{\pi}) &= \operatorname{minv}(\phi(\widehat{\sigma})) + \operatorname{inv}(\gamma_U) + \operatorname{inv}(\gamma_M) + \binom{j}{2} \\ &= \operatorname{minv}(\phi(\widehat{\pi})), \end{aligned} \tag{3.24}$$

as desired. □

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