Investigations of Graph Polynomials

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ABSTRACT

Investigations of graph polynomials

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This thesis consists of two parts. The first part is a brief introduction to graph polynomials. We define the matching, rook and hit polynomials, reveal the connection between them and show necessary conditions that imply that all roots of these polynomials are real.

In the second part, we focus on the closely related Monotone Column Permanent (MCP) conjecture of Haglund, Ono, and Wagner [HOW99]. This conjecture is known to be true for $n \leq 3$ [Hag00] and is open for $n \geq 4$. We present our following new results on this conjecture. First, we give an alternative proof for the $n = 2$ case. Then, we generalize the MCP conjecture for non-square matrices. Further developing the idea of the proof used in the $n = 2$ case, we show that the conjecture holds for matrices of size $n \times 2$ and $2 \times m$ as well. Finally, we investigate a different approach in attempt to prove the conjecture. We obtain partial results by proving that the MCP conjecture for $n \times n$ matrices implies the MCP conjecture for $n \times m$ matrices for $n \leq m$. We also show a conditional result that if the MCP conjecture is true for $n \times n$ then it is also true for the $(n+1) \times (n+1)$ case under the assumption that the permanents of certain minors of the matrix have interlacing roots.

Introduction

Graph polynomials are polynomials assigned to graphs. Interestingly, they also arise in many areas outside Graph Theory as well. The matching polynomial, for example, was studied under different names in Combinatorics, Statistical Physics, and Theoretical Chemistry. The matching polynomial – the generating function of the matching numbers – provides a compact representation and, furthermore, allows us to infer information about these numbers using tools from Generatingfunctionology, Linear Algebra, Analysis, and Geometry.

In this thesis, we focus on graph polynomials that have only real roots. Polynomials with only real roots arise in various applications in Control Theory and Computer Science, but also admit interesting mathematical properties on their own. Newton noted that the sequence of coefficients of such polynomials form a log-concave (and hence unimodal) sequence. These polynomials also have strong connections to totally positive matrices.

Organization

In section 1, we start by introducing graph polynomials. We define the matching, rook and hit polynomials, and their weighted analogues. We review some classic results of Heilmann and Lieb [HL72], and Nijenhuis [Nij76] showing that the roots of the weighted matching polynomials and weighted rook polynomials, respectively, are all real. Then, we continue with the recent works of Haglund, Ono, and Wagner [HOW99] and Haglund [Hag00] which generalize the above results for the hit polynomials of certain classes of graphs.

In section 2, we continue by exploring possible further generalizations of the above theorems. In particular, we investigate the Monotone Column Permanent (MCP) Conjecture of Haglund, Ono, and Wagner [HOW99] that asserts that the hit polynomial of a general class of graphs related to the generalization of Ferrers boards have only real roots. This conjecture is known to be true for $n \leq 3$ [Hag00] and is open for $n \geq 4$.

We prove the following new results on this conjecture. First, we give an alternative proof of the $n = 2$ case. Then, we generalize the conjecture for non-square matrices and further developing the idea used in our proof, we prove the $n \times 2$ and $2 \times m$ cases as well. We continue by proving some necessary conditions for the 3×3 case. We conclude with outlining a different approach in attempt to prove the conjecture, and also prove some new results which provide further evidence that the conjecture holds for larger matrices.

1 Graph polynomials with real roots

In this section, we introduce the graph polynomials which will be the subject of our investigation. We start with the matching polynomials of graphs. We state the Heilmann-Lieb theorem which shows that all roots of the matching polynomials are real. Then, we define the closely related rook polynomials of boards and state the theorem of Nijenhuis, that all roots of the rook polynomial are real. We show the explicit correspondence, noted by E. Bender, between the rook polynomials of boards and matching polynomials of graphs, namely that the latter are special cases of the matching polynomials for bipartite graphs. We also define the hit polynomials of boards (or equivalently bipartite graphs) and we state a theorem of Haglund, Ono, and Wagner that shows that the roots of the hit polynomials of the Ferrers boards are all real. This result generalizes the above theorems for Ferrers boards, because it implies the roots of their rook polynomials are all real. We also note, however, that this theorem of Haglund, Ono, and Wagner does not generalize to all bipartite graphs. In particular, we conclude the section by giving an example of a graph that has a hit polynomial with complex roots.

1.1 Matching polynomial

Matching polynomials play an important role in Combinatorics. They are related to various other polynomials such as the chromatic polynomial, Chebyshev polynomials, and Hermite polynomials and they have been extensively studied in the past decades. We start by providing the basic definition of the matching polynomials and state the surprising theorem of Heilmann and Lieb that shows that all roots of these polynomials are real.

Definition 1.1 (Matching polynomial)**.** Let $G = (V, E)$ be a simple, undirected graph. A k-matching of G is a set of k disjoint edges of E. Let $\mathcal{M}_k(G)$ denote the set of all k-matchings of G and $m_k = |\mathcal{M}_k|$ denote their number. We define the matching polynomial of G as the generating polynomial of $m_k(G)$:

$$\mu(G,x) = \sum_{k=0}^{\lfloor n/2 \rfloor} m_k(G) x^k \ .$$

Here, $n = |V|$ is the number of vertices in G. We adopt the convention that $m_0(G) = 1$. Note, that this polynomial is sometimes called the matching generating polynomial to disambiguate it from the so-called matching defect polynomial (sometimes also referred to as the matchings polynomial [God93]). The matching defect polynomial is usually defined as $\sum_{k=0}^{\lfloor n/2 \rfloor} m_k(G) x^{n-2k} = \mu(G, \frac{1}{x}) x^n$.

Often we deal with weighted graphs, graphs where we assign a weight to each edge. Hence, it is useful to consider the following generalization of the matching polynomial.

Definition 1.2 (Weighted matching polynomial)**.** Given a graph G and a weight function $w : E \to (0, \infty)$, define the weight of a matching $M \in \mathcal{M}_k(G)$ to be $w(M) = \prod_{e \in M} w(e)$. Then, the weighted matching polynomial of G can be written as (see Figure 1 for an example):

$$\phi(G,x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \left(\sum_{M \in \mathcal{M}_k(G)} w(M) \right) x^k.$$

A rather surprising theorem on the location of the roots of weighted matching polynomials is the following.

Theorem 1.3 (Heilmann-Lieb [HL72])**.** *If $G = (V, E)$ is a simple weighted graph with nonnegative edge weights, then the roots of $\phi(G, x)$ are real.*

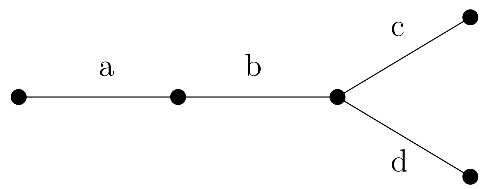

Figure 1: A weighted graph G with edge weights a, b, c, d. The weighted matching polynomial of G is $\phi(G, x) = (ac + ad)x^2 + (a + b + c + d)x + 1$. By setting $a = b = c = d = 1$, we get that the matching polynomial of G is $\mu(G, x) = 2x^2 + 4x + 1$.

We now define *interlacing* – a property that we will use throughout the rest of the thesis.

Definition 1.4 (Interlacing)**.** Let $f(x)$ and $g(x)$ denote two polynomials with real roots of degree n and $n - 1$, respectively. Denote the roots of $f(x)$ by $x_1, x_2, \ldots, x_n$ and the roots of $g(x)$ by $\xi_1, \xi_2, \ldots, \xi_{n-1}$. We say that $g(x)$ interlaces $f(x)$, and denote it by $g(x) \prec f(x)$ if their roots obey the following relation:

$$x_1 \le \xi_1 \le x_2 \le \xi_2 \le \cdots \le \xi_{n-1} \le x_n \ .$$

If $f(x)$ and $g(x)$ are both of degree n, then using the above notation, $g(x) \prec f(x)$ if

$$\xi_1 \le x_1 \le \xi_2 \le x_2 \le \cdots \le x_{n-1} \le \xi_n \le x_n \ .$$

(Sometimes this second relation is called *left alternating* and handled separately, but for our purposes it is sufficient to handle them together.)

For sake of completeness, we provide a sketch of Heilmann-Lieb's original proof.

Sketch of proof of Theorem 1.3. We will prove the version here which uses the assumption that the edge weights are strictly positive. The nonnegative edge weight version follows from this by applying a continuity argument. The key observation in the proof is that the matching polynomial of a graph G satisfies the following two-step recursion, i.e., for any vertex i in G:

$$\phi(G,x) = \phi(G-i,x) + x \sum_{j \in \Gamma(i)} w(i,j)\phi(G-i-j,x).$$

Here, $G-i$ denotes the graph obtained by removing the vertex i and its adjacent edges from G. $\Gamma(i)$ denotes the set of vertices that are adjacent to vertex i, and $w(i,j)$ is the positive weight assigned to edge (i,j). In fact, we prove the theorem by proving the following stronger statement.

1. The roots of $\phi(G,x)$ are negative real numbers.

2. For all vertices $i \in V(G) : \phi(G-i,x) \prec \phi(G,x)$.

We prove this statement by induction on n, the number of vertices in G. The statement holds trivially for the empty graph and if G contains only a single vertex. Assume it is true for all subgraphs G'. The second part implies that for any pair of vertices $i,j : \phi(G-i-j,x) \prec \phi(G-i,x)$. Because the weights are positive, and the roots of $\phi(G-i-j,x)$ are negative by the first part of the inductive hypothesis this also implies that $\sum_{j\in\Gamma(i)} w(i,j)\phi(G-i-j,x) \prec \phi(G-i,x)$. Now consider the sign of the value the sum polynomial takes on at the location of the roots of $\phi(G-i,x)$ and as x tends to positive and negative infinity. Due to the interlacing, the sign changes at all neighboring positions and we can count $\lfloor n/2 \rfloor$ sign changes. Hence, we can deduce that $\phi(G,x)$ has $\lfloor n/2 \rfloor$ real roots. As $\phi(G,x)$ has degree $\lfloor n/2 \rfloor$, it follows that it has only real roots. These roots are all negative, since the sign changes occur in the negative intervals. We are left to show that $\phi(G-i,x)$ interlaces $\phi(G,x)$. For this, observe that $\phi(G-i,x) \prec x\sum_{j\in\Gamma(i)} w(i,j)\phi(G-i-j,x)$ and since $f \prec g$ implies that $f \prec f+g$ as shown in [HOW99] (see also [Wag92]) the statement follows. □

1.2 Rook polynomial

We now turn to a related class of polynomials that originates from counting the number of possible ways of placing non-attacking rooks on a chessboard. As we will see it later, the rook polynomials are in fact special cases of the matching polynomials. In Rook Theory, the so-called *rooks problem* was formulated in the following way: In how many ways can we place k non-attacking rooks on a (general) chessboard?

In this section, we define the rook polynomials of boards, and their weighted version. We show the connection to the matching polynomials of graphs and state the theorem of Nijenhuis, showing that all roots of the weighted rook polynomials are real. First of all, we define formally what we mean by a generalized chessboard which we simply call a board.

Definition 1.5 (Board)**.** Let A be an $n \times m$ array. We define a board $\mathcal{B}$ to be a subset of squares in A, denoted by $\mathcal{B} \subseteq A$. (See Figure 2 for an example.)

Now we can define the rook polynomials.

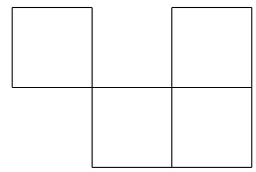

Figure 2: Example of a board $\mathcal{B} \subset A_{2\times 3}$ with four squares.

Definition 1.6 (Rook polynomial)**.** Let A be an $n \times m$ array and $\mathcal{B} \subseteq A$ a board. We say that two rooks are non-attacking if they are not in the same row nor in the same column. Denote the number of ways k non-attacking rooks can be placed on $\mathcal{B}$ by $r_k(\mathcal{B})$. The rook polynomial is the generating polynomial of $r_k(\mathcal{B})$:

$$\rho(\mathcal{B}, x) = \sum_{k=0}^{\min(n,m)} r_k(\mathcal{B}) x^k \ .$$

For example, the rook polynomial of the board shown in Figure 2 is $3x^2 + 4x + 1$.

Similarly to the matching polynomial, the rook polynomial can also be generalized to a weighted version. We can associate each square of the board with a weight. The weight information can be conveniently encoded in a matrix.

Definition 1.7 (Weighted rook polynomial)**.** Let A be an $n \times m$ matrix. Instead of simply counting the number of ways k rooks can be placed, we associate each placement with a weight. A weight of a placement of rooks is the product of the entries of A which are under the rooks. We define the (weighted) kth rook number, $r_k(A)$, to be the sum of the weights over all non-attacking placements of k rooks on board A. The weighted rook polynomial of A is the generating function:

$$\rho(A, x) = \sum_{k=0}^{\min(n,m)} r_k(A) x^k \ .$$

(a) Weighted board (b) Weighted bipartite graph

Figure 3: Correspondence between a board (a) and a bipartite graph (b). The rook polynomial of the board is the same as the matching polynomial of the graph, namely: $(ac + ad + ae + bd + be)x^2 + (a + b + c + d + e)x + 1$.

Note that it is sufficient to define the weighted rook polynomial for $n \times m$ matrices only since since any general weighted board can be represented by putting zeros in the corresponding position where the square is not part of the board.

E. Bender noted that the rook polynomial can be viewed as a matching polynomial of a complete bipartite graph [Nij76]. Consider the following bijection between weighted boards and weighted bipartite graphs. Let $\mathcal{B}$ be an $n \times m$ board and consider a complete bipartite graph $K_{n,m}$ with two vertex partitions R and C with cardinalities n and m, respectively. The partition R will correspond to the rows of the board and C to the columns. The squares of the board correspond to the edges of the graph. More precisely, the entry in the ith row and jth column of the board will correspond to the weight $w(i, j)$, where $i \in R$ and $j \in C$. So, $\rho(\mathcal{B}, x) = \mu(K_{n,m}, x)$, with the appropriate weight function, i.e., $w(i, j) = \mathcal{B}_{ij}$. See Figure 3 for an example.

Given this correspondence, the following result of Nijenhuis [Nij76] on the location of the roots of weighted rook polynomials is in fact a special case of Theorem 1.3.

Theorem 1.8 (Nijenhuis)**.** *If A is a matrix with nonnegative real entries, then all roots of $\rho(A, x)$ are real.*

1.3 Hit polynomial

Another important polynomial in Rook Theory is the hit polynomial. There is a strong connection between the rook and hit numbers (see Theorem 1.11), and hence the rook and hit polynomials. As a consequence of this connection and a theorem of Laguerre (see Theorem 1.14) the following observation was made in [HOW99]:

Proposition 1.9 (Haglund–Ono–Wagner)**.** *If all the roots of the hit polynomial of a bipartite graph are real, then all the roots of its rook polynomials are also real.*

Therefore, it is of great interest to investigate for which bipartite graphs (or equivalently, matrices) are all the roots of the hit polynomial real. Since, by proving that the hit polynomial of a matrix is real, one could obtain a generalization of Nijenhuis' theorem for a class of graphs (or matrices).

In this section, we define the hit polynomials and show their connection to the rook polynomials. Then, we consider a class of boards, called Ferrers boards. We state a theorem of Haglund, Ono, and Wagner that shows that the hit polynomials of these boards have only real roots. We continue by exploring the boundaries of the above theorem and show that the theorem of Haglund, Ono, and Wagner does not extend to all boards (or equivalently bipartite graphs). We give a simple example of a graph whose hit polynomial has complex roots.

Definition 1.10 (Hit polynomial)**.** The hit polynomial is the generating polynomial for the k-hit numbers. The k-hit numbers can be thought of in the following way. Given a spanning subgraph G of $K_{n,n}$, what is the number of perfect matchings in a $K_{n,n}$ which contain exactly k edges from G [God93]. Formally, let G be a subgraph of $K_{n,n}$ and denote the k-hit number of G by $h_k(G)$. Then the hit polynomial of G is defined as:

$$\tau(G, x) = \sum_{k=0}^{n} h_k(G)x^k \ .$$

Analogously to the rook polynomials, we also define the hit polynomial of a bipartite graph $G \subseteq K_{n,n}$ to be the hit polynomial of its corresponding $n \times n$ board. Note that the definition of the hit polynomials can be easily extended to the weighted case as well. Again, we can use a matrix to encode the weights and the zero weight represents that there is no edge, and a zero column or row represents an isolated point in the corresponding graph.

The following theorem of Kaplansky and Riordan [KR46] shows the intricate connection between the rook numbers and hit numbers.

Theorem 1.11 (Kaplansky–Riordan)**.** *Let A be a $n \times n$ matrix. Then the hit polynomial of A can be written as*

$$\tau(A, x) = \sum_{k=0}^{n} r_k(A)(x-1)^k(n-k)! \ .$$

Now we define an important class of boards, called Ferrers boards.

Definition 1.12 (Ferrers board)**.** Let A be an $n \times n$ board (or $0-1$ matrix) with the property that if $a_{ij} = 1$, then $a_{k\ell} = 1$ whenever $k \geq i$ and $\ell \geq j$.

Theorem 1.13 (Haglund–Ono–Wagner)**.** *All the roots of the hit polynomial of a Ferrers boards are real.*

Haglund, Ono, and Wagner [HOW99] also showed that, among other results, this theorem implies Theorem 1.8 for Ferrers boards (see Proposition 1.9). We provide a proof of the proposition for sake of completeness.

Proof of Proposition 1.9. Note that the hitting polynomial $\tau(G, x)$ has only real roots if and only if $\tau(G, x+1)$ has, and equivalently if and only if $\tau(G, \frac{1}{x+1})$ has only real roots (since the roots are negative, hence nonzero). Since $\tau(G, \frac{1}{x+1}) = \sum_{k=0}^{n} r_k(A)x^k k!$ has only real roots so does $\rho(G, x) = \sum_{k=0}^{n} r_k(A)x^k$ by the following theorem. □

Theorem 1.14 (Laguerre)**.** *If $\sum_k a_k x^k$ is a polynomial with real roots, then so is $\sum_k a_k x^k / k!$.*

However, as noted in [HOW99], the converse of Laguerre's theorem in general does not hold. Here, we show that Theorem 1.13 does not extend to all graphs. Consider the path graph P_5 on five vertices: v_1, v_2, v_3, v_4, v_5 with edges (v_1, v_2), (v_2, v_3), (v_3, v_4), and (v_4, v_5). The rook numbers are: $r_2(P_5) = 3$, $r_1(P_5) = 4$, and $r_0(P_5) = 1$. Note the rook polynomial $\rho(P_5, x) = 3x^2 + 4x + 1 = (3x+1)(x+1)$ has real roots, however the hit polynomial $\tau(P_5, x) = 3(x-1)^2 \cdot 0! + 4(x-1) \cdot 1! + 1 \cdot 2! = 3x^2 - 2x + 1$ does not, since its discriminant is negative, $D = 4 - 12 < 0$.

2 The Monotone Column Permanent Conjecture

In this section, we will discuss a Monotone Column Permanent (MCP) conjecture by Haglund, Ono, and Wagner which, if true, would be a generalization of their result on the hit polynomials of Ferrers boards (see Theorem 1.13). In Rook Theory terminology, the conjecture would extend the results from Ferrers boards to the so-called Monotone Column matrices. In contrast to Ferrers boards, the entries in a Monotone Column matrix are real numbers that are weakly increasing down column.

The conjecture has been shown to be true for $n \leq 3$ and for several special cases, and it is also supported by computational evidence [HOW99, Hag00].

We start this section by showing the connection between the hit polynomials and permanents. This will allow us to state the conjecture in its permanent form. Then, we continue by providing an alternative proof for the $n = 2$ case. We generalize the conjecture for non-square matrices and prove the $n \times 2$ and $2 \times m$ cases. We also provide some evidence for the 3×3 case. Finally, we investigate a different approach in attempt to prove the conjecture. We obtain partial results by proving that the MCP conjecture for $n \times n$ matrices implies the MCP conjecture for $n \times m$ matrices for $n \leq m$. We also show a conditional result that if the MCP conjecture is true for $n \times n$, then it is also true for the $(n+1) \times (n+1)$ case under the assumption that the permanents of certain minors of the matrix have interlacing roots.

Let us begin with noting an interesting connection, namely the nth rook number, $r_n(A)$ is in fact the permanent of A. Recall the definition of the permanent.

Definition 2.1 (Permanent)**.** Let A be an $n \times n$ matrix. The permanent of A, denoted by $\mathrm{per}(A)$, is defined as

$$\mathrm{per}(A) = \sum_{\sigma \in \mathfrak{S}_n} a_{1\sigma(1)} a_{2\sigma(2)} \dots a_{n\sigma(n)} \ ,$$

where $\mathfrak{S}_n$ denotes the set of all permutations on n letters.

This connection is important because it allows us to rewrite the hit polynomial using the permanent, as shown in [HOW99]:

$$\mathrm{per}(A(z-1) + J_n) = \tau(A, z) \ .$$

Here J_n denotes the matrix of all ones. Now, we can state the conjecture posed by Haglund, Ono, and Wagner [HOW99] in its permanent form:

Conjecture 2.2 (Monotone Column Permanent (MCP))**.** *Let A be a real $n \times n$ matrix with entries weakly increasing down the columns, i.e., $\forall i, j : a_{i,j} \leq a_{i+1,j}$. Then, the roots of* $\mathrm{per}(zA + J_n)$ *are real, where J_n is an $n \times n$ matrix of all ones.*

The MCP conjecture was shown to be true for $n \leq 3$ [Hag00] and for the following special cases [HOW99]: Let A be an $n \times n$ Ferrers matrix, i.e., a matrix with real entries weakly increasing down columns and down rows. If in addition one of the following conditions holds, then the polynomial $\mathrm{per}(A + zJ_n)$ has only real roots.

U: $a_{ij} \in \{0, 1\}$ for all $1 \leq i, j \leq n$, or

$\prod$: There exist real numbers p_i for $1 \leq i \leq n$ and q_j for $1 \leq j \leq n$ such that $a_{ij} = p_i q_j$ for all $1 \leq i, j \leq n$, or

$\sum$: There exist real numbers p_i for $1 \le i \le n$ and q_j for $1 \le j \le n$ such that $a_{ij} = p_i + q_j$ for all $1 \le i, j \le n$, or

staircase: If $a_{1,n} \le a_{2,1}, a_{2,n} \le a_{3,1}, \ldots, a_{n-1,n} \le a_{n,1}$.

In the remaining parts of the thesis we describe our new results on the MCP conjecture. We start with considering the $n = 2$ case first exhibiting a slightly different proof than in [HOW99].

2.1 An alternative proof of the MCP conjecture for $n = 2$

Lemma 2.3. *Let* $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ *be a matrix with the weakly increasing column property, i.e.,* $a_{11} \le a_{21}$*,* $a_{12} \le a_{22}$*. Then, both roots of* $p(z) = \mathrm{per}(A + zJ_2)$ *are real.*

Proof. We have

$$\begin{aligned} p(z) &= (a_{11} + z)(a_{22} + z) + (a_{12} + z)(a_{21} + z) = \\ &= 2z^2 + (a_{11} + a_{12} + a_{21} + a_{22})z + a_{11}a_{22} + a_{12}a_{21}. \end{aligned}$$

In order for the roots to be real, p(z) must have a nonnegative discriminant:

$$\begin{aligned} D &= (a_{11} + a_{12} + a_{21} + a_{22})^2 - 4 \cdot 2 \cdot (a_{11}a_{22} + a_{12}a_{21}) = \\ &= (a_{11} - a_{12} + a_{21} - a_{22})^2 + 4(a_{11} - a_{21})(a_{12} - a_{22}) \ge 0. \end{aligned}$$

Note that the inequality holds due to the weakly increasing column property. □

A similar proof was shown in [HOW99]. There, the authors proved the $n = 2$ case also by showing that the discriminant is nonnegative. We would like to point out two differences. Our proof only assumes the weakly increasing property by column. Furthermore, we can extend it to matrices of size $n \times 2$ and $2 \times m$. To show this, first we need to introduce the generalized notion of a permanent of a (not necessarily square) matrix [Min78]:

Definition 2.4 (Permanent of a non-square matrix)**.** Let $A = (a_{ij})$ be an $n \times m$ matrix with $n \leq m$. The permanent of A is defined to be:

$$\mathrm{per}(A) = \sum_{\pi} a_{1\pi(1)} a_{2\pi(2)} \dots a_{n\pi(n)} \ ,$$

where the summation extends over all injective functions from $\{1, \dots, n\}$ to $\{1, \dots, m\}$. The definition for $n > m$ is analogous.

Note that this definition is a natural generalization of Definition 2.1 because it maintains the same interpretation in terms of rook numbers (cf. Definitions 1.6, 1.7), and hence can be naturally extended for hit polynomials, too. We would also like to point out that due to the monotone column property, the $n \times 2$ and $2 \times m$ cases are quite different. For example, we cannot simply argue about using the symmetry as we might not have the monotone row property.

2.2 Proof of the MCP conjecture for matrices of size $n \times 2$

Using the generalized notion of the permanent, we extend the previous result in a similar fashion to matrices of size $n \times 2$ whose entries are weakly increasing by columns.

Lemma 2.5. *Let* $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ \vdots & \vdots \\ a_{n,1} & a_{n,2} \end{pmatrix}$ *be a matrix with the weakly increasing column property, i.e.,* $a_{i,1} \leq a_{i+1,1}$ *and* $a_{i,2} \leq a_{i+1,2}$ *for all* $1 \leq i \leq n-1$*. Then, both roots of the polynomial* $p(z) = \mathrm{per}(A + zJ_{n,2})$ *are real. Here* $J_{n,2}$ *represents the* $n \times 2$ *size matrix of all ones.*

Proof. First observe that:

$$p(z) = n(n-1)z^2 + (n-1)z\sum_{i=1}^{n}(a_{1,i} + a_{2,i}) + \mathrm{per}(A) \ .$$

The roots of $p(z)$ are real if and only if the discriminant D of $p(z)$ is nonnegative.

By rearranging the terms we get:

$$D = (n-1)^2\left(\sum_{i=1}^{n}(a_{i,1} + a_{i,2})\right)^2 - 4n(n-1)\mathrm{per}(A) = \quad (2.1)$$

$$= (n-1)^2\left(\sum_{i=1}^{n}(a_{i,1} - a_{i,2})\right)^2 + 4(n-1)\sum_{k<\ell}(a_{k,1} - a_{\ell,1})(a_{k,2} - a_{\ell,2}). \quad (2.2)$$

One can check the equality by comparing the number of terms on each side (see Table 1). Note that the first summand in (2.2) is clearly nonnegative. The second one is nonnegative due to the weakly increasing assumption. Hence, $D \geq 0$. $\square$

	Equation (2.1)		Equation (2.2)	
$a_{k,\ell}^2$	$(n-1)^2$	0	$(n-1)^2$	0
$a_{k,1}a_{\ell,1}$	$2(n-1)^2$	0	$2(n-1)^2$	0
$a_{k,1}a_{k,2}$	$2(n-1)^2$	0	$-2(n-1)^2$	$4(n-1)^2$
$a_{k,1}a_{\ell,2}$	$2(n-1)^2$	$-4n(n-1)$	$2(n-1)^2$	$-4n(n-1)$

Table 1: The coefficients of each term are displayed to aid the verification of the rearrangement of terms in Equation (2.1) and (2.2). For the second and fourth row, we assume $k \neq \ell$.

2.3 Proof of the MCP conjecture for matrices of size $2 \times m$

The other natural generalization of the 2×2 case is to consider matrices of size $2 \times m$. Recall that this case is not equivalent to the $n \times 2$ case, because we only require the columns (and not the rows) to be weakly increasing.

Lemma 2.6. *Let* $A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1,m} \\ a_{21} & a_{22} & \dots & a_{2,m} \end{pmatrix}$ *is a* $2 \times m$ *matrix with weakly increasing column property, i.e.,* $\forall j : a_{1j} \leq a_{2j}$*, where* $J_{2,m}$ *denotes the* $2 \times m$ *size matrix of all ones. Then, both roots of the polynomial* $p(z) = \mathrm{per}(zA + J_{2,m})$ *are real.*

Proof. Observe that the discrimant can be written in the following form:

$$\begin{aligned} D &= (m-1)^2 \left(\sum_{j=1}^{m} (a_{1,j} + a_{2,j}) \right)^2 - 4m(m-1)\mathrm{per}(A) = \\ &= (m-1) \sum_{k<\ell} \left((a_{1,k} + a_{2,k} - a_{1,\ell} - a_{2,\ell})^2 + 4(a_{1,k} - a_{2,k})(a_{1,\ell} - a_{2,\ell}) \right) \ . \end{aligned}$$

Again, using the weakly increasing property we have that $D \geq 0$. □

2.4 The MCP conjecture for matrices of size 3×3

A natural next step is to try to continue this approach for 3×3 matrices: compute the discriminant and show that it is nonnegative. This approach, however, seems difficult as the discriminant of a cubic equation is more complex and, in addition, we have already more terms in a 3×3 permanent to begin with. For now, we can prove the following necessary condition. If a cubic polynomial $az^3 + bz^2 + cz + d$ has 3 real roots, then $b^2 \geq 3ac$. This can be viewed as a consequence of Rolle's Theorem since if a cubic polynomial has 3 real roots than its derivative $3az^2 + 2bz + c$ has 2 real roots, which is equivalent to the above condition.

Lemma 2.7. *Let* $\mathrm{per}(A + zJ_3) = az^3 + bz^2 + cz + d$*, where* A *is a* 3×3 *matrix with entries weakly increasing down columns, then* $b^2 \geq 3ac$*.*

Proof. By the expansion of the permanent we get that: $a = 6$, $b = 2\sum_i \sum_j a_{ij}$, and $c = \sum_i \sum_j \mathrm{per}(M_{ij})$, where the summation is over all 2×2 minors M_{ij} of the matrix A. For a 2×2 matrix $M = (m_{ij})$ with $m_{11} \leq m_{21}$ and $m_{12} \leq m_{22}$, let us define $\mathrm{posprod}(M) = (m_{11} - m_{21})(m_{12} - m_{22})$. Note, this term is nonnegative by the monotone column property.

Now we can write:

$$\begin{aligned} b^2 - 3ac &= 4\left(\sum_{i=1}^{3}\sum_{j=1}^{3} a_{ij}\right)^2 - 18\sum_{i=1}^{3}\sum_{j=1}^{3}\mathrm{per}(M_{ij}) = \\ &= 2\sum_{i<k}\left(\sum_{j=1}^{3} a_{ij} - \sum_{j=1}^{3} a_{kj}\right)^2 + 6\sum_{i=1}^{3}\sum_{j=1}^{3}\mathrm{posprod}(M_{ij}) \geq 0 \ . \end{aligned}$$

We follow the similar idea of rearranging the terms as before. Since the terms we arrive at are all nonnegative, we have that $b^2 \geq 3ac$. $\square$

In order to show that all three roots are real, we would need to show that the following necessary and sufficient condition holds (using the notation of Lemma 2.7):

$$D = b^2c^2 - 4b^3d - 4ac^3 + 18abcd - 27a^2d^2 \geq 0 \ .$$

As noted in [HOW99] there is no single condition analogous to discriminants for polynomials of degree $n > 3$. There are several other more complicated criteria involving Sturm sequences, minors of Toeplitz matrices, and determinants which we do not discuss here.

In the following sections, we present our main results. We consider a different approach by trying to prove the conjecture in an inductive fashion. We show some conditional results as a supporting evidence for the MCP conjecture.

2.5 An inductive approach to the MCP conjecture

Lemma 2.8. *Assume the MCP holds for $n \times n$ matrices. Then, it also holds for $n \times m$ for all $m \geq n$.*

We will make use of the following lemma by Chudnovsky and Seymour [CS07] (see also [BHH88] for a stronger version). First, we define the notion of compatibility and pairwise compatibility of polynomials as in [CS07].

Definition 2.9 (Compatibility)**.** Let $f_1(x), f_2(x), \ldots, f_k(x)$ be polynomials in one variable with real coefficients. We say that they are compatible if all roots of the polynomial $\sum_{i=1}^{k} c_i f_i(x)$ are real, for arbitrary $c_1, c_2, \ldots c_k \geq 0$.

Definition 2.10 (Pairwise compatibility)**.** Let $f_1(x), f_2(x), \ldots, f_k(x)$ be polynomials in one variable with real coefficients. We say that they are pairwise compatible if for all $i, j \in \{1, 2, \ldots, k\}$: $f_i(x)$ and $f_j(x)$ are compatible.

Lemma 2.11 (Chudnovsky–Seymour)**.** *Let $f_1(x), f_2(x), \ldots, f_k(x)$ be pairwise compatible polynomials with positive leading coefficients. Then, $f_1(x), f_2(x), \ldots, f_k(x)$ are compatible.*

Proof of Lemma 2.8. The $n = 1$ case is trivial, so fix n to be any integer larger than 1. We prove the lemma by induction on m. The $m = n$ case is true by assumption. Now assume the MCP conjecture holds for $n \times m$ matrices, and let A be a matrix of size $n \times (m + 1)$ with the monotone column property. Denote the $n \times m$ size submatrices of A by A_i, where A_i is the submatrix obtained from A by removing the ith column, and let $p_i(z) = \text{per}(zJ_{n,m} + A_i)$, for $i = \{1, 2, \ldots, m + 1\}$. By the induction hypothesis $p_1(z), p_2(z), \ldots, p_{m+1}(z)$ have only real roots. We will show that $p_1(z), p_2(z), \ldots, p_{m+1}(z)$ are pairwise compatible.

Consider the polynomials $p_i(z)$ and $p_j(z)$, for some i, j. The permanent function is invariant of permuting the columns of the matrix, hence we can arrange A_i and A_j in a form that they agree on their first $m - 1$ columns, which we denote by B. Then by denoting the columns in which the two matrices differ by v_i and v_j, respectively, and using the $A_i = B|v_i$ notation, it is easy to see that:

$$c_i p_i(z) + c_j p_j(z) = c_i \text{per}(zJ_{n,m} + B|v_i) + c_j \text{per}(zJ_{n,m} + B|v_j) = 2\text{per}(zJ_{n,m} + B|w) \ ,$$

where $w = (c_i v_i + c_j v_j)/2$. Since v_i and v_j are monotone columns and $c_i, c_j \geq 0$, w is a monotone column as well. Note that $B|w$ is a matrix of size $n \times m$ with the monotone column property, so by the induction hypothesis the polynomial $\text{per}(zJ + B|w)$ and hence $c_i p_i(z) + c_j p_j(z)$ have only real roots. The choice of i, j was arbitrary, therefore, $p_1(z), p_2(z), \ldots, p_{m+1}(z)$ are pairwise compatible.

Since each $p_i(z)$ is monic, we can apply Lemma 2.11. We get that the polynomial

$$\text{per}(zJ + A) = \frac{1}{m - n + 1} \sum_{i=1}^{m+1} p_i(z)$$

has only real roots. □

Corollary 2.12. *Using the result of R. Mayer [May] for the* 3×3 *case we have that the MCP conjecture is true for* $3 \times m$ *matrices (for all* $m \geq 3$*).*

Remark 2.13. In the proof of Lemma 2.8 we strongly relied on the monotone column property. The same proof would not work for $n \times m$ case with $n > m$, since we do not necessarily have the monotone row property. For example, consider the following matrix:

$$A = \begin{pmatrix} -1 & -2 \\ 0 & 1 \\ 3 & 2 \end{pmatrix} .$$

Both per $\begin{pmatrix} z-1 & z-2 \\ z & z+1 \end{pmatrix} = 2z^2 - 2z - 1$ and per $\begin{pmatrix} z & z+1 \\ z+3 & z+2 \end{pmatrix} = 2z^2 + 6z + 3$ have real roots, but their sum $4z^2 + 4z + 2 = (2z+1)^2 + 1$ has no real roots.

In order to prove the MCP conjecture in an inductive fashion, we would need another lemma to complement Lemma 2.8. We would need to show that the $n \times m$ case with $n < m$ implies the $m \times m$ one. Next, we prove a slightly different statement.

2.6 A conditional result for the MCP conjecture

Here, we prove a conditional result which relies on the assumption that the permanents of certain minors of the matrix have interlacing roots. We note that in [HOW99] a similar idea was used to prove the conjecture for the special cases mentioned in the beginning of the section 2 (see page 14).

Lemma 2.14. *Let A be an $n \times n$ matrix with the weakly increasing column property. Denote the $(n-1) \times (n-1)$ size minors of A by $A_{11}, A_{12}, \ldots, A_{nn}$, where A_{ij} is the matrix obtained by removing the ith row and jth column from A. Assume that for all $1 \le i \le n$, the polynomials $p_i(z) = \operatorname{per}(zJ_{n-1} + A_{i,n})$ have only real roots and that they interlace, i.e., $\forall i < j : p_i(z) \prec p_j(z)$. Then, $\operatorname{per}(zJ_n + A)$ has only real roots.*

We will use Lemma 2.11 and the following theorem of Wang and Yeh [WY05].

Theorem 2.15 (Wang–Yeh)**.** *Let $f(x)$ and $g(x)$ be two polynomials whose leading coefficients have the same sign. Suppose $f(x)$ and $g(x)$ have only real roots, and $g \prec f$. If $ad \le bc$, then $F(x) = (ax+b)f(x) + (cx+d)g(x)$ has only real roots.*

Proof of Lemma 2.14. Given an $n \times n$ matrix A with the monotone column propery, let $p_i(z) = \text{per}(zJ_{n-1} + A_{i,n})$, for all $1 \le i \le n$. We also define $q_i(z) = (z + a_{i,n})p_i(z)$, so that $\sum_{i=1}^{n} q_i(z) = \text{per}(zJ_n + A)$. This corresponds to the expansion of the permanent along the last column.

We show that $q_i(z)$ and $q_j(z)$ are compatible, for $1 \le i < j \le n$. For any $\alpha, \beta \ge 0$, let

$$F_{ij}(z) = \alpha q_j(z) + \beta q_i(z) = (\alpha z + \alpha a_{j.n})p_j(z) + (\beta z + \beta a_{i,n})p_i(z) \ .$$

We have $\alpha \cdot \beta a_{i,n} \le \alpha a_{j,n} \cdot \beta$ by the monotone column property of A. We also have that $p_i(z)$ and $p_j(z)$ have real roots and $p_i(z) \prec p_j(z)$ by assumption, so we can apply Theorem 2.15 and conclude that $F_{ij}(z)$ has only real roots. Since for all $i < j$ we get that $q_i(z)$ and $q_j(z)$ are compatible by Lemma 2.11 $q_1(z), q_2(z), \ldots, q_n(z)$ are also compatible. Hence, all roots of $\text{per}(zJ_n + A) = \sum_{i=1}^{n} q_i(z)$ are real. $\square$

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