RESEARCH SUMMARY

HYPERGEOMETRIC FUNCTIONS AND DIFFERENTIAL EQUATIONS

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The main theme of my recent research is the study of the multivariate hypergeometric systems introduced by Gelfand, Graev, Kapranov and Zelevinsky in the 1980s. These systems of partial differential equations, known as A-hypergeometric systems, arise naturally from the study of semigroup rings and their toric varieties. As the Gauss hypergeometric equations they generalize, A-hypergeometric systems depend on complex parameters. However, allowing the parameters of A-hypergeometric systems to vary will produce behavior that is never encountered in the univariate case. An example is the possibility of rank variation. The rank of a system of differential equations is defined to be the dimension of its space of complex holomorphic solutions. For an ordinary differential equation, the rank is given by the order. Thus, for the classical hypergeometric equations, where the order is not altered when the parameters change, rank is constant. This is not the case for A-hypergeometric systems, or for Horn systems, another multivariate generalization of the hypergeometric paradigm.

We are thus led to study the behavior of rank in parametric families of systems of differential equations. Ezra Miller, Uli Walther and I have proved that, under mild hypotheses satisfied by A-hypergeometric systems, rank is an upper semicontinuous function in the sense of algebraic geometry. In particular, the set where rank attains its minimum is a Zariski dense open subset of the parameter space. Partial results in this direction were known for A-hypergeometric systems (due to myself and others) and for Horn systems (obtained jointly with Alicia Dickenstein and Timur Sadykov). What is really surprising is that upper semicontinuity is not a feature of the very special hypergeometric situation, but holds in great generality.

In the hypergeometric situation, of course, much more detailed information is available, including explicit formulas for the generic rank. These were obtained for A-hypergeometric systems by Gelfand-Kapranov-Zelevinsky, and Adolphson. I provided a combinatorial rank formula for Horn systems in two variables in collaboration with Alicia Dickenstein and Timur Sadykov. In the general case of Horn systems, upcoming joint work with Alicia Dickenstein shows that the generic rank can be computed in terms of the multiplicities of the associated primes of the underlying binomial complete intersection.

I have recently settled (in collaboration with Ezra Miller and Uli Walther) the fundamental question of precisely when A-hypergeometric ranks are constant, by giving an explicit description of the locus of rank-jumping parameters for A-hypergeometric systems in terms of the local cohomology of the corresponding semigroup rings. As a consequence, we show that an A-hypergeometric system has constant rank if and only if the underlying semigroup ring is Cohen–Macaulay. Our techniques are completely homological, and do not depend on the type of singularities our A-hypergeometric system has. This contrasts with previous partial results (obtained by myself and others) which needed the singularities to be regular.

My strongest motivation for the study of A-hypergeometric systems comes from combinatorics. The solutions of these systems are power series whose coefficients satisfy simple recursions, and are thus very attractive as a candidate class for multivariate generating functions. However the tools that made the classical hypergeometric functions so useful in this context, such as contiguity relations and summation formulas, have not been developed yet for A-hypergeometric functions. Towards this goal, my collaborators and I plan to study the solution sheaf of an A-hypergeometric system, a mysterious object so far, in the hope that sheaf cohomology will provide enough control to investigate the combinatorial properties of solutions. In the same vein, information about the Gröbner degenerations of A-hypergeometric systems is highly desirable, since solutions can be constructed in the regular case using the Gröbner techniques of Saito, Sturmfels and Takayama. Since, for generic parameters, degenerating the A-hypergeometric system is equivalent to degenerating the underlying toric ideal, I propose to investigate the generalized A-hypergeometric system associated to the toric Hilbert scheme of a fixed toric variety. This will allow me to study all of the degenerations of the toric ideal at once from the hypergeometric point of view. Of especial interest will be the behavior for non-generic parameters. I also plan to study the Weyl closure properties of A-hypergeometric systems, which are related to the kind of singularities these systems exhibit. In particular, such a study will shed light on the singularities of the related Horn systems.

In a different direction, I plan to continue my collaboration with F. Alberto Grünbaum on inverse problems on directed graphs. In our recent work, we identified a large class of graphs for which a very general inverse problem can be solved explicitly. We propose to classify all the graphs for which our methods provide solutions, and to develop different tools for more general graphs. Potential applications of this work include modeling of packet traffic on the Internet and optical tomography, which refers to medical imaging using low-energy sources, such as infrared laser, which are less harmful than X-rays. Of course, in the case of optical tomography we need to take into account the scattering effect, which makes the corresponding inverse problem nonlinear and thus hard for effective numerical solution.