DEVELOPMENTS FROM BARR'S THESIS

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In 1962, Barr exhibited an idempotent in the rational group algebra of the symmetric group which split the Hochschild cochain complex of a commutative algebra into the direct sum of the Harrison complex and a complement. This discovery has led to significant developments both in the cohomology of commutative algebras and in understanding the relations between Hochschild cohomology and simplicial and sheaf cohomology.

This note is based on a talk given at a festival in honor of Michael Barr's 60th birthday, McGill University, Montreal, May 29, 1997. It is mainly historical; I hope that readers will help me to correct errors and to fill in unrecorded details. My personal thanks go to Tom Fox, Bill Lawvere, Robert Seely, to all who helped to organize this stimulating festival, as well as to the referee whose careful reading has led to significant improvements in the present paper.

1. Origins. Mike Barr was born in Philadelphia, received his bachelor's degree from the University of Pennsylvania in 1959 and Ph.D. in 1962, writing on "Cohomology of Commutative Algebras". He was David Harrison's last Ph.D. student at the University of Pennsylvania, and Harrison was in turn Emil Artin's last student in the United States. Mike's thesis was written in the same year as my first paper on the deformation theory of algebras; although unconnected at the time, a quarter-century later there would be a significant joining of ideas. Abstract algebraic cohomology theory was relatively new and not well understood in 1962. [Hochschild, 1945] had written his first paper on the subject in 1944 while a World War II draftee serving at the Aberdeen Proving Ground. Although the homology and cohomology of abstract groups had evolved earlier, it was from geometric ideas [Hopf, 1942], while the older special case of dimension 2 arose through the classical theory of factor sets for crossed product algebras [Schur, 1904], [Brauer, 1926], [Schreier, 1926]. The [Chevalley-Eilenberg, 1948] cohomology theory for Lie algebras was developed shortly after Hochschild's work, but again had a strong geometric motivation, calculating the cohomology of Lie groups. A year prior to Barr's work, [Harrison, 1962] formulated a cohomology theory for commutative algebras which seemed to have no motivation other than to show that one could define a natural cohomology theory for that category. Today we can define cohomology groups for practically anything by means of operads, cf. [Ginzburg-Kapranov 1994],

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although computing them is generally not an easy matter. Hochschild's work was seminal. Here is his basic definition:

Suppose that A is an associative algebra over a coefficient ring k (which need not be a field) and that M is an A bimodule. An n-cochain F of A with coefficients in M is then a k-multilinear map $F : A \times \cdots \times A(n \text{ times}) \to M$ (or equivalently a k-linear map $A^{\otimes n} \to M$). The k module of all these will be denoted $C^n(A, M)$. There is a coboundary map $\delta : C^n(A, M) \to C^{n+1}(A, M)$ defined by

$$(\delta F)(a_1, \dots, a_{n+1}) = a_1 F(a_2, \dots, a_{n+1}) + \sum_{i=1}^{n} (-1)^i F(a_1, \dots, a_i a_{i+1}, \dots, a_{n+1}) + (-1)^{n+1} F(a_1, \dots, a_n) a_{n+1}$$

It is easy to verify that $\delta^2 = 0$, so denoting the kernel of δ on C^n by Z^n (n-cocycles) and δC^{n-1} by B^n (n-coboundaries), we define the nth cohomology group $H^n(A, M)$ of A with coefficients in M to be Z^n/B^n. (Let $C^0(A, M)$ be M itself and $Z^0 = H^0 = \{m \in M | am - ma = 0\}$.) Hochschild's definition does not require that A have a unit, but our algebras always will, and bimodules will be unital, i.e., multiplication by the unit element of the algebra on either side will be the identity map. However, some morphisms may not be unital, e.g., the inclusion of a matrix algebra as a block of a larger one; the image of the unit is then only an idempotent in the larger algebra.

For the bimodule M we can take A itself. The direct sum $H^*(A, A) = \bigoplus H^n(A, A)$ then has a rich structure, part of which, the cup product, was already in Hochschild's first paper: writing C^m for $C^m(A, A)$, if $F \in C^m, G \in C^n$, define $F \smile G \in C^{m+n}$ by $F \smile G(a_1, \dots, a_{m+n}) = F(a_1, \dots, a_m)G(a_{m+1}, \dots, a_{m+n})$. It is immediate that $\delta(F \smile G) = \delta F \smile G + (-1)^m F \smile \delta G$, so the cup product on cochains induces one on the cohomology. Less obvious ([Gerstenhaber, 1963]), the cup product on $H^*(A, A)$ is graded commutative (supercommutative) and there is a graded Lie product on the cochains which descends to the cohomology and at the cohomology level acts as graded derivations of the associative cup product. Here are the relations: if ζ^m, η^n, χ^p are cohomology classes of dimensions m, n, p, repectively then

(1) $$\zeta^m \smile \eta^n = (-1)^{mn} \eta^n \smile \zeta^m$$

(2) $$(-1)^{(m-1)(p-1)}[\eta, [\zeta, \chi]] + (-1)^{(p-1)(n-1)}[\zeta, [\chi, \eta]] + (-1)^{(n-1)(m-1)}[\chi, [\eta, \zeta]] = 0$$

(3) $$[\eta \smile \zeta, \chi] = [\eta, \chi] \smile \zeta + (-1)^{m(p-1)} \eta \smile [\zeta, \chi]$$

Such a structure is now called a Gerstenhaber algebra. With hindsight we should have recognized their prevalence earlier. The simplest example is the exterior algebra on a Lie algebra. A closely related example in geometry is the multivectors (exterior algebra on the Lie algebra of tangent vector fields) on a manifold. Here the exterior product serves as the supercommutative associative multiplication and the Nijenhuis–Schouten bracket is the graded Lie bracket.

2. Simplicial cohomology is a special case of Hochschild cohomology. Hochschild's theory originally seemed to have little to do with topology, but we know now that simplicial cohomology is actually a special case of Hochschild cohomology [Gerstenhaber–Schack, 1983]. (There are deeper geometric connections, §7.) From a small category $\mathcal{C}$ one can build both a simplicial complex $\Sigma = \Sigma(\mathcal{C})$ called its *nerve* and, once we are given a commutative, unital coefficient ring k, an algebra $A = A(\mathcal{C}, k)$. The 0-simplices of Σ are just the objects of $\mathcal{C}$, while the r-simplices σ for $r > 0$ are r-tuples $(\phi_r, \phi_{r-1}, \ldots, \phi_1)$ of composable morphisms in $\mathcal{C}$, i.e., where $\phi_{j+1}\phi_j$ is defined for $j = 1, \ldots, r-1$. A 0-simplex has no faces. If ϕ is a morphism from j to i in $\mathcal{C}$, then the 1-simplex (ϕ) has 0-th face the object i and 1-st face the object j (which may be equal to i). For $r > 1$ the r-th face is $(\phi_{r-1}, \phi_{r-2}, \ldots, \phi_0)$, the 0-th face is $(\phi_r, \ldots, \phi_1)$, and if $0 < j < r$ then the j-th face is obtained by replacing the successive morphisms ϕ_{j+1}, ϕ_j by their product. One can now define the cohomology groups $H^r(\Sigma(\mathcal{C}), k)$ with coefficients in k in the usual way. These will not be changed if one works modulo degenerate simplices, i.e., those in which some morphism is the identity, so it is convenient to consider only non-degenerate ones and to discard any faces which happen to be degenerate. A partially ordered set or poset I with partial order $\prec$ is a special case of a small category; here a non-degenerate r-simplex is just a linearly ordered subset or chain $i_0 \precneqq i_1 \precneqq \cdots \precneqq i_r$ of objects of I. (The order is reversed because we used $\prec$ instead of $\succ$.) The j-th face is now obtained simply by deleting i_j. One can assemble the "solid" simplices of $\Sigma(\mathcal{C})$ into a topological space called the geometric realization of $\mathcal{C}$, details of which we omit, but which is easy to visualize when the $\mathcal{C}$ is a finite poset, I. Represent the objects of I as linearly independent points (vertices) in a real vector space whose dimension is $\#I$, and take the union of the convex hulls of those sets of vertices which form chains. A group may also be viewed as a category with a single object in which all morphisms (the group elements) are isomorphisms. For the 2-element group the geometric realization is just the infinite real projective space, i.e., quotient of the infinite sphere by the antipodal map.

The algebra $A(\mathcal{C}, k)$ consists of column-finite matrices whose rows and columns are indexed by the objects of $\mathcal{C}$, where the entry in the row indexed by i and column indexed by j is an element of the free k-module with basis $\mathrm{Hom}_{\mathcal{C}}(j, i)$. If $\phi : j \to i$ is a morphism in $\mathcal{C}$ then E_ϕ will denote the matrix all of whose entries are 0 except for the one in row i and coluumn j, where the entry is ϕ. Setting $E_\phi E_\psi = E_{\phi\psi}$ if $\phi\psi$ is defined and the zero matrix otherwise one can extend the multiplication naturally to the entire algebra. (This is just another description of the algebra denoted $\mathbb{k}!$ in the paper cited above; we will encounter a generalization in §6. The set of formal finite linear combinations of the matrices E_ϕ with coefficients in the ring k is too small unless $\mathcal{C}$ is finite; although already an algebra, it generally has no unit element. Note also that $A(\mathcal{C}, k)$ is not a functor from $\mathcal{C}$ to unital algebras. A functor $\mathcal{C} \rightsquigarrow \mathcal{C}'$ which is one-to-one on the objects of $\mathcal{C}$ will induce an algebra morphism $A(\mathcal{C}, k) \to A(\mathcal{C}', k)$, but this will not be unital unless the functor is a bijection on objects.) When the category $\mathcal{C}$ is a finite poset I with $\#I = n$ the algebra $A(I, k)$ is particularly easy to describe. Without loss of generality we may assume that I is just the set of integers $\{1, \ldots, n\}$ with a (reflexive) partial order $\prec$ in which $i \prec j$ implies $i \leq j$. The algebra A is just the linear span over k of those $n \times n$ matrices E_{ij} with $i \prec j$, i.e., having a single non-zero entry equal to 1 in the (i,j) place, and so it is an algebra of upper triangular matrices. For a group G viewed as a category, the algebra $A(G,k)$ is just the group algebra with coefficients in k.

Just as with simplicial complexes, every small category $\mathcal{C}$ has a barycentric subdivision $\mathcal{C}'$ whose simplicial cohomology $H^*(\Sigma(\mathcal{C}'),k)$ is canonically isomorphic to that of $\Sigma(\mathcal{C})$ with the same coefficient ring (cf. §6), [Gerstenhaber–Schack, 1988]. The second barycentric subdivision $\mathcal{C}''$ is always a poset. Therefore, to show that for every small $\mathcal{C}$ there is an algebra A whose Hochschild cohomology coincides with the simplicial cohomology of $\mathcal{C}$, it is sufficient to know the following special case of the Cohomology Comparison Theorem (§6).

Theorem. *For any poset I and (commutative, unital) coefficient ring k there is a canonical isomorphism $H^m(A(I,k),A(I,k)) \cong H^m(\Sigma(I),k)$ for all $m \geq 0$.* □

It follows, in particular, that for every simplicial complex Σ there is an algebra with the same cohomology: Just make the simplices of Σ into a partially ordered set $I = I(\Sigma)$ using the face relation. Then $\Sigma(I(\Sigma))$ is just the barycentric subdivision Σ' of the original Σ and has the same simplicial cohomology, so given Σ, the partially ordered set we want is just Σ itself with the face relation. The familiar fact that the cup product in simplicial cohomology is graded commutative now follows from the fact that it is already so in the more general Hochschild cohomology. However, the graded Lie product on $H^*(A,A)$ with A of the form $A(I,k)$ vanishes identically, which is not the case for an arbitrary associative algebra.

While the proof of the theorem is technically complicated it depends on three basic ideas. The first is that algebraic cohomology can be defined or computed (depending on one's point of view) using the so-called bar resolution. The half-century since Hochschild's original work has given us exact sequences, the Yoneda formulation, projective resolutions, and many sophisticated tools, but the essence of what we have learned is this: All reasonable definitions of the cohomology of an algebra give the same result. The cohomology is somehow inherent in the algebra every bit as much as the cohomology of, say, a space which has a simplicial triangulation is inherent in that space, although we may choose to compute it using simplicial cochains. The equivalent but different approaches give powerful theoretical and computational tools, since in a particular instance some may be much easier to apply than others. For some purposes, like algebraic deformation theory, Hochschild's original definition is essential.

The second basic idea, due again to [Hochschild, 1956] is that of the cohomology of an algebra A (with coefficients in a bimodule M) *relative* to a subalgebra S. Hochschild observed that one can define a subcomplex of his cochain complex whose m-th module $C^m(A,S;M)$ consists of S-relative cochains F defined by the following properties (where $s \in S$):

(1) $F(a_1,\dots,a_is,a_{i+1},\dots,a_m) = F(a_1,\dots,a_i,sa_{i+1},\dots,a_m)$ for all $s \in S$ and all i,

(2) $F(sa_1,\dots,a_m) = sF(a_1,\dots,a_m), \quad F(a_1,\dots,a_ms) = F(a_1,\dots,a_m)s.$

Moreover, there is a still smaller complex consisting of the reduced S-relative cochains F which by definition vanish when any argument lies in S. The third basic idea is that of the separability of a k-algebra S over an arbitrary coefficient ring k [Auslander–Goldman, 1960]. One of at least six equivalent properties is that

$H^1(S, M) = 0$ for all S bimodules M. By a standard dimension-shifting argument this implies that S is homologically trivial, i.e., that $H^n(S, M) = 0$ for all $n > 0$. If k is an algebraically closed field, this is equivalently to saying that S is a finite direct sum in algebraists' terminology (really a categorical direct product) of matrix algebras of varying dimensions. What draws these ideas together is the following

Theorem. *If S is a separable subalgebra of A, then the inclusion of the complex of reduced S-relative cochains into the full Hochschild complex induces an isomorphism of cohomology.* □

Although fundamental and trivial to prove using the bar resolution, this theorem seems to have been overlooked until [Gerstenhaber–Schack, 1986]. It is easiest to see how it applies when the small category $\mathcal{C}$ is a finite poset I. If $\#I = n$ then we may suppose that $I = \{1, \ldots, n\}$ and that the partial order is compatible with the natural order of the integers. The algebra $A(I, k)$ is then an algebra of $n \times n$ upper triangular matrices and contains all the diagonal matrices. These form a separable subalgebra S isomorphic to the direct product of k with itself n times. The reduced S-relative complex of A is then identical with the simplicial cochain complex of $\Sigma(I)$. To see this, observe that an m-cochain F is determined by its values $F(E_{i_1 j_1}, \ldots, E_{i_m j_m})$. Now if E_{ij}, E_{rs} are successive arguments in F, note that $E_{ij} = E_{ij}E_{jj}$, while $E_{jj}E_{rs} = 0$ unless $j = r$, so by (1), $F(\ldots, E_{i_p j_p}, E_{i_{p+1} j_{p+1}} \ldots)$ vanishes if $j_p \neq i_{p+1}$. Moreover, its value, by (2), must lie in $E_{i_1 i_1} A E_{j_m j_m}$ which is just a copy of k. That is, F simply assigns an element of k to the m-simplex $(i_1, \ldots, i_m, j_m)$. We may thus view it as a simplicial cochain, and it is now trivial to show that the Hochschild and simplicial coboundaries coincide. So it becomes a tautology in this case that the Hochschild cohomology coincides with the simplicial cohomology.

One can recover I from $A(I)$, so if we start with a simplicial complex we can at least recover its barycentric subdivision from the algebra asociated to it. Now $A(I)$ generally has non-trivial deformations which are not of the form $A(I)$ but are sufficiently like it that we can still recover I (cf. [Gerstenhaber–Schack, 1988b]). In a sense this broadens the concept of a topological simplicial complex.

3. The commutative case, Harrison, and Barr. Suppose now that A is a commutative k algebra and that M is a symmetric A bimodule, i.e., that $am = ma$ for all $m \in M, a \in A$. (Since A is commutative, any left A module may be viewed as a symmetric bimodule, but note that resolutions are different in the two categories.) Harrison had the insight that one could then define a special subcomplex of the Hochschild complex. It is not his original definition, however, which appears in his ground-breaking paper, but one suggested by the referee, Mac Lane. Write $a_1 \otimes \cdots \otimes a_n \in A^{\otimes n}$ simply as $(a_1, \ldots, a_n)$, let the symmetric group S_n operate by setting $\sigma(a_1, \ldots, a_n) = (a_{\sigma^{-1}1}, \ldots, a_{\sigma^{-1}n})$ and define the shuffle product by

$$(a_1, \ldots, a_m) * (a_{m+1}, \ldots, a_{m+n}) = \sum_{\sigma} \operatorname{sgn}(\sigma) \cdot \sigma(a_1, \ldots, a_{m+n})$$

where the sum is over all permutations σ of $1, \ldots, m+n$ such that $\sigma 1 < \sigma 2 < \cdots < \sigma m$ and $\sigma(m+1) < \sigma(m+2) < \cdots < \sigma(m+n)$. For example, $(a_1, a_2) * (a_3, a_4) = (a_1, a_2, a_3, a_4) - (a_1, a_3, a_2, a_4) + (a_1, a_3, a_4, a_2) + (a_3, a_1, a_2, a_4) - (a_3, a_1, a_4, a_2) +$

(a_3, a_4, a_1, a_2). Call $F \in C^n(A, M)$ a Harrison cochain if F vanishes when evaluated on any shuffle. The k module of these will be denoted by $C^n_H(A, M)$. (For $n = 0, 1$ all cochains are Harrison.) Since $(a) * (b) = (a, b) - (b, a)$, a 2-cochain F is Harrison precisely when $F(a, b) = F(b, a)$ for all a, b, i.e., when F is symmetric. When A is commutative and M symmetric the Harrison cochains form a subcomplex of the Hochschild cochain complex, so one can define the Harrison cohomology groups $\mathrm{Har}^n(A, M)$. Now the inclusion of a subcomplex into the full complex can do strange things to cohomology; in particular it seldom induces an inclusion of the cohomology of the subcomplex into that of the full complex. Nevertheless, in the lowest case, $n = 2$, if k contains $1/2$ then we can write every 2-cochain F as a sum $F = F_+ + F_-$ of a symmetric part and a skew part, with $F_+(a, b) = \frac{1}{2}(F(a, b) + F(b, a)), F_-(a, b) = \frac{1}{2}(F(a, b) - F(b, a))$. Moreover, it is easy to check that if F is a cocycle then so are $F_+$and F_-. Every Harrison 2-cochain is equal to its symmetric part, so $\mathrm{Har}^2(A, M)$ is actually a direct summand of Hochschild's $H^2(A, M)$.

With this slim evidence, Barr conjectured that if k contains the rationals then Harrison's cochain complex is actually a direct summand of Hochschild's. Barr's 1962 thesis at first seems pedestrian and was never published; pages of computation show that the Harrison groups are direct summands of the Hochschild groups in dimensions 2, 3, and 4. But in a brilliant paper [Barr, 1968] completed the program and much more. The key is that S_n operates on n-cochcains: If $F \in C^n(A, M), \sigma \in S_n$, define $(\sigma F)(a_1, \ldots, a_n) = (F\sigma^{-1})(a_1, \ldots, a_n) = F(a_{\sigma 1}, \ldots, a_{\sigma n})$. So the rational group algebra $\mathbb{Q}S_n$ operates, and in this there is a non-central idempotent e_n with the following property: when A is commutative and M symmetric then $\delta(e_n F) = e_{n+1}(\delta F)$. So the Hochschild complex is a direct sum of the two subcomplexes $e_n C^n(A, M)$ and $(1 - e_n)C^n(A, M)$, and the latter, it happens, is just Harrison's complex! The thesis laboriously computed e_2, e_3, and e_4; the paper showed conceptually that Barr's idempotent e_n exists for all n. The subject then lay dormant until the fall of 1985, when Mike invited me to visited Montreal.

4. The BGS idempotents and BGS decomposition of cohomology. The summer before Mike's invitation I spent studying module deformations with Don Schack, but in Montreal, Mike drew me back to the topic of commutative algebras. It had been clear almost from the beginning of algebraic deformation theory ([Gerstenhaber, 1964]) that when A is commutative, the infinitesimal deformations of A to other commutative algebras are precisely the elements of Harrison's $\mathrm{Har}^2(A, A)$. Mike recounted the history of Harrison's paper and gave me a copy of the original version, preserved from his days as Harrison's student.

Classically, the de Rham cohomology in dimension n of a complex projective manifold has a Hodge decomposition into a direct sum of $n + 1$ summands (§7). Algebraic deformation theory had paralleled precisely that for analytic manifolds (and within a year we knew that all the infinitesimal aspects of the analytic theory are indeed special cases of the algebraic one). It was therefore puzzling that while the Hodge decomposition had $n+1$ summands, that for the Hochschild cohomology of a commutative algebra so far had but two. Mike and I talked discussed his 1968 paper and the possibility of there being a finer decomposition of Hochschild cohomolgy for commutative algebras, but time ran out. On returning to Philadelphia and comparing notes from the summer with Harrison's original manuscript, I realized that Barr's 1968 paper had already created all the basic tools one needed. With little additional effort, one could see that $\mathbb{Q}S_n$ in fact contains n mutually orthogonal idempotents $e_n(1), \ldots, e_n(n)$ which sum to unity and have the property that if A is commutative and M a symmetric A-bimodule, then for all $F \in C^n(A, M)$ one has $\delta(e_n(k)F) = e_{n+1}(k)(\delta F)$. Moreover, Barr's original e_n is just $e_n(1)$. Setting $C^{k,n-k} = e_n(k)C^n(A, M)$, if we fix k then these form a subcomplex $C^{k,*-k}$ of C^*. The Hochschild cochain complex thus splits into a direct sum of infinitely many subcomplexes, the first of which is the Harrison subcomplex, but in dimension n there are only n non-zero summands. The idempotents therefore operate on the cohomology groups, and setting $H^{k,n-k} = e_n(k)H^n$ we have $H^n = \oplus_{k=1}^n H^{k,n-k}$. So H^n indeed has a decomposition into a direct sum of n summands, although from the analytic case one would have expected $n + 1$. (The mystery of the missing summand was solved soon after, §6.) The idempotents and decompostion will (following the referee's suggestion) be labeled BGS (Barr-Gerstenhaber-Schack).

No finer decomposition of the Hochschild cohomology is possible: In the original paper [Gerstenhaber–Schack, 1987] it was already shown that the $e_n(k)$ are universal in the sense that if we have a sequence of idempotents e'_n commuting with the Hochschild coboundary then e'_n is a linear combination of the $e_n(k)$. Ways have been found to compute the BGS idempotents more efficiently. For $\sigma \in S_n$ denote by $d(\sigma)$ the number of its *descents*, i.e., of i such that $\sigma i > \sigma(i + 1)$; [Garsia 1990] then gives the following generating function:

$$\sum_{k=1}^{n} x^k e_n(k) = \frac{1}{n!} \sum_{\sigma \in \mathcal{S}_n} (x - d(\sigma))(x - d(\sigma) + 1) \cdots (x - d(\sigma) + n - 1) \operatorname{sgn}(\sigma)\sigma.$$

(One could use ascents, since #ascents + #descents $= n - 1$; the number of σ in $\mathcal{S}_n$ with k ascents is the Eulerian number $\left\langle {n \atop k} \right\rangle$.) Of the $e_n(k)$ with fixed n, only $e_n(n)$ is central; it is the skew-symmetrizer $\frac{1}{n!}\sum \operatorname{sgn}(\sigma)$, denoted ϵ_n by Barr. No element of $Z^{n,0} = \epsilon_n Z^n$ can be a coboundary, so each represents a unique cohomology class. These cocycles are the skew multiderivations, i.e., multilinear maps $F : A \times \cdots \times M(n \text{ times}) \to A$ which are skew symmetric in the variables and a derivation as a function of each individually. The theorem of Hochschild– Kostant–Rosenberg (HKS) asserts, in particular, that if A is a polynomial ring in a finite number of variables then $H^n(A, M)$ is just the module of skew multiderivations regardless of the nature of the coefficient ring. So in characteristic 0 we have $H^n = H^{n,0}$ for polynomial rings.

5. Some complements. So far we have spoken only of Hochschild cohomology and of commutative algebras in characteristic zero, but there are some useful generalizations. First, one can deduce from the decomposition of the Hochschild cohomology of a commutative algebra that there is an identical decomposition in cyclic cohomology, cf. [Natsume–Schack, 1989], [Loday, 1989]. Second, the decomposition holds also for the cohomology of supercommutative algebras. Here one must be careful with the definition of the shuffle product, since it now takes into account the signs introduced when elements of various degrees move past each other. As the Hochschild cohomology $H^*(A,A)$ of an algebra A is itself supercommutative under the cup product one gets a decomposition of the cohomology of the cohomology which may be of some use for Koszul algebras, where the duality asserts that the cohomology of the cohomology is the original algebra. More interesting, there are closely related idempotents $\rho_n(k)$ due to [Reutenauer, 1986] which are connected to the natural decomposition of the symmetric algebra on a free Lie algebra. Denote by θ the involution of $\mathbb{Q}\mathcal{S}_n$ sending every σ to $(\operatorname{sgn}\sigma)\sigma$. Then $\rho_n(k) = \theta e_n(k)$. This is probably no accident but a consequence of the Ginzburg–Kapranov operadic duality between the commutative associative theory and the Lie theory. (The BGS idempotents $e_n(k)$ have sometimes been called Eulerian, but this could apply equally well to the Reutenauer idempotents.)

Note that to define the idempotents $e_n(k)$ with a fixed n it is necessary only to be able to divide by $n!$, so even in characteristic p there is a decomposition of cohomology in dimensions less than p, but more is true. For all n and all $i = 1,\dots,p-1$, the idempotents

$$\bar{e}_n(i) = \sum_{m=1}^{\lfloor (n-i)/(p-1) \rfloor} e_n(i + m(p-1))$$

remain well-defined in characteristic p. Hence there is a decomposition of the cohomology in all dimensions, but for $n > p-1$, instead of having n components, we only have $p-1$.

Unfortunately, the cup and graded Lie products generally do not preserve the decomposition of the cohomology $H^*(A,A)$ of a commutative algebra. That is, in general $H^{k,n-k} \smile H^{l,m-l} \not\subset H^{k+l,m+n-k-l}$, and similarly for the Lie multiplication. However, if we set $H^n_{\text{even}} = \sum_{k \text{ even}} H^{n-k,k}$ and similarly for H^n_{odd}, then these are defined whenever we can divide by 2 and one does have

$$H^*_{\text{even}} \smile H^*_{\text{even}}, H^*_{\text{odd}} \smile H^*_{\text{odd}} \subset H^*_{\text{even}}; \qquad H^*_{\text{even}} \smile H^*_{\text{odd}} \subset H^*_{\text{odd}}$$

So, for example, if $F, G \in H^{2,0}$ then $F \smile G \in H^{4,0} + H^{2,2}$. For a detailed analysis of what does hold, cf. [Bergeron and Wolfgang,1995]. They call the component in $H^{2,2}$ an "error" term. The HKS theorem implies that for a polynomial ring there can be no error terms.

The existence of the idempotents $e_n(k)$ raises interesting questions in combinatorics. One is, what are the dimensions of the left $\mathbb{Q}\mathcal{S}_n$ modules $(\mathbb{Q}\mathcal{S}_n)e_n(k)$? [Hanlon, 1990] shows that these are the Stirling numbers of the first kind, namely, the coefficients of the powers of x^k in the polynomial $p_n(x) = x(x+1)(x+2)\cdots(x+n-1)$. A very readable account of relations with other work including that of [Bayer and Diaconis, 1992] (reported in the New York Times because it calculated that to randomize a deck of cards required seven riffle shuffles) can be found in [Hanlon, 1992].

6. The GS double complex. Suppose again that $\mathcal{C}$ is a small category, k the coefficient ring, and that we now have both a presheaf $\mathbb{A}$ of unital, associative k-algebras over $\mathcal{C}$ and a presheaf $\mathbb{M}$ of unital $\mathbb{A}$ bimodules. Explicitly, $\mathbb{A}$ is a contravariant functor from $\mathcal{C}$ to k-algebras and $\mathbb{M}$ is a contravariant functor to k-modules such that 1) for every object i of $\mathcal{C}$ the k-module $\mathbb{M}(i)$ is an $\mathbb{A}(i)$ bimodule, and 2) whenever we have a morphism $\phi : i \to j$ in $\mathcal{C}$ then $\mathbb{M}(\phi) : \mathbb{M}(j) \to \mathbb{M}(i)$ is an $\mathbb{A}(j)$ bimodule morphism, where $\mathbb{M}(i)$ is viewed as an $\mathbb{A}(j)$ bimodule by means of the morphism $\mathbb{A}(\phi) : \mathbb{A}(j) \to \mathbb{A}(i)$. We can then construct a generalized simplicial (GS) double complex combining simplicial and Hochschild cohomology in the following way. Recall that an r-simplex σ of the nerve of $\mathcal{C}$ is just a sequence of composable morphisms $\sigma : i_0 \xrightarrow{\phi_1} i_1 \xrightarrow{\phi_2} \cdots \xrightarrow{\phi_r} i_r$, which we denoted simply $(\phi_r, \phi_{r-1}, \dots, \phi_1)$. The faces $\partial_p\sigma, p = 0, \dots, r$ have already been defined. Denote the composite morphism $\phi_r\phi_{r-1}\cdots\phi_1$ by $|\sigma|$ and set $d\sigma = i_0, c\sigma = i_r$ (the domain and codomain of $|\sigma|$, respectively). Then $\mathbb{M}(d\sigma)$ is an $\mathbb{A}(c\sigma)$ bimodule by virtue of the morphism $\mathbb{A}(|\sigma|)$. For all integers $r, s, \geq 0$, we can now define $C^{r,s}$ to be the k-module of functions Γ which assign to every r simplex σ of the nerve of $\mathcal{C}$ a Hochschild s cochain $\Gamma^\sigma \in C^s(\mathbb{A}(c\sigma), \mathbb{M}(d\sigma))$. The "horizontal" coboundary $\delta_{\mathrm{h}} : C^{r,s} \to C^{r,s+1}$ is just the Hochschild coboundary, while the "vertical" coboundary $\delta_{\mathrm{v}} : C^{r,s} \to C^{r+1,s}$ is essentially dual to the simplicial boundary: If $\sigma = (\phi_{r+1}, \phi_r, \dots, \phi_1)$ is an $r+1$-simplex, then

$$(\delta_{\mathrm{v}}\Gamma)^\sigma = \mathbb{M}(\phi_0) \circ \Gamma^{\partial_0\sigma} + \sum_{p=1}^{r}(-1)^p\Gamma^{\partial_p\sigma} + (-1)^{r+1}\Gamma^{\partial_{r+1}\sigma}\mathbb{A} \circ (\phi_{r+1}).$$

(Note that $|\partial_p\sigma| = |\sigma|$ for $p \neq 0, r+1$ since the intermediate faces are obtained simply by composing two successive morphisms in σ, but the domain of $\partial_0\sigma$ is i_1, which is why we need $\mathbb{M}(\phi_0)$ in the first term. We similarly need to adjust for the changed codomain in the last.)

The horizontal and vertical coboundaries commute, so we have a double complex $C^{*,*}$; its total complex is $\{C^*, \delta\}$ where $C^n = \oplus_{r+s=n}C^{r,s}$, and the restriction of δ to $C^{r,s}$ is $\delta_{\mathrm{v}} + (-1)^r\delta_{\mathrm{h}}$. The total cohomology in dimension n will be denoted $H^n(\mathbb{A}, \mathbb{M})$. It generalizes both the Hochschild cohomology of a single algebra, the case where $\mathcal{C}$ is reduced to a single object with only the identity morphism, and the simplicial cohomology of the nerve of $\mathcal{C}$, the case where $\mathbb{A} = \mathbb{M} = \mathbb{k}$, the presheaf in which every algebra or module is just the coefficient ring k and all morphisms are necessarily the identity. (To see that this gives the usual simplicial cohomology, use the fact that we would get the same cohomology by using reduced Hochschild cochains. Then all $C^{r,s}$ with $s > 0$ vanish, so we just get simplicial cochains with coefficients in k.) As remarked earlier, there are many ways to define (or compute) cohomology. The Yoneda method here yields the same $H^*(\mathbb{A}, \mathbb{M})$.

As a special case, take $\mathbb{A} = \mathbb{k}$, so the $\mathbb{M}(i)$ are simply k modules. Since we may compute using reduced Hochschild cohains (which vanish when any argument is in k), in the foregoing we need take only Hochschild cochains of dimension 0, which are simply module elements. This gives the cohomology $H^*(\mathcal{C}, \mathbb{M})$ of the nerve of $\mathcal{C}$ with local coefficients $\mathbb{M}$. Suppose now that we have a vector bundle $\mathcal{F}$ over a manifold X and an open covering I of X which is closed under taking intersections. View I as a poset with inclusion mappings as morphisms. For every $i \in I$, the sections of $\mathcal{F}$ over i form a k module. (We do not need here that they form a module over the real continuous functions defined on i.) Corresponding to every inclusion $i \to j$ there is a restriction morphism from sections over j to sections over i, so we have a presheaf, still denoted by $\mathcal{F}$ when there can be no confusion. We can therefore form $H^*(I, \mathcal{F})$. If I' is a refinement of I, i.e., if I' is again an open covering and for every $i' \in I'$ there is fixed an inclusion of i' into some $i \in I$, then there is a natural map $H^*(I, \mathcal{F}) \to H^*(I', \mathcal{F})$. By definition, $H^*(X, \mathcal{F})$ is the inverse limit of these cohomology groups. It is useful to know that if we can find a covering I of X (closed under intersections) the open sets i of which are homologically trivial, i.e., which have the property that $H^n(i, \mathcal{F}) = 0$ for all $\mathcal{F}$ and all $n > 0$, then we can avoid passing to the limit, for then $H^*(X, \mathcal{F}) = H^*(I, \mathcal{F})$ for all $\mathcal{F}$. For real manifolds, it is sufficient to take an open covering by sets which are homeomorphic to solid balls (and whose intersections have the same property). For the complex manifolds of the next section we will be concerned with holomorphic bundles (whose transition functions by definition are holomorphic), and a manifold will be homologically trivial if its cohomology with coefficients in any of these vanishes in all positive dimensions. Suppose now that k is a field of characteristic zero and that all $\mathbb{A}(i)$ are commutative and all $\mathbb{M}(i)$ are symmetric. Each horizontal complex then has a BGS decomposition which is preserved by the vertical coboundaries, so we can still define $H^{i,n-i}(\mathbb{A}, \mathbb{M})$, but the sum of these for $i = 1, \dots, n$ need no longer be all of H^n. Because the BGS idempotents annihilate zero-dimensional cochains a direct summand of the cohomology is omitted, namely the cohomology of the nerve of $\mathcal{C}$ using $\mathbb{M}$ as local coefficients, denoted $H^n(\mathcal{C}, \mathbb{M})$; this is the missing $(n+1)$-st component. When $\mathbb{A} = \mathbb{M} = \mathbb{k}$ then all $H^{i,n-i}$ with $i > 0$ vanish and all that remains is $H^{0,n}(\mathbb{k}, \mathbb{k})$, the usual simplicial cohomology of the nerve of $\mathcal{C}$ with constant coefficients in k. The missing component $H^{0,n}$ vanished as long as we were dealing with a single algebra, since the nerve of $\mathcal{C}$ in that case was just a point.

Remarkably, $H^*(\mathbb{A}, \mathbb{A})$ can be exhibited as the cohomology of a single algebra with coefficients in itself, and therefore also carries a Gerstenhaber algebra structure. To begin, we define a single algebra $\mathbb{A}!$, a generalization of $A(\mathcal{C}, k)$, which was the special case where $\mathbb{A} = \mathbb{k}$. This is again an algebra of column-finite matrices with rows and columns indexed by the objects of $\mathcal{C}$. Now, however, for every morphism $\phi : i \to j$ of $\mathcal{C}$ take a copy denoted $\mathbb{A}(i)\phi$ of $\mathbb{A}(i)$ and view this copy as a left $\mathbb{A}(i)$ and right $\mathbb{A}(j)$ module (using the morphism ϕ). The entries in the (i, j) place of $\mathbb{A}!$ will now be taken from the direct sum over all ϕ of these $\mathbb{A}(i)\phi$. Performing an analogous construction with $\mathbb{M}$ to produce an $\mathbb{A}!$ bimodule $\mathbb{M}!$, we have the

Cohomology Comparison Theorem. *If $\mathcal{C}$ is a poset, then there is a canonical isomorphism $H^*(\mathbb{A}, \mathbb{M}) \cong H^*(\mathbb{A}!, \mathbb{M}!)$.* □

This settles the problem when $\mathcal{C}$ is a poset, but one can reduce the general case to that. As mentioned, every small category $\mathcal{C}$ has a barycentric subdivision $\mathcal{C}'$. An intrinsic part of its structure is a functor $\mathcal{C}' \to \mathcal{C}$ such that, if we denote by $\mathbb{A}'$ and $\mathbb{M}'$ the pull-backs to $\mathcal{C}'$ of $\mathbb{A}$ and $\mathbb{M}$, respectively, then $H^*(\mathbb{A}', \mathbb{M}')$ is canonically isomorphic to $H^*(\mathbb{A}, \mathbb{M})$. The second barycentric subdivision $\mathcal{C}''$ is always a poset, so the algebra and module we seek are $\mathbb{A}''!$ and $\mathbb{M}''!$ (denoted $\mathbb{A}!!, \mathbb{M}!!$ in [Gerstenhaber–Schack 1988a]). This raises a new question: Although $\mathbb{A}''!$ is generally not commutative, it is easy to check that the BGS decomposition carries over to $H^*(\mathbb{A}''!, \mathbb{M}''!)$, mirroring that of $H^*(\mathbb{A}, \mathbb{M})$. So there exist non-commutative algebras whose cohomologies have natural BGS decompositions, but we do not know, in general, when this happens.

We have used presheaves to allow our rings to be non-commutative but could, of course, have started with a sheaf of commutative rings $\mathcal{O}_X$ on a space X, a sheaf of (coherent) $\mathcal{O}_X$ modules $\mathcal{F}$ and an open covering of X closed under taking intersections. Taking sections of $\mathcal{O}_X$ over the opens i of I gives a presheaf $\mathbb{A}$ of algebras, and one similarly obtains a presheaf of $\mathbb{A}$ modules.

7. Complex geometry. This brings us back to the starting point. The BGS decomposition was discovered because of the Barr idempotent. Its existence suggested that there should be a decomposition of the cohomology of a commutative algebra with coefficients in a symmetric module analogous to the Hodge decomposition of the cohomology of a projective manifold. This is more than an analogy. Suppose that X is a complex analytic manifold of dimension m, so by definition, every point has a neighborhood U in which there are complex coordinates $z_1, \dots, z_m$ mapping the neighborhood homeomorphically onto an open neighborhood of the origin in $\mathbb{C}^m$, and where the transformation of coordinates in overlapping neighborhoods is complex analytic. In U the 1-forms $dz_1, \dots, dz_m, d\bar{z}_1, \dots, d\bar{z}_m$ form a basis for the module of C^∞ forms over the ring of complex-valued C^∞ functions. Denote the module of 1-forms on all of X by $\mathcal{A}$. Since the coordinate transformations are complex analytic, it follows that we can write $\mathcal{A} = \mathcal{A}^{1,0} \oplus \mathcal{A}^{0,1}$, where $\mathcal{A}^{1,0}$ consists of those forms which locally (i.e., in every coordinate neighborhood) are linear combinations only of $dz_1, \dots, dz_n$ and $\mathcal{A}^{0,1}$ involves only $d\bar{z}_1, \dots, d\bar{z}_n$. Therefore $\wedge^n \mathcal{A} = \oplus_{r+s=n} \mathcal{A}^{r,s}$ where $\mathcal{A}^{r,s}$ consists of those forms which locally have the form $f(z) dz_{i_1} \wedge \cdots \wedge dz_{i_r} \wedge d\bar{z}_{j_1} \wedge \dots d\bar{z}_{j_s}$. The de Rham coboundary decomposes, $d = \partial + \bar{\partial}$, where ∂ involves differentiation with respect only to the z coordinates and $\bar{\partial}$ only to $\bar{z}$. One has $\partial : \mathcal{A}^{r,s} \to \mathcal{A}^{r+1,s}, \bar{\partial} : \mathcal{A}^{r,s} \to \mathcal{A}^{r,s+1}, \partial^2 = \bar{\partial}^2 = \partial\bar{\partial} + \bar{\partial}\partial = 0$, so for each r we have the Dolbeault complex $\cdots \to \mathcal{A}^{r,s} \xrightarrow{\bar{\partial}} \mathcal{A}^{r,s+1} \to \dots$ and corresponding cohomology group $H^{r,s}_{\bar{\partial}}(X)$. A form $\omega \in \mathcal{A}^{p,0}$ is called holomorphic if $\bar{\partial}\omega = 0$; the space of these will be denoted Ω. Dolbeault's theorem then asserts that $H^{r,s}_{\bar{\partial}}(X) = H^s(X, \wedge^r \Omega)$. When X is projective, i.e., a submanifold of complex projective space of some dimension (or more generally, a compact Kähler manifold), then the beautiful but deep Hodge theory of harmonic forms implies that these spaces span the cohomology of X:

$$H^n(X, \mathbb{C}) = \oplus_{r+s=n} H^s(X, \wedge^r \Omega)$$

.

Now let $\mathcal{O}_X$ denote the sheaf of germs of holomorphic functions on the complex manifold X. If $\mathcal{F}$ is a holomorphic vector bundle on X then its sheaf of germs of sections, which we continue to denote simply by $\mathcal{F}$, is a sheaf of modules over $\mathcal{O}_X$. As particular cases, we have the sheaf Ω and the sheaf $\mathcal{T}$ of holomorphic tangent vectors, its dual. Now take a covering I of X which is closed under taking intersections. Then, as in the previous section, we have a presheaf $\mathbb{A}$ of commutative algebras. (We could dispense with passing through sheaves by simply taking for each open set i of I the ring $\mathbb{A}(i)$ of functions which are defined and holomorphic in i.) Suppose now that X is projective and that we have chosen I to consist of the affine opens of X, i.e., those open subsets which are intersections of X with an affine subspace of the projective space. These are not only closed under intersections but are homologically trivial in the sense that that their cohomology with coefficients in any holomorphic vector bundle vanishes. Then from [Gerstenhaber–Schack 1988a] we have the following natural isomorphism,

$$H^{r,s}(\mathbb{A},\mathcal{F}) \cong H^s(X, \wedge^r \mathcal{T} \otimes_{\mathbb{A}} \mathcal{F}).$$

Summing over $r+s=n$ gives $H^n(\mathbb{A},\mathcal{F}) \cong \oplus_{r=0}^n H^{n-r}(X,\wedge^r\mathcal{T}\otimes_{\mathbb{A}}\mathcal{F})$. Now taking $\mathcal{F}=\mathbb{A}$ itself and summing over n gives a natural isomorphism

$$H^*(\mathbb{A},\mathbb{A}) \cong H^*(X,\wedge\mathcal{T}).$$

Both sides are Gerstenhaber algebras (that on the right coming from the fact that $\wedge\mathcal{T}$ is the exterior algebra on a Lie algebra), and this is a Gerstenhaber algebra isomorphism. This implies that

$$[H^{r,s}(\mathbb{A},\mathbb{A}), H^{r',s'}(\mathbb{A},\mathbb{A})] \subset H^{r+r',s+s'-1}(\mathbb{A},\mathbb{A})$$
$$H^{r,s}(\mathbb{A},\mathbb{A}) \smile H^{r',s'}(\mathbb{A},\mathbb{A}) \subset H^{r+r',s+s'}(\mathbb{A},\mathbb{A})$$

This shows, incidentally, that in this case $H^*(\mathbb{A},\mathbb{A})$ is bigraded in both the cup and bracket products, and there is no "error term". Now let $\mathcal{K}=\wedge^m\Omega$ (where $m=\dim X$) be the canonical line bundle on X, and take $\mathcal{F}=\mathcal{K}$. Then we have, in particular, that the component $H^{r,s}(\mathbb{A},\mathcal{K})$ of the BGS decomposition is naturally isomorphic to the component $H^s(X,\wedge^{m-r})$ of the analytic Hodge decomposition of the usual cohomology with complex coefficients, $H^{m-r+s}(X,\mathbb{C})$. In this case, therefore, the BGS decomposition reduces to the Hodge decomposition which inspired it, although one gets the components of the decomposition in a different order.

Recent work shows that one can state the foregoing results directly in terms of sheaves rather than presheaves, and the hypothesis of being projective can also be weakened, [Kontsevich, 1995] and (more explicitly) [Swan, 1996]. Let X be a quasiprojective complex manifold (i.e., an open subset of a projective manifold). Note that the presheaf $\mathbb{A}$ of the previous paragraph is just the restriction of $\mathcal{O}_X$ to the category of affine opens and that because of the duality of Ω and $\mathcal{T}$ we have $\bigwedge^q\mathcal{T}\otimes\mathcal{K}=\bigwedge^{m-q}\Omega$. Then we have

$$H^n(\mathcal{O}_X,\mathcal{K}\otimes\mathcal{F}) = \bigoplus_{r+s=n} H^r(X,\wedge^{m-s}\Omega\otimes\mathcal{F}).$$

The right side is sheaf cohomology and the left Hochschild, but one must define what one means by the Hochschild cohomology here. Swan's definition is $H^n(\mathcal{O}_X,\mathcal{F}) = \mathrm{Ext}^n_{\mathcal{O}_{X\times X}}(\mathcal{O}_X,\mathcal{F})$, where $\mathcal{F}$ is viewed as an $\mathcal{O}_{X\times X}$ module through the diagonal map $X\to X\times X$, and he shows that this agrees with the Gerstenhaber-Schack definition. Swan calls this the absolute Hodge decomposition. Kontsevich, using a different method of proof has shown that it is not even necessary to require X to be quasiprojective. As a particular case, one has

$$H^n(\mathcal{O}_X,\mathcal{O}_X) = \bigoplus_{r+s=n} H^r(X,\wedge^s\mathcal{T})$$

For $n = 2$ the left side is now just the purely algebraic infinitesimal deformations of the structure sheaf, while the right is the sum of three components, $H^2(X, \mathcal{O}_X) \oplus H^1(X, \mathcal{T}) \oplus H^0(X, \wedge^2 \mathcal{T})$. The middle summand here is just the classical module of infinitesimal deformations of X as a complex analytic manifold. The discovery that there was such a space of infinitesimal deformations by [Frölicher–Nijenhuis, 1957] set in motion the classsical work of [Kodaira–Spencer, 1958, 1960] summarized in [Kodaira, 1986]. The last summand can be interpreted as the deformations of the structure sheaf which point in the direction of a deformation to a "non-commutative variety". The first term is still mysterious. Thus the purely algebraic BGS decomposition has opened new questions even in the analytic theory. These results are certainly not yet in final form because the decomposition of the Hochschild cohomology exists even when X has singularities; there will be infinitesimals corresponding to deformations of the singularities, cf. [Gerstenhaber, 1969]. We also don't yet know the corresponding assertion in characteristic p, where we still have a decomposition of the Hochschild cohomology into $p - 1$ parts. Finally the single non-commutative algebra $\mathbb{A}!$ seems to encode much (perhaps all) of the information contained in the analytic structure of X; studying it may reveal more information about X itself.

8. Epilogue. So Barr's pedestrian-looking thesis has actually taken us a long way: to the full BGS decomposition of the cohomology of a commutative algebra in characteristic zero and a partial decomposition in characteristic p, up into geometry, and back down in an unlikely way to questions about a single algebra which are still unanswered. On the way we have seen some surprising things, including that simplicial cohomology is a special case of Hochschild cohomology, that the classical Hodge decomposition of the cohomology of a complex manifold is a special case of the purely algebraic BGS decomposition, and that there is a single non-commutative ring which mysteriously encodes much of the structure of a complex variety. (One would like to express the whole deformation theory of analytic manifolds in a purely algebraic way; a major problem is to interpret convergence questions.) Although we are now 35 years from its writing, it is reasonable to expect even more surprising developments from Barr's thesis.

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