COMPATIBLE DEFORMATIONS

Murray Gerstenhaber and Anthony Giaquinto

August 31, 1998

Abstract. Two deformations of the same algebra A are *compatible* if one can interpolate a continuous (in some sense) family of deformations between them. When A has finite dimension this is equivalent to having the deformed algebras lie on a common irreducible component of the variety of algebras of dimension that of A. If the infinitesimals of the deformations are $\alpha, \beta \in H^2(A, A)$ then a necessary condition for compatibility is the vanishing of the Gerstenhaber bracket $[\alpha, \beta] \in H^3(A, A)$; the bracket is thus an obstruction to compatibility. An analogous assertion holds for complex manifolds (and is a consequence of the algebraic case). In general, there are higher obstructions. As an application we show that while the quantum plane has a deformation to the Weyl algebra, the analog of the quantum plane in characteristic 2 can not be deformed to a separable algebra.

If we have two deformations of the same algebra, then one can ask if there is, in some sense, a continuous family of deformations (a 'variety of algebras') joining them; when that is the case we will call the deformations *compatible.* We show that there is a natural obstruction to compatibility, namely, the Gerstenhaber bracket of the infinitesimals of the deformations. In the case of finite dimensional algebras over a field, for example, this can be used to show that two deformations of a single algebra A lie on different components of the variety of algebras of the given dimension. It follows, in particular, that neither deformed algebra can be deformed to the other.

This paper grew out of an attempt to decide whether the Donald– Flanigan conjecture held for the eight-element quaternion group $\mathcal{Q} = \{\pm 1, \pm i, \pm j, \pm k\}$, where $ij = k$ and $i^2 = j^2 = k^2 = -1$. The conjecture asserts that if p is a prime dividing the order of a finite group G, then the group algebra $\mathbb{F}_p G$ (which always has a nonzero radical) can be deformed to a separable algebra. A proof that the Donald–Flanigan conjecture fails for $\mathcal{Q}$ will be given in a separate note, using the obstruction defined here. (There is a sketch at the end of this note.)

Compatibility of deformations is meaningful also in the category of complex analytic manifolds, and one finds an analogous obstruction. Although the direct proof is just as trivial, it is worth noting that one really does not need an independent proof, since all the infinitesimal aspects of the analytic deformation theory have been algebraized.

Key words and phrases. deformations, varieties of algebras, Donald-Flanigan conjecture.

1. Components of the variety of algebras. Suppose that A is an algebra over a ring k and that we have two deformations of A, with multiplications $*$ and $*'$ of the form

$$a * b = ab + tF_1(a,b) + t^2F_2(a,b) + \ldots, \quad a *' b = ab + uG_1(a,b) + u^2G_2(a,b) + \ldots.$$

Recall that the condition for associativity of $*$ is that for all $n > 0$ we have $\sum_{i=1}^{n-1} F_i \circ F_{n-i} = \delta F_n$. (Here δ is the Hochschild coboundary, and if F, G are any 2-cochains of A with coefficients in itself, then $F \circ G(a,b,c) = F(G(a,b),c) - F(a,G(b,c)))$. In particular, $F_1 \circ F_1$ is a coboundary. Now if F is a 2- cocycle of A (with coefficients in itself), whose cohomology class we denote by α, then the class of $F \circ F$ (which is always a 3-cocycle) depends only on that of F and may be denoted $\alpha \circ \alpha$ (the Gerstenhaber square of α). It is the obstruction to finding an F_2 such that the multiplication $ab + tF(a,b) + t^2F_2(a,b)$ is associative modulo t^3 (the coefficient ring now being $k[t]/t^3$, where k is the original coefficient ring). Now denote the deformed algebras defined by $*$ and $*'$ by A_t and A_u, respectively. If they lie, in any reasonable sense, on the same connected component of the variety of algebras containing A, then not only are the cohomology classes of $F = F_1$ and $G = G_1$ tangent to that component, but the same must hold for any linear combination of F and G. It follows, in particular, that the primary obstruction to $F+G$, viewed as an infinitesimal deformation of A, must vanish. This obstruction is the cohomology class of $(F+G)\circ(F+G)$. Now $(F+G)\circ(F+G) = F\circ F + G\circ G + F\circ G + G\circ F = F\circ F + G\circ G + [F,G]$ where $[F,G] = F\circ G + G\circ F$ is the Gerstenhaber bracket, cf [G1]. Since we started with two full deformations it follows that the classes of $F\circ F$ and $G\circ G$ already vanish, for we have $F\circ F = \delta F_2, G\circ G = \delta G_2$. So we have

Theorem 1. *If two deformations of an algebra A lie on the same irreducible component of a variety of algebras, and if the infinitesimals of these deformations are, respectively, the classes α, β of the 2-cocycles F and G, then $[\alpha,\beta] = 0$, i.e., $[F,G]$ must be a coboundary.* □

It follows that the class of $[F,G]$ may be viewed as an obstruction to deforming A_t to an algebra in the component of A_u and hence to the compatibility of these deformations.

The main ambiguity in what has been said is the meaning of a 'variety of algebras'. This is straightforward when A is a finite-dimensional algebra, say of dimension N, over an algebraically closed field k. For if we choose a basis for A, then it can be represented by its structure constants (relative to this basis) as a point in affine N^3 space over k. (Of course, many points may represent the same algebra; the general linear group $Gl(N,k)$ operates on the space and the isomorphism classes of algebras are its orbits.) The space of structure constants $\mathcal{C}$ is an algebraic variety which is, in general, a union of various irreducible components. Having chosen a basis for A, a deformation of A like A_t above defines a new set of structure constants which are now functions of t and may be viewed as defining a generic point of some irreducible subvariety of $\mathcal{C}$. This subvariety must then be entirely contained in one or more of the irreducible components of $\mathcal{C}$. (It may be contained in more than one.) The theorem then gives a necessary condition that there exist a single component of $\mathcal{C}$ containing the subvarieties definend by the two deformations A_t, A_u. Conversely, if such a component exists then the deformations are compatible.

In the infinite dimensional case, or over an arbitrary coefficient ring k, it may be that deformations provide the best way to define irreducible components of the 'variety of algebras'. The deformations A_t and A_u may be said to lie on the same component if there is a deformation with two parameters, t and u, of the form $a * b = ab + \Phi_1(a, b; t, u) + \Phi_2(a, b; t, u) + \dots$, where $\Phi_1 = tF_1 + uG_1$ and each Φ_i is a homogeneous polynomial of total degree i in t and u which reduces to F_i when $u = 0$ and to G_i when $t = 0$. This is, in effect, an algebraic way of asserting that there is a continuous family of deformations connecting the two original ones, i.e., that they are compatible.

The ideas above hold not only for associative algebras, but also for Lie algebras and commutative associative algebras, except that one must use, respectively, the Hochschild, Chevalley–Eilenberg and Harrison cohomologies. They probably carry over to many other categories, where one may have to use more exotic means, such as triples or operads, to define the appropriate cohomology. (The basic operation in the definition of an operad, namely composition, arose first in algebraic deformation theory, [G1, G2, G4]. The formal definition due to May comes from homotopy theory, which in some aspects is not far removed from algebraic deformation theory.)

There exist, in general, higher obstructions to compatibility of deformations. These follow precisely the pattern of obstructions to constructing a deformation of a single algebra. Suppose that we have the deformations given by $*$ and $*'$ above and that we have constructed a deformation with a single parameter t of the form $ab + t(F + G)(a, b) + t^2\Phi_2(a, b) + \dots$. Then $(F + G) \circ (F + G) = \delta\Phi_2$ and $[F + G, \Phi_2] = \delta\Phi_3$, etc.. Writing $\Phi_2 = F_2 + G_2 + \Theta_2$, it follows that $[F, G_2] + [G, F_2] + [F + G, \Theta_2]$ must be a coboundary, so the class of this element (which is guaranteed to be a cocycle) is the next obstruction to the compatibility of $*$ and $*'$. As with the deformations of a single algebra, we now have the following problem: The obstruction class depends on the choice of Θ, which is determined only up to a cocycle, and not merely on the original infinitesimals (the classes of) F and G. Making the wrong choice could produce a non-zero obstruction where a cleverer one might succeed.

2. The complex analytic case. The analogue of Theorem 1 holds for complex analytic manifolds without separate proof because 1) all of the infinitesimal concepts in that deformation theory can be captured purely algebraically using the deformation theory of presheaves of algebras, [GS] and 2) the Cohomology Comparison Theorem of that paper in turn reduces that theory to the theory of a single algebra. Compact complex analytic manifolds behave very much like finite dimensional algebras. The deformations of a single such manifold $\mathcal{M}$ form, by Kuranishi's Theorem, a finite dimensional variety which may have several irreducible components. Let T denote the sheaf of germs of holomorphic tangent vector fields on $\mathcal{M}$ and $\bigwedge T$ be its exterior algebra. The graded Lie algebra structure on $\bigwedge T$ induces a graded Lie structure on the total cohomology $H^*(\mathcal{M}, \bigwedge T)$ analogous to the Gerstenhaber bracket for the total Hochschild cohomology $H^*(A, A)$ of an associative algebra A with coefficients in A itself. Infinitesimal deformations of the complex structure are now elements of $H^1(\mathcal{M}, T) = H^1(\mathcal{M}, \bigwedge^1 T)$, these being in an algebraic sense, of total degree 2. If we have two full deformations of the complex structure, with infnitesimals the classes of cocycles F and G, respectively, then the class of $[F,G]$, which lies in $H^2(\mathcal{M},T)$, may be viewed as an obstruction to having both deformations lie on a common component containing $\mathcal{M}$ of the variety of complex manifolds.

3. Differential deformations. If A is an algebra with differential operators then one can define 'differential cochains' to be ones which are cup products of differential operators. These are typically closed under the Hochschild coboundary, so one can define 'differential Hochschild cohomology'. Since differential operators are also normally closed under composition, one can in these favorable cases define 'differential deformations'. One can then also define differential compatibility. This raises the question of whether two deformations which are compatible are differentially compatible. This will, of course, be the case if all cochains are differential.

The operators may be purely algebraic. One can, for example, always take as differential operators the polynomials in the derivations of A into itself. As a special case, on the polynomial ring $A = k[x_1,\dots,x_n]$ in n variables over an arbitrary coefficient ring k we have the differential operators $\partial_i = \partial/\partial x_i$. In characteristic zero, all cochains are differential if one is willing to allow formal infinite sums of monomials in the ∂_i and the variables. Moreover, inclusion of the complex of differential cochains into the Hochschild complex induces an isomorphism on cohomology. The cohomology of a polynomial ring A in characteristic zero is given by the theorem of Hochschild-Kostant-Rosenberg (HKR) which asserts, in this case, that the cohomology of A is just the exterior algebra on the derivations of A (linear combinations of the ∂_i with coeficients in A) where the cup product replaces the exterior product. In finite characteristics there are more cohomology classes. For example, in characteristic zero the class of $F = \partial_x \smile \partial_x$ vanishes, since there $F = -\frac{1}{2}\delta(\partial_x^2)$ but in characteristic 2 it is not zero. Similarly, the class of $\partial_x^2 \smile \partial_x + \partial_x \smile \partial_x^2$ vanishes in characteristic zero but not in characteristic 3. However, in characteristic p the cochains of a truncated polynomial algebra $k[x_1,\dots,x_n]/(x_1^p,\dots,x_n^p)$ are all still differential (and now one only needs finite sums). Note that the HKR theorem implies, in particular, that if $A = \mathbb{C}[x_1,\dots,x_n]$, then $H^m(A,A) = 0$ for $m > n$, but this no longer holds in positive characteristics.

4. The quantum plane in characteristic zero and the Weyl algebra.. The complex polynomial ring $A = \mathbb{C}[x,y]$ in two variables has two natural deformations. For the first, set $a * b = ab + \sum_{i=1}^{\infty}(t^i/i!)\partial_x a \partial_y b$. (Denoting the original multiplication by $\pi : A \otimes A \to A$, tacitly extended to be bilinear over $\mathbb{C}[[t]]$, this can be written $* = \pi \exp(\partial_x \otimes \partial_y)$. Although the series is infinite, the deformed product of two polynomials is a polynomial in t.) With this we have $x * y = xy + t, y * x = xy$, so $x * y - y * x = t$. The deformed algebra is thus just the smallest Weyl algebra. It is simple, but being infinite dimensional, it is not be separable, and its cohomology does not vanish identically. (A separable algebra over an algebraically closed field is just a finite sum of matrix algebras or 'multimatrix algebra'; its cohomology in positive dimensions with coefficients in any bimodule is identically zero.) However, like any deformation to a separable algebra, the deformation from the polynomial algebra to the Weyl algebra is a jump deformation: The value of t is not important as all non-zero complex values of t give isomorphic algebras. The infinitesimal of the deformation is the class of $F = \partial_x \smile \partial_y$.

For the second deformation, set $*' = \exp(x\partial_x \otimes y\partial_y)$. Here we have $x *' y = e^t xy$

and $y *' x = xy$. This is the standard quantum plane with $q = e^t$. Different complex values of t in general give non-isomorphic algebras. The infinitesimal of the deformation is the class of $G = x\partial_x \smile y\partial_y$. Surprisingly, but for trivial reasons, we have

Proposition. *The deformations of $A = \mathbb{C}[x, y]$ to the Weyl algebra and to the quantum plane are compatible.*

Proof. By the HKR Theorem, $H^3(A, A) = 0$, so all obsructions vanish. □

In some sense, therefore, the Weyl algebra and the standard quantum plane lie on a common variety of algebras. In fact, the same proof shows that all deformations of $\mathbb{C}[x, y]$ form a single variety of algebras. The analogous assertion does not hold for the finite analogs in positive characteristics.

5. Deformations of a truncated polynomial algebra in positive characteristic. Suppose now that the characteristic is p and let $A = k[x, y]/(x^p, y^p)$. This has deformations analogous to the Weyl algebra and to the quantum plane, as well as others. The simplest is to a direct sum of p^2 copies of the rational function field $k(t)$. (The coefficient ring must be enlarged by adjoining the deformation parameter.) For note that $k[x]/x^p$ can be deformed to $k(t)[x]/(x^p - t^{p-1}x)$, which is just a direct sum of p copies of $k(t)$, so $k[x]/x^p \otimes k[y]/y^p$ deforms to direct sum of p^2 copies. For the analog of the deformation to the Weyl algebra, it is simplest to define the deformation by setting $x * y = xy + t, y * x = xy$. It is then not difficult but here not necessary to write a general deformed product $a * b$ as a polynomial in t. All we will need later is that the infinitesimal F of the deformation again has the form $F = \partial_x \smile \partial_y$. We again have have $x * y - y * x = t$. Now, however, we have an isomorphism to the algebra M_p of $p \times p$ matrices: Send x to the matrix $X = (e_{21}+e_{32}+\cdots+e_{p,p-1})$ and y to the matrix $Y = t(e_{12}+2e_{23}+\cdots+(p-1)e_{p-1,p})$. These generate all of M_p. The Weyl algebra may be viewed as one form of limit of matrix algebras; letting $p \to \infty$, its generators can be represented by two infinite dimensional matrices. Unlike the Weyl algebra, M_p is separable. Similarly, it is easiest to define the deformation of the truncated polynomial algebra A in characteristic p to the finite 'quantum plane' simply by setting $x *' y = qy *' x$ with $q = 1 + t$. The infinitesimal of this deformation is then $G = x\partial_x \smile y\partial_y$. We no longer have an HKR Theorem. In fact, using only the primary obstruction, one can show that in characteristic 2 the deformations $*$ and $*'$ are not compatible.

Recall that $F \circ G(a, b, c) = F(G(a, b, c) - F(a, G(b, c))$and $[F, G] = F \circ G + G \circ F$. In our case, a simple computation independent of the coefficient ring gives

$$[F, G] = [\partial_x \smile \partial_y, x\partial_x \smile y\partial_y] = \partial_x \smile (y\partial_y - x\partial_x) \smile \partial_y. \tag{1}$$

Note that separable algebras are rigid because their cohomology is trivial; in particular they have no non-trivial infinitesimal deformations. A separable or more generally rigid algebra A (over an algebraically closed field) lies on a unique irreducble component of the variety of structure constants. This component generally will have subvarieties representing algebras which are not rigid but which deform to algebras isomorphic to A; we have just seen such an example. But the points of this component representing algebras isomorphic to A form a Zariski open subset.

Proposition. *In characteristic 2, the deformations from the truncated polynomial algebra $A = k[x,y]/(x^2,y^2)$ to separable algebras and to the quantum plane are all mutually incompatible. In particular, one can not deform the quantum plane in characteristic 2 to a separable algebra.*

Proof. Since A has dimension four, the only multimatrix algebras to which it could deform would be 1) the direct sum of four copies of k(t) and $M_2 = M_2(k(t))$. Clearly, deformations to distinct rigid algebras are incompatible, and a deformation to a commutative rigid algebra can not be compatible with any deformation to a non- commutative algebra. Therefore, we only have to show the incomopatibility of the deformations of A to M_2 and to the quantum plane. We show that the obstruction cocycle $[F,G]$ of (1) is not a coboundary.

Suppose, if possible, that $[F,G] = \delta f$ for some 2-cochain f. The 2-cocycle $a\partial_x \smile \partial_x + b\partial_x \smile \partial_y + c\partial_y \smile \partial_x + d\partial_y \smile \partial_y$ takes the values a,b,c,d on $(x,x),(x,y),(y,x),(y,y)$ respectively, so adding to f a suitable 2-cocycle of the given form (which does not change its coboundary), we may assume that f vanishes on these arguments. Further, $[F,G]$ vanishes whenever its first argument is y, and in particular on (y,x,x), so we have $\delta f(y,x,x) = 0$. The left side is just $f(yx,x)$, which therefore vanishes, and we have similarly $f(xy,y) = f(x,xy) = f(y,xy) = 0$. Finally, $\delta f(y,x,xy) = 0$. In view of the preceding, the left side is just $f(xy,xy)$. So all values of f vanish, which is impossible. □

In characteristics greater than two, the primary obstruction to compatibility (of the deformations of the truncated polynomial algebra to the quantum plane and to M_p) becomes a coboundary, namely, of $\frac{1}{2}(x\partial_x^2 \smile \partial_y - y\partial_x \smile \partial_y^2)$. However, by using the secondary obstruction one can show in characteristic 3 that the deformations are still incompatible. Our direct calculation is tedious but it what one expects is this: In characteristic p one can prolong the infinitesimal deformation $F+G$ through the term in t^{p-1}, but then, no matter how this prologation is done, one meets a non-trivial obstruction. This would be consistent with the fact that in the limit, when one gets to characteristic zero, there no longer is any obstruction, and the deformations become compatible.

6. The Donald–Flanigan conjecture.

Let G be a finite group whose order we denote by $\#G$. If k is a (commutative, unital) ring in which $\#G$ is invertible, then Maschke's Theorem asserts that the group ring kG is separable. When k is an algebraically closed field this says that kG is a direct sum of matrix algebras of various dimensions. If, however, p is a prime which divides $\#G$ then $\mathbb{F}_pG$ has a non-zero radical. (The sum of all the group elements is a central element of $\mathbb{F}_p$ and has square equal to zero.) The Donald–Flanigan conjecture is a "modular form" of Maschke's Theorem: it asserts that if k is a field of characteristic p dividing $\#G$ then kG can be deformed to a separable algebra. It has further been conjectured that the deformation can be so chosen that passing to the algebraic closure of the coefficient field, the deformed algebra becomes a direct sum of matrix algebras of the same sizes, and in natural correspondence with, those of the complex group algebra $\mathbb{C}G$. Such a deformation is called a "strong" solution to the Donald–Flanigan problem for the group G, while a deformation which is separable but which does not decompose as $\mathbb{C}G$ does is called a "weak" one. It is possible for there to be both a strong and a weak solution (or conceivably several weak solutions) for a single group G. In fact, we have just seen this. For if we denote by C_n the cyclic group of n elements, then $\mathbb{F}_2[C_2 \times C_2]$ is the truncated polynomial algebra in two variables, which can be deformed both to a direct sum of four copies of the (algebraically closed) coefficient field, the strong solution, and to an algebra of 2×2 matrices, a weak solution. (As noted before, the "coefficient field" has actually been enlarged to contain the deformation parameter.) For a bibliography on the Donald–Flanigan conjecture, cf. [GSps]

Unfortunately, the conjecture fails for the 8-element quaternion group Q. Here is the idea of our proof. Filter the group algebra $B = \mathbb{F}_2 Q$ by the powers of its radical and let A be the associated graded algebra. Then B is a deformation of A, [G3]. One can show rather readily that A can not be deformed to a direct sum of M_2 plus four copies of the coefficient field, so neither can the group algebra B. In view of the dimension, the only other separable deformation of B would be to M_4. However, one can also readily deform the associated graded algebra A to M_4 and one can show, using only the primary obstruction, that the deformations of the associated graded algebra A to the group algebra B and to M_4 are not compatible. Therefore, the group algebra itself can not be deformed to M_4. The reson for reserving this to a separate note is that the calculations are so far both tedious and not transparent.

This leaves us with the paradox that the prediction from the Donald–Flanigan conjecture given in [GG] and proven in [FJL] is true even though the conjecture, as stated, is not. Something subtler and making the same prediction may hold, but at present we have no idea what it might be.

References

[FJL] P. Fleischmann, I. Janiszczak,and W. Lempken, *Finite groups have non-Schur centralizers*, Manuscripta Mathematika **80** (1993), 213–224.

[G1] M. Gerstenhaber, *The cohomology structure of an associative ring*, Annals of Math. **78** (1963), 267–288.

[G2] ———, *On the deformation of rings and algebras*, Annals of Math. **79** (1964), 59–103.

[G3] ———, *On the deformation of rings and algebras. II*, Annals of Math. **84** (1966), 1–19.

[G4] ———, *On the deformation of rings and algebras. III*, Annals of Math. **88** (1968), 1–34.

[GG] M. Gerstenhaber and D. J. Green, *A group-theoretic consequence of the Donald–Flanigan conjecture*, Jour. of Algebra **166** (1994), 356–363.

[GS] M. Gerstenhaber and S. D. Schack, *Algebraic cohomology and deformation theory*, in Deformation Theory of Algebras and Structures and Applications (. M. Hazewinkel and M. Gerstenhaber, eds.), Kluwer, Dordrecht, 1988, pp. 11–264.

[GSps] M. Gerstenhaber and Mary E. Schaps, *Hecke algebras, $U_q sl_n$, and the Donald–Flanigan conjecture for S_n*, Trans. Amer. Math. Soc. **349** (1997), 3353–3371.

Dept. of Mathematics, University of Pennsylvania, Philadelphia, PA 19104-6395
E-mail address: mgersten@math.upenn.edu

E-mail address: