

WHY IS $\pi < 2\phi$?

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ABSTRACT. We give a combinatorial proof of the inequality in the title in terms of *Fibonacci numbers* and *Euler numbers*. The result is motivated by Sidorenko's theorem on the number of linear extensions of the poset and its complement. We conclude with some open problems.



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1. INTRODUCTION

We start with the inequality

$$(*) \quad \pi < 2\phi, \quad \text{where } \phi = \frac{1 + \sqrt{5}}{2}$$

is the *golden ratio*. The question in the title may seem rather innocent. Of course, $\pi \approx 3.141593 < 2\phi \approx 3.236068$. How deep can this be? Turns out, inequality (*) has a conceptual proof in terms of two classical combinatorial sequences. Let us set this up first.

Our first sequence $\{F_n\}$ is the *Fibonacci numbers*, defined by $F_0 = F_1 = 1$, $F_{n+1} = F_n + F_{n-1}$ for $n \geq 1$. This is perhaps best known integer sequence which begins

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \dots$$

See [Ko] and [OEIS, A000045] for a trove of information about this wonderful sequence.

Our second sequence $\{E_n\}$ is the sequence of *Euler numbers*. This is a sequence which begins

$$1, 1, 1, 2, 5, 16, 61, 272, 1385, 7936, 50521, \dots$$

Our favorite definition of the sequence is via the *Euler–Bernoulli triangle*:

$$\begin{array}{cccccccc} & & & & & & & \mathbf{1} \\ & & & & & & & 0 \rightarrow \mathbf{1} \\ & & & & & & & \mathbf{1} \leftarrow 1 \leftarrow 0 \\ & & & & & & & 0 \rightarrow 1 \rightarrow 2 \rightarrow \mathbf{2} \\ & & & & & & & \mathbf{5} \leftarrow 5 \leftarrow \mathbf{4} \leftarrow 2 \leftarrow 0 \\ & & & & & & & 0 \rightarrow 5 \rightarrow \mathbf{10} \rightarrow \mathbf{14} \rightarrow 16 \rightarrow \mathbf{16} \end{array}$$

Here one alternates direction,¹ start the row with zero, and each new number equal to the previous number plus the number above. For example, $14 = 10 + 4$ as in the last row of the triangle above. The last number in each row is the Euler number. We refer to [S2] for an extensive survey and to [OEIS, A000111] for numerous result and further references.

Theorem 1. *For all $n \geq 1$, we have:*

$$E_n \cdot F_n \geq n!$$

For example, $F_3 \cdot E_3 = 2 \cdot 3 = 3!$, $F_4 \cdot E_4 = 5 \cdot 5 = 25 > 4! = 24$, $F_5 \cdot E_5 = 8 \cdot 16 = 128 > 5! = 120$, etc. To understand the connection, recall the classical generating functions for each sequence:

$$\mathcal{F}(t) = \sum_{n=0}^{\infty} F_n t^n = \frac{1}{1-t-t^2} \quad \text{and}$$

$$\mathcal{E}(t) = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!} = \tan(t) + \sec(t) = \frac{1 + \sin(t)}{\cos(t)}.$$

¹This procedure is also called the *ox-plowing* and *boustrophedon* order.

These formulas imply the following (also classical) asymptotics of the numbers:

$$F_n \sim \frac{1}{\sqrt{5}} \phi^{n+1} \quad \text{and} \quad \frac{E_n}{n!} \sim \frac{4}{\pi} \left(\frac{2}{\pi}\right)^n.$$

Here we use $a_n \sim b_n$ as a notation for $a_n/b_n \rightarrow 1$ as $n \rightarrow \infty$.

In fact, we do not need the constants upfront, but only the base of exponent. Here ϕ is the smallest root of $1 - t - t^2 = 0$. Similarly, $\pi/2$ is the smallest (in absolute value) solution of $\cos(t) = 0$. While the formula for Fibonacci numbers is written in most Combinatorics textbooks, the asymptotic formula for Euler numbers is not as well known. We refer to a marvelous monograph [FIS] where this is one of the first motivating examples.

Now, the theorem and the asymptotics above give

$$1 \leq \frac{F_n \cdot E_n}{n!} \sim \frac{4\phi}{\sqrt{5}\pi} \left(\frac{2\phi}{\pi}\right)^n.$$

This implies inequality (*). See below why the inequality has to be strict.

The rest of the paper is structured as follows. First, we give a combinatorial proof of the theorem in the next section. We then discuss the origin of the problem and state some curious open problems (Section 3).

Remark 2. The picture on the first page shows part of the *Fibonacci spiral* that approximates the *golden spiral* (see e.g. [Ma, §1.2]). If the height of the big rectangle is r then the width of the rectangle is $r \cdot \phi$. The quarter-circle on the left has length $r\pi/2$. This length is smaller than the width of the rectangle.

2. COMBINATORIAL PROOF OF THEOREM 1

We start with classical combinatorial interpretations of Euler and Fibonacci numbers. These will be used to obtain a combinatorial proof of Theorem 1.

First, consider words in $\{\diamond, \subset, \supset\}$, where each open bracket is followed by a closed bracket. Denote by \mathcal{B}_n the set of such sequences. For example,

$$\mathcal{B}_4 = \{\diamond\diamond\diamond\diamond, \diamond\diamond\subset\supset, \diamond\subset\supset\diamond, \subset\supset\diamond\diamond, \subset\supset\subset\supset\}$$

Proposition 3. *We have $|\mathcal{B}_n| = F_n$, for all $n \geq 1$.*

Let S_n denote the set of all permutations of $\{1, 2, \dots, n\}$, so $|S_n| = n!$. Permutation $\sigma \in S_n$ is called *alternating* if $\sigma(1) < \sigma(2) > \sigma(3) < \sigma(4) > \dots$. Let \mathcal{A}_n be the set of alternating permutations in S_n .

Proposition 4. *We have $|\mathcal{A}_n| = E_n$, for all $n \geq 1$.*

These results are well known (see e.g. [GJ, S1]). The first is an easy exercise on induction. The second is similar; one needs to define $\mathcal{A}_{n,k} = \{\sigma \in \mathcal{A}_n, \sigma(1) = k\}$ and realize that $|\mathcal{A}_{n,k}|$ are the numbers in the Euler–Bernoulli triangle.

We can now reformulate Theorem 1 as follows:

$$|\mathcal{A}_n| \cdot |\mathcal{B}_n| \geq |S_n|$$

Consider now the map $\Phi : \mathcal{A}_n \times \mathcal{B}_n \rightarrow S_n$ defined as follows: $\Phi(\sigma, w) = \omega$, where ω is a permutation obtained from $\sigma \in \mathcal{A}_n$ by swapping numbers in the same pair of brackets in $w \in \mathcal{B}_n$. For example,

$$\Phi((3, 6, 2, 5, 4, 7, 1, 8), \diamond\diamond\subset\supset\diamond\subset\supset\diamond) = (3, 6, 5, 2, 4, 1, 7, 8).$$

The theorem now follows from the following lemma.

Lemma 5. *The map $\Phi : \mathcal{A}_n \times \mathcal{B}_n \rightarrow S_n$ is a surjection.*

Proof. We need to show that for every $\omega \in S_n$ there exist $\sigma \in \mathcal{A}_n$ and $w \in \mathcal{B}_n$ such that $\omega = \Phi(\sigma, w)$. Denote by $J = \{\omega(2), \omega(4), \dots\}$ the set of entries in even positions, and let $b = \omega(i)$ be the smallest entry in J . Locally, permutation ω looks as follows:

$$\omega = (\dots, x, a, b, c, y \dots).$$

Now, if $b > a, c$, do nothing. Since $x, y > b$, locally we have the desired inequalities $x > a < b > c < y$. Then repeat the procedure by induction for sub-permutations $\sigma_1 = (\dots, x, a)$ and $\sigma_2 = (c, y, \dots)$.

If $b < \max\{a, c\}$, swap b with the largest of these elements. Say, this is a . Again, locally we have the desired inequalities $x > b < a > c$. Make the word w have a pair of brackets $\subset \supset$ indicating that a and b are swapped. Then repeat the procedure by induction for sub-permutations $\sigma_1 = (\dots, x)$ and $\sigma_2 = (c, y, \dots)$. In the case when $\max\{a, c\} = c$, proceed symmetrically with permutations $\sigma_1 = (\dots, x, a)$ and $\sigma_2 = (b, y, \dots)$. Let σ denote the resulting permutation at the end of the process.

Observe that elements which move (b and possibly a/c) move at most once, so the bracket sequence w is well defined. Note also that at every move elements at even positions could only increase and at odd – decrease, and that the parity of positions translates to σ_1 and σ_2 . At the end we obtain alternating inequalities at every place in σ_1, σ_2 , and the last element of σ_1 /first element of σ_2 , decrease or increase depending on the parity of their position and does not violate the inequalities with the fixed elements in the middle (b and possibly a or c). Thus σ is alternating, as desired. Finally, note that $\Phi(\sigma, w) = \omega$, by construction. This completes the proof. \square

Exercise 6. *Find a pair of permutations $\sigma, \sigma' \in S_4$ such that $\Phi(\sigma) = \Phi(\sigma')$. Use the proof above to show that $E_n \cdot F_n > n!(1 + \varepsilon)^n$ for some explicit $\varepsilon > 0$.*

3. LINEAR EXTENSIONS OF POSETS

Let \mathcal{P} be a poset on a set X of $n = |X|$ elements, with linear ordering denoted by \prec . Let $e(\mathcal{P})$ be the number of *linear extensions* of \mathcal{P} , defined as bijections $f : X \rightarrow \{1, \dots, n\}$ such that $f(u) < f(v)$ for all $u, v \in X$. For example, if the poset \mathcal{P} forms a single n -chain (every two elements are comparable), we have $e(\mathcal{P}) = 1$. On the other hand, if the poset \mathcal{P} forms a single n -antichain (no two elements are comparable), we have $e(\mathcal{P}) = n!$. We refer to [T1, T2] for standard definitions and notation.

The following geometric construction is our main source of examples. Let $S \subset \mathbb{R}^2$ be a finite set of points. Define an ordering $(x_1, y_1) \preceq (x_2, y_2)$ when $x_1 \leq x_2$ and $y_1 \leq y_2$. The resulting poset \mathcal{P}_S is called *two-dimensional*. For example a poset $\mathcal{H}_{p,q}$

with $p + q + 1$ elements forming a *hook* (two incomparable chains with p and q elements with an extra minimal element) has $\binom{p+q}{p}$ linear extensions. Similarly, poset \mathcal{U}_n forming a zigzag pattern with n points as in in Figure 1, has Euler number $e(\mathcal{U}_n) = E_n$ of linear extensions.

Another notable example is the poset \mathcal{C}_k with $2 \times k$ elements forming a grid. It has *Catalan number* of linear extensions:

$$e(\mathcal{C}_k) = \frac{1}{k+1} \binom{2k}{k}$$

(see e.g. [S1, S3] and [OEIS, A000108]).

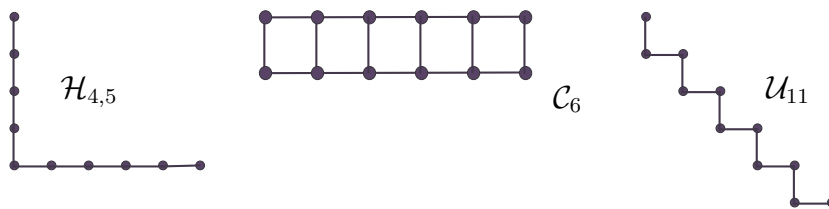


FIGURE 1. Two-dimensional posets $\mathcal{H}_{4,5}$, \mathcal{C}_6 and \mathcal{U}_{11} .

For a poset \mathcal{P} on set S , denote by $C(\mathcal{P})$ the comparability graph of \mathcal{P} . A poset $\overline{\mathcal{P}}$ on S is called a *complement* if its comparability graph $C(\overline{\mathcal{P}})$ is the complement of $C(\mathcal{P})$. Note that a poset can have more than one complement.

Proposition 7. *Every two-dimensional poset \mathcal{P} has a complement poset $\overline{\mathcal{P}}$.*

We leave the proof of the proposition to the reader with a hint given in Figure 2.

Exercise 8. *Describe the complement poset $\overline{\mathcal{H}}_{p,q}$. Show that $e(\overline{\mathcal{H}}_{p,q}) = (p + q + 1)p!q!$.*

Exercise 9. *Similarly to the previous exercise describe the complement poset $\overline{\mathcal{U}}_n$. Use induction to prove that $e(\overline{\mathcal{U}}_n) = F_n$.*

Exercise 10. *Describe the complement poset $\overline{\mathcal{C}}_k$. Prove that $Q_k := e(\overline{\mathcal{C}}_k)$ is the number of permutations $(a_1, \dots, a_k, b_1, \dots, b_k) \in S_{2k}$, such that $a_i < b_j$ for all $1 \leq i < j \leq k$.*

Remark 11. The problem of computing $e(\mathcal{P})$ is known to be $\#P$ -complete [BW], and is difficult even in some seemingly simple cases (see e.g.[BBS, ERZ, MPP]).

We are now getting to the heart of the motivation behind Theorem 1.

Theorem 12 (Sidorenko [Sid]). *Let \mathcal{P} be a two-dimensional poset with n elements, and let $\overline{\mathcal{P}}$ be a complement of \mathcal{P} . We have:*

$$e(\mathcal{P})e(\overline{\mathcal{P}}) \geq n!$$

Clearly, when \mathcal{P} is an n -chain, we have $\overline{\mathcal{P}}$ is an n -antichain, and the inequality is tight. Similarly, by Exercise 8, we have $e(\mathcal{H}_{p,q})e(\overline{\mathcal{H}}_{p,q}) = n!$ since $n = |\mathcal{H}_{p,q}| = p + q + 1$ in this case, so the inequality is tight again.

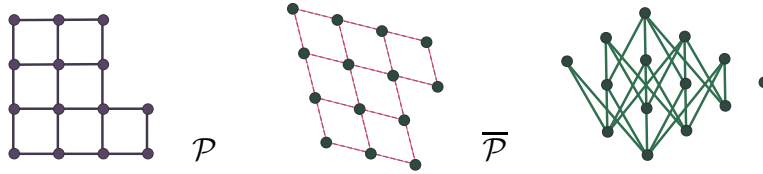


FIGURE 2. Two-dimensional poset \mathcal{P} , its complement $\overline{\mathcal{P}}$, and the Hasse diagram of $\overline{\mathcal{P}}$.

Observe that Exercise 9 and Sidorenko's theorem immediately imply Theorem 1. Note that the proof of Sidorenko's theorem is non-bijective and uses Stanley's interpretation of $e(\mathcal{P})$ as volumes of certain polytopes. Our proof suggests that there might be a direct combinatorial proof for all two-dimensional posets. If this is too much to hope for, perhaps the following problem can be resolved.

Open Problem 13. Give a combinatorial proof that $Q_k C_k \geq (2k)!$, where $Q_k = e(\overline{\mathcal{C}}_k)$. A direct computation shows that the sequence $\{Q_k\}$ starts with 12, 150, 3192, 106290, etc. Find the generating function

$$\mathcal{Q}(t) = 1 + \sum_{n=1}^{\infty} Q_n \frac{t^n}{n!}$$

and exact asymptotics for Q_k . Note that by Sidorenko's theorem and Exercise 10, we have $Q_k \geq n!/4^k$.

Remark 14. We should mention a counterpart to Sidorenko's theorem in [BBS], giving the following upper bound:

$$e(\mathcal{P}) e(\overline{\mathcal{P}}) \leq n! \left(\frac{\pi}{2}\right)^n \quad \text{as } n \rightarrow \infty.$$

The proof uses *Santaló's inequality* for polar polytopes, which is sharp for convex bodies. The authors of [BBS] suggest that this bound can be further improved, although not by much.

Open Problem 15. Denote by \mathcal{R}_k the poset corresponding to $[k \times k]$ square of points in the grid. It is known that

$$\log e(\mathcal{R}_k) = \frac{1}{2} n \log n + \left(\frac{1}{2} - 2 \log 2\right) n + O(\sqrt{n} \log n).$$

(see e.g. [MPP] and [OEIS, A039622]). Find the asymptotics of $e(\overline{\mathcal{R}}_k)$. Note that since $e(\mathcal{R}_k) \leq \sqrt{n!}$, we have $e(\overline{\mathcal{R}}_k) \geq \sqrt{n!}$, where $n = k^2$. Note also that by the remark above we have:

$$\log e(\overline{\mathcal{R}}_k) = \frac{1}{2} n \log n + \Theta(n).$$

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