ASYMPTOTICS OF SYMMETRIC POLYNOMIALS WITH APPLICATIONS TO STATISTICAL MECHANICS AND REPRESENTATION THEORY

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Abstract. We develop a new method for studying the asymptotics of symmetric polynomials of representation–theoretic origin as the number of variables tends to infinity. Several applications of our method are presented: We prove a number of theorems concerning characters of infinite–dimensional unitary group and their q–deformations. We study the behavior of uniformly random lozenge tilings of large polygonal domains and find the GUE–eigenvalues distribution in the limit. We also investigate similar behavior for Alternating Sign Matrices (equivalently, six–vertex model with domain wall boundary conditions). Finally, we compute the asymptotic expansion of certain observables in $O(n = 1)$ dense loop model.

Contents

1. Introduction

1.1. **Overview.** In this article we study the asymptotic behavior of symmetric functions of representation–theoretic origin, such as Schur rational functions or characters of symplectic or orthogonal groups, etc, as their number of variables tends to infinity. In order to simplify the exposition we stick to Schur functions in the introduction where it is possible, but most of our results hold in a greater generality.

The rational Schur function $s_\lambda(x_1,\dots,x_n)$ is a symmetric Laurent polynomial in variables $x_1,\dots,x_n$. They are parameterized by N–tuples of integers $\lambda=(\lambda_1\geq\lambda_2\geq\dots\geq\lambda_N)$ (we call such N–tuples *signatures*, they form the set $\mathbb{GT}_N$) and are given by Weyl's character formula as

$$s_\lambda(x_1,\dots,x_N)=\frac{\det\left[x_i^{\lambda_j+N-j}\right]_{i,j=1}^N}{\prod_{i<j}(x_i-x_j)}.$$

Our aim is to study the asymptotic behavior of the *normalized* symmetric polynomials

$$S_\lambda(x_1,\dots,x_k;N,1)=\frac{s_\lambda(x_1,\dots,x_k,\overbrace{1,\dots,1}^{N-k})}{s_\lambda(\underbrace{1,\dots,1}_{N})} \tag{1.1}$$

and also

$$S_\lambda(x_1,\dots,x_k;N,q)=\frac{s_\lambda(x_1,\dots,x_k,1,q,q^2,\dots,q^{N-k-1})}{s_\lambda(1,\dots,q^{N-1})}, \tag{1.2}$$

for some $q>0$. Here $\lambda=\lambda(N)$ is allowed to vary with N, k is any fixed number and $x_1,\dots,x_k$ are complex numbers, which may or may not vary together with N, depending on the context. Note that there are explicit expressions (Weyl's dimension formulas) for the denominators in formulas (1.1) and (1.2), therefore, their asymptotic behavior is straightforward.

The asymptotic analysis of expressions (1.1), (1.2) is important because of the various applications in representation theory, statistical mechanics and probability, including:

- For any k and any *fixed* $x_1,\dots,x_k$, such that $|x_i|=1$, the convergence of $S_\lambda(x_1,\dots,x_k;N,1)$ (from (1.1)) to some limit and the identification of this limit can be put in representation–theoretic framework as the approximation of indecomposable characters of the infinite–dimensional unitary group $U(\infty)$ by normalized characters of the unitary groups $U(N)$, the latter problem was first studied by Kerov and Vershik [VK].
- The convergence of $S_\lambda(x_1,\dots,x_k;N,q)$ (from (1.2)) for any k and any *fixed* $x_1,\dots,x_k$ is similarly related to the *quantization* of characters of $U(\infty)$, see [G1].
- The asymptotic behavior of (1.1) can be put in the context of Random Matrix Theory as the study of the Harish–Chandra-Itzykson–Zuber integral

$$\int_{U(N)}\exp(\mathrm{Trace}(AUBU^{-1}))dU, \tag{1.3}$$

where A is a fixed Hermitian matrix of finite rank and $B=B(N)$ is an $N\times N$ matrix changing in a regular way as $N\to\infty$. In this formulation the problem was thoroughly studied by Guionnet and Maïda [GM].

- A normalized Schur function (1.1) can be interpreted as the expectation of a certain observable in the probabilistic model of uniformly random lozenge tilings of planar domains. The asymptotic analysis of (1.1) as $N \to \infty$ with $x_i = \exp(y_i/\sqrt{N})$ and fixed y_is gives a way to prove the local convergence of random tiling to a distribution of random matrix origin — the GUE–corners process (the name GUE–minors process is also used). Informal argument explaining that such convergence should hold was suggested earlier by Okounkov and Reshetikhin in [OR1].
- When λ is a *staircase Young diagram* with $2N$ rows of lengths $N-1, N-1, N-2, N-2, \dots, 1, 1, 0, 0$, (1.1) gives the expectation of an observable(closely related to the Fourier transform of the number of vertices of type a on a given row) for the uniformly random configurations of the six–vertex model with domain wall boundary conditions (equivalently, Alternating Sign Matrices). Asymptotic behavior as $N \to \infty$ with $x_i = \exp(y_i/\sqrt{N})$ and fixed y_i gives a way to study the local limit of the six–vertex model with domain wall boundary conditions near the boundary.
- For the same staircase λ the expression involving (1.1) with $k = 4$ and Schur polynomials replaced by the characters of symplectic group gives the mean of the boundary-to-boundary current for the completely packed $O(n = 1)$ dense loop model, see [GNP]. The asymptotics (now with fixed x_i, not depending on N) gives the limit behavior of this current, significant for the understanding of this model.

In the present article we develop a new unified approach to study the asymptotics of normalized Schur functions (1.1), (1.2) (and also for more general symmetric functions like symplectic characters and polynomials corresponding to the root system BC_n), which gives a way to answer all of the above limit questions. There are 3 main ingredients of our method:

(1) We find simple contour integral representations for the normalized Schur polynomials (1.1), (1.2) with $k = 1$, i.e. for

$$\frac{s_\lambda(x, 1, \dots, 1)}{s_\lambda(1, \dots, 1)} \quad \text{and} \quad \frac{s_\lambda(x, 1, q, \dots, q^{N-2})}{s_\lambda(1, \dots, q^{N-1})}, \tag{1.4}$$

and also for more general symmetric functions of representation–theoretic origin.

(2) We study the asymptotics of the above contour integrals using the *steepest descent* method.

(3) We find formulas expressing (1.1), (1.2) as $k \times k$ determinants of expressions involving (1.4), and combining the asymptotics of these formulas with asymptotics of (1.4) compute limits of (1.1), (1.2).

In the rest of the introduction we provide a more detailed description of our results. In Section 1.2 we briefly explain our methods. In Sections 1.3, 1.4, 1.5, 1.6, 1.7 we describe the applications of our method in asymptotic representation theory, probability and statistical mechanics. Finally, in Section 1.8 we compare our approach for studying the asymptotics of symmetric functions with other known methods.

In the next papers we also apply the techniques developed here to the study of other classes of lozenge tilings [Pa] and to the investigation of the asymptotic behavior of decompositions of tensor products of representations of classical Lie groups into irreducible components [BuG].

1.2. **Our method.** The main ingredient of our approach to the asymptotic analysis of symmetric functions is the following integral formula, which is proved in Theorem 3.8. Let $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N)$, and let $x_1, \ldots, x_k$ be complex numbers. Denote

$$S_\lambda(x_1, \ldots, x_k; N, 1) = \frac{s_\lambda(x_1, \ldots, x_k, 1, \ldots, 1)}{s_\lambda(1, \ldots, 1)}$$

with $N - k$ 1s in the numerator and N 1s in the denominator.

Theorem 1.1 (Theorem 3.8)**.** *For any complex number x other than 0 and 1 we have*

$$S_\lambda(x; N, 1) = \frac{(N-1)!}{(x-1)^{N-1}} \frac{1}{2\pi \mathbf{i}} \oint_C \frac{x^z}{\prod_{i=1}^N (z - (\lambda_i + N - i))} dz, \tag{1.5}$$

where the contour C encloses all the singularities of the integrand.

We also prove various generalizations of formula (1.5): one can replace 1s by the geometric series $1, q, q^2, \ldots$ (Theorem 3.6), Schur functions can be replaced with characters of symplectic group (Theorems 3.15 and 3.18) or, more, generally, with multivariate Jacobi polynomials (Theorem 3.22). In all these cases a normalized symmetric function is expressed as a contour integral with integrand being the product of elementary factors. The only exception is the most general case of Jacobi polynomials, where we have to use certain hypergeometric series.

Recently (and independently of the present work) a formula similar to (1.5) for the characters of orthogonal groups $O(n)$ was found in [HJ] in the study of the mixing time of certain random walk on $O(n)$. A close relative of our formula (1.5) can be also found in Section 3 of [CPZ].

Using formula (1.5) we apply tools from complex analysis, mainly the method of steepest descent, to compute the limit behavior of these normalized symmetric functions. Our main asymptotic results along these lines are summarized in Propositions 4.1, 4.2, 4.3 for real x and in Propositions 4.7 and 4.8 for complex x.

The next important step is the formula expressing $S_\lambda(x_1, \ldots, x_k; N, 1)$ in terms of $S_\lambda(x_i; N, 1)$ which is proved in Theorem 3.7:

Theorem 1.2 (Theorem 3.7)**.** *We have*

$$S_\lambda(x_1, \ldots, x_k; N, 1) = \frac{1}{\prod_{i<j}(x_i - x_j)} \prod_{i=1}^k \frac{(N-i)!}{(x_i - 1)^{N-k}} \times \det\left[D_{x_i}^{k-j}\right]_{i,j=1}^k \left(\prod_{j=1}^k S_\lambda(x_j; N, 1) \frac{(x_j - 1)^{N-1}}{(N-1)!} \right), \tag{1.6}$$

where D_x is the differential operator $x\frac{\partial}{\partial x}$.

Formula (1.6) can again be generalized: 1s can be replaced with geometric series $1, q, q^2, \ldots$ (Theorem 3.5), Schur functions can be replaced with characters of the symplectic group (Theorems 3.14, 3.17) or, more, generally, with multivariate Jacobi polynomials (Theorem 3.21). Formulas similar to (1.6) can be found in the literature, see e.g. [GP, Proposition 6.2], [KuSk].

The advantage of formula (1.6) is its relatively simple form, but it is not straightforward that this formula is suitable for the $N \to \infty$ limit. However, we are able to rewrite this formula in a different form (see Proposition 3.9), from which this limit transition is immediate. Combining the limit formula with the asymptotic results for $S_\lambda(x; N, 1)$ we get the full asymptotics for $S_\lambda(x_1, \ldots, x_k; N, 1)$. As a side remark, since

we deal with analytic functions and convergence in our formulas is always (at least locally) uniform, the differentiation in formula (1.6) does not introduce any problems.

Theorems 1.1 and 1.2 allow us to study the asymptotic behavior of normalized Schur functions in various settings, which are motivated by the current applications:

- As $\lambda_i(N)/N \to f(i/N)$ in a sufficiently regular fashion for a monotone piecewise continuous function f on $[0,1]$ (used in the statistical mechanics applications of Section 5) or as $\lambda(N)$ grows in certain sub-linear regimes (used in the representation theoretic applications of Section 6).
- As the variables $x_1, \dots, x_k$ are fixed, or as they depend on N, e.g. $x_i = e^{y_i/\sqrt{N}}$ for fixed y_i (used in Sections 5.1 and 5.2).

We believe that the combination of Theorems 1.1, 1.2 with the well-developed steepest descent method for the analysis of complex integral, paves the way to study the delicate asymptotics of Schur polynomials (and more general symmetric functions of representation-theoretic origin) in numerous limit regimes which might go well beyond the applications presented in this paper.

1.3. **Application: asymptotic representation theory.** Let $U(N)$ denote the group of all $N \times N$ unitary matrices. Embed $U(N)$ into $U(N+1)$ as a subgroup acting on the space spanned by first N coordinate vectors and fixing $N+1$st vector, and form the infinite–dimensional unitary group $U(\infty)$ as an inductive limit

$$U(\infty) = \bigcup_{N=1}^{\infty} U(N).$$

Recall that a (normalized) *character* of a group G is a continuous function $\chi(g)$, $g \in G$ satisfying:

(1) χ is constant on conjugacy classes, i.e. $\chi(aba^{-1}) = \chi(b)$,
(2) χ is positive definite, i.e. the matrix $\left[\chi(g_i g_j^{-1})\right]_{i,j=1}^{k}$ is Hermitian non-negative definite, for any $\{g_1, \dots, g_k\}$,
(3) $\chi(e) = 1$.

An *extreme character* is an extreme point of the convex set of all characters. If G is a compact group, then its extreme characters are normalized matrix traces of irreducible representations. It is a known fact (see e.g. the classical book of Weyl [W]) that irreducible representations of the unitary group $U(N)$ are parameterized by signatures, and the value of the trace of the representation parameterized by λ on a unitary matrix with eigenvalues $u_1, \dots, u_N$ is $s_\lambda(u_1, \dots, u_N)$. Using these facts and applying the result above to $U(N)$ we conclude that the normalized characters of $U(N)$ are the functions

$$\frac{s_\lambda(u_1, \dots, u_N)}{s_\lambda(1, \dots, 1)}.$$

For "big" groups such as $U(\infty)$ the situation is more delicate. The study of characters of this group was initiated by Voiculescu [Vo] in 1976 in connection with *finite factor representations* of $U(\infty)$. Voiculescu gave a list of extreme characters, later independently Boyer [Bo] and Vershik-Kerov [VK] discovered that the classification theorem for the characters of $U(\infty)$ follows from the result of Edrei [Ed] on the characterization of totally positive Toeplitz matrices. Nowadays, several other proofs of Voiculescu–Edrei classification theorem is known, see [OO], [BO], [Pe2]. The theorem itself reads:

Theorem 1.3. *The extreme characters of $U(\infty)$ are parameterized by the points ω of the infinite-dimensional domain*

$$\Omega \subset \mathbb{R}^{4\infty+2} = \mathbb{R}^\infty \times \mathbb{R}^\infty \times \mathbb{R}^\infty \times \mathbb{R}^\infty \times \mathbb{R} \times \mathbb{R},$$

where Ω is the set of sextuples

$$\omega = (\alpha^+, \alpha^-, \beta^+, \beta^-; \delta^+, \delta^-)$$

such that

$$\alpha^\pm = (\alpha_1^\pm \geq \alpha_2^\pm \geq \cdots \geq 0) \in \mathbb{R}^\infty, \quad \beta^\pm = (\beta_1^\pm \geq \beta_2^\pm \geq \cdots \geq 0) \in \mathbb{R}^\infty,$$

$$\sum_{i=1}^\infty (\alpha_i^\pm + \beta_i^\pm) \leq \delta^\pm, \quad \beta_1^+ + \beta_1^- \leq 1.$$

The corresponding extreme character is given by the formula

$$\chi^{(\omega)}(U) = \prod_{u \in \mathrm{Spectrum}(U)} e^{\gamma^+(u-1)+\gamma^-(u^{-1}-1)} \prod_{i=1}^\infty \frac{1+\beta_i^+(u-1)}{1-\alpha_i^+(u-1)} \frac{1+\beta_i^-(u^{-1}-1)}{1-\alpha_i^-(u^{-1}-1)}, \tag{1.7}$$

where

$$\gamma^\pm = \delta^\pm - \sum_{i=1}^\infty (\alpha_i^\pm + \beta_i^\pm).$$

Our interest in characters is based on the following fact.

Proposition 1.4. *Every extreme normalized character χ of $U(\infty)$ is a uniform limit of extreme characters of $U(N)$. In other words, for every χ there exists a sequence $\lambda(N) \in \mathbb{GT}_N$ such that for every k*

$$\chi(u_1, \dots, u_k, 1, \dots) = \lim_{N\to\infty} S_\lambda(u_1, \dots, u_k; N, 1)$$

uniformly on the torus $(S_1)^k$, where $S_1 = \{u \in \mathbb{C} : \ |u| = 1\}$.

In the context of representation theory of $U(\infty)$ this statement was first observed by Kerov and Vershik [VK]. However, this is just a particular case of a very general convex analysis theorem which was reproved many times in various contexts (see e.g. [V], [OO], [DF2]).

The above proposition raises the question which sequences of characters of $U(N)$ approximate characters of $U(\infty)$. Solution to this problem was given by Kerov and Vershik [VK].

Let μ be a Young diagram with row lengths μ_i, column lengths μ_i' and whose length of main diagonal is d. Introduce *modified Frobenius coordinates*:

$$p_i = \mu_i - i + 1/2, \quad q_i = \mu_i' - i + 1/2, \quad i = 1, \dots, d.$$

Note that $\sum_{i=1}^d p_i + q_i = |\mu|$.

Given a signature $\lambda \in \mathbb{GT}_N$, we associate two Young diagrams λ^+ and λ^- to it: The row lengths of λ^+ are the positive λ_i's, while the row lengths of λ^- are minus the negative ones. In this way we get two sets of modified Frobenius coordinates: p_i^+, q_i^+, $i = 1, \dots, d^+$ and p_i^-, q_i^-, $i = 1, \dots, d^-$.

Theorem 1.5 ([VK], [OO], [BO], [Pe2])**.** *Let $\omega = (\alpha^\pm, \beta^\pm; \delta^\pm)$ and suppose that the sequence $\lambda(N) \in \mathbb{GT}_N$ is such that*

$$p_i^+(N)/N \to \alpha_i^+, \quad p_i^-(N)/N \to \alpha_i^-, \quad q_i^+(N)/N \to \beta_i^+, \quad q_i^-(N)/N \to \beta_i^+,$$
$$|\lambda^+|/N \to \delta^+, \quad |\lambda^-|/N \to \delta^-.$$

Then for every k

$$\chi^{\omega}(u_1,\dots,u_k,1,\dots)=\lim_{N\to\infty} S_{\lambda(N)}(u_1,\dots,u_k;N,1)$$

uniformly on the torus $(S_1)^k$.

Theorem 1.5 is an immediate corollary of our results on asymptotics of normalized Schur polynomials, and a new short proof is given in Section 6.1.

Note the remarkable multiplicativity of Voiculescu–Edrei formula for the characters of $U(\infty)$: the value of a character on a given matrix (element of $U(\infty)$) is expressed as a product of the values of a single function at each of its eigenvalues. There exists an independent representation–theoretic argument explaining this multiplicativity. Clearly, no such multiplicativity exists for finite N, i.e. for the characters of $U(N)$. However, we claim that the formula (1.6) should be viewed as a manifestation of *approximate multiplicativity* for (normalized) characters of $U(N)$. To explain this point of view we start from $k=2$, in this case (1.6) simplifies to

$$S_\lambda(x,y;N,1)=S_\lambda(x;N,1)S_\lambda(y;N,1) + \frac{(x-1)(y-1)}{N-1}\frac{(x\frac{\partial}{\partial x}-y\frac{\partial}{\partial y})[S_\lambda(x;N,1)S_\lambda(y;N,1)]}{y-x}.$$

More generally Proposition 3.9 claims that for any k formula (1.6) implies that, informally,

$$S_\lambda(x_1,\dots,x_k;N,1)=S_\lambda(x_1;N,1)\cdots S_\lambda(x_k;N,1)+O(1/N),$$

therefore, (1.6) states that normalized characters of $U(N)$ are approximately multiplicative and they become multiplicative as $N\to\infty$. This is somehow similar to the work of Diaconis and Freedman [DF] on *finite exchangeable sequences*. In particular, in the same way as results of [DF] immediately imply de Finetti's theorem (see e.g. [A]), our results immediately imply the multiplicativity of characters of $U(\infty)$.

In [G1] a q–deformation of the notion of character of $U(\infty)$ was suggested. Analogously to Proposition 1.4, a q–character is a limit of Schur functions, but with different normalization. This time the sequence $\lambda(N)$ should be such that for every k

$$\frac{s_{\lambda(N)}(x_1,\dots,x_k,q^{-k},q^{-k-1},\dots,q^{1-N})}{s_{\lambda(N)}(1,q^{-1},\dots,q^{1-N})} \tag{1.8}$$

converges uniformly on the set $\{(x_1,\dots,x_k)\in\mathbb{C}^k \mid |x_i|=q^{1-i}\}$. An analogue of Theorem 1.5 is the following one:

Theorem 1.6 ([G1])**.** *Let $0<q<1$. Extreme q–characters of $U(\infty)$ are parameterized by the points of set $\mathcal{N}$ of all non-decreasing sequences of integers:*

$$\mathcal{N}=\{\nu_1\le\nu_2\le\nu_3\le\dots\}\subset\mathbb{Z}^\infty,$$

Suppose that a sequence $\lambda(N)\in\mathbb{GT}_N$ is such that for any $j>0$

$$\lim_{i\to\infty}\lambda_{N+1-j}(N)=\nu_j, \tag{1.9}$$

then for every k

$$\frac{s_{\lambda(N)}(x_1,\dots,x_k,q^{-k},q^{-k-1},\dots,q^{1-N})}{s_{\lambda(N)}(1,q^{-1},\dots,q^{1-N})} \tag{1.10}$$

converges uniformly on the set $\{(x_1,\dots,x_k)\in\mathbb{C}^k \mid |x_i|=q^{1-i}\}$ and these limits define the q–character of $U(\infty)$.

Using the q–analogues of formulas (1.5) and (1.6) we give in Section 6.2 a short proof of the second part of Theorem 1.6, see Theorem 6.5. This should be compared with [G1], where the proof of the same statement was quite involved. We go beyond the results of [G1], give new formulas for the q–characters and explain what property replaces the multiplicativity of Voiculescu–Edrei characters given in Theorem 1.3.

1.4. **Application: random lozenge tilings.** Consider a tiling of a domain drawn on the regular triangular lattice of the kind shown at Figure 1 with rhombi of 3 types, where each rhombus is a union of 2 elementary triangles. Such rhombi are usually called *lozenges* and they are shown at Figure 2. The configuration of the domain is encoded by the number N which is its width and N integers $\mu_1 > \mu_2 > \cdots > \mu_N$ which are the positions of *horizontal lozenges* sticking out of the right boundary. If we write $\mu_i = \lambda_i + N - i$, then λ is a signature of size N, see left panel of Figure 1. Due to combinatorial constraints the tilings of such domain are in correspondence with tilings of a certain polygonal domain, as shown on the right panel of Figure 1. Let Ω_λ denote the domain encoded by a signature λ.

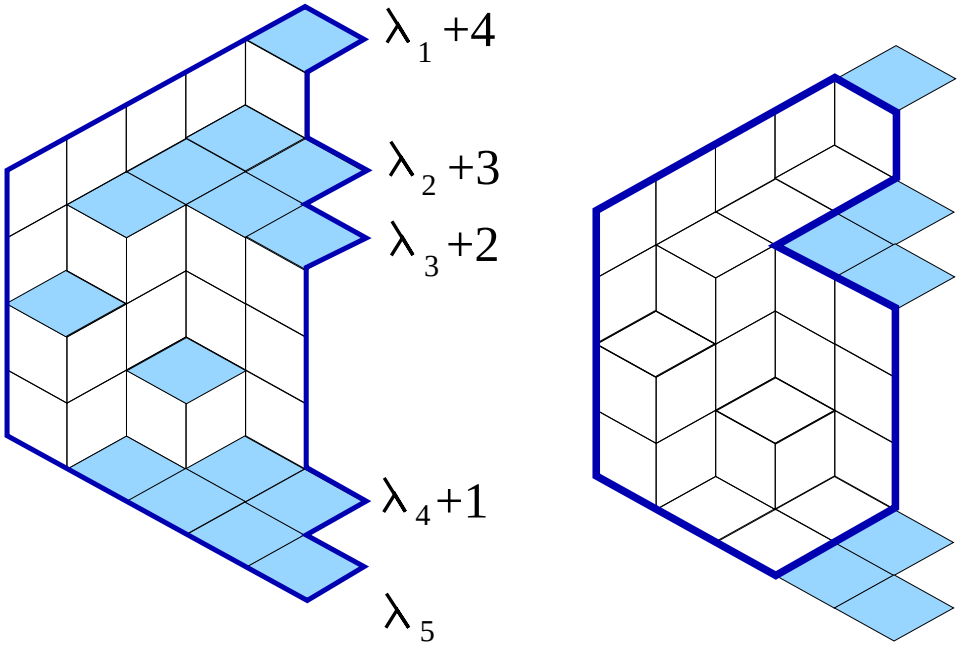

FIGURE 1. Lozenge tiling of the domain encoded by signature λ (left panel) and of corresponding polygonal domain (right panel).

It is well–known that each lozenge tiling can be identified with a stepped surface in $\mathbb{R}^3$ (the three types of lozenges correspond to the three slopes of this surface) and with a perfect matching of a subgraph of a hexagonal lattice, see e.g. [Ke]. Note that there are finitely many tilings of Ω_λ and let Υ_λ denote a uniformly random lozenge tiling of Ω_λ. The interest in lozenge tilings is caused by their remarkable asymptotic behavior. When N is large the rescaled stepped surface corresponding to Υ_λ concentrates near a deterministic limit shape. In fact, this is true also for more general domains, see [CKP]. One feature of the limit shape is the formation of so–called *frozen regions*; in terms of tilings, these are the regions where asymptotically with high probability only single type of lozenges is observed. This effect is visualized in Figure 3, where a sample from the uniform measure on tilings of the simplest tilable domain — hexagon

— is shown. It is known that in this case the boundary of the frozen region is the inscribed ellipse, see [CLP], for more general polygonal domains the frozen boundary is an inscribed algebraic curve, see [KO] and also [Pe1].

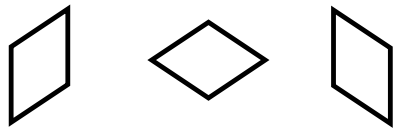

FIGURE 2. The 3 types of lozenges, the middle one is called "horizontal".

In this article we study the local behavior of lozenge tiling near a *turning point* of the frozen boundary, which is the point where the boundary of the frozen region touches (and is tangent to) the boundary of the domain. Okounkov and Reshetikhin gave in [OR1] a non-rigorous argument explaining that the scaling limit of a tiling in such situation should be governed by the *GUE–corners* process (introduced and studied by Baryshnikov [Bar] and Johansson–Nordenstam [JN]), which is the joint distribution of the eigenvalues of a Gaussian Unitary Ensemble (GUE–)random matrix (i.e. Hermitian matrix with independent Gaussian entries) and of its top–left corner square submatrices. In one model of tilings of infinite polygonal domains, the proof of the convergence can be based on the determinantal structure of the correlation functions of the model and on the double–integral representation for the correlation kernel and it was given in [OR1]. Another rigorous argument, related to the asymptotics of *orthogonal polynomials* exists for the lozenge tilings of hexagon (as in Figure 3), see [JN], [N].

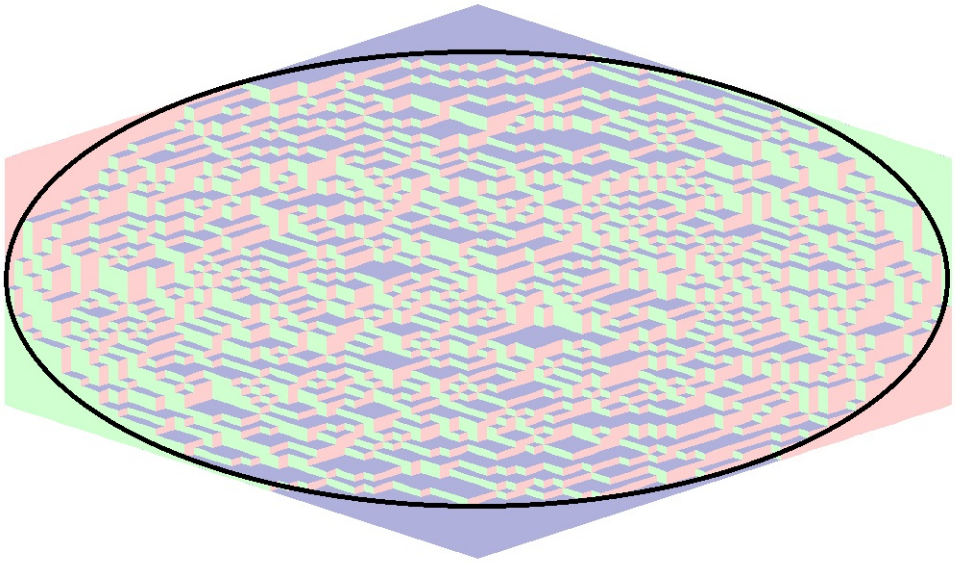

FIGURE 3. A sample from uniform distribution on tilings of $40 \times 50 \times 50$ hexagon and corresponding theoretical frozen boundary. The three types of lozenges are shown in three distinct colors.

Given Υ_λ let $\nu_1 > \nu_2 > \cdots > \nu_k$ be the horizontal lozenges at the kth vertical line from the left. (Horizontal lozenges are shown in blue in the left panel of Figure 1.) We set $\nu_i = \kappa_i + k - i$ and denote the resulting random signature κ of size k as Υ_λ^k. Further,

let $\mathbb{GUE}_k$ denote the distribution of k (ordered) eigenvalues of a random Hermitian matrix from a Gaussian Unitary Ensemble.

Theorem 1.7 (Theorem 5.1)**.** *Let $\lambda(N) \in \mathbb{GT}_N$, $N = 1, 2, \dots$ be a sequence of signatures. Suppose that there exist a non-constant piecewise-differentiable weakly decreasing function $f(t)$ such that*

$$\sum_{i=1}^{N} \left| \frac{\lambda_i(N)}{N} - f(i/N) \right| = o(\sqrt{N})$$

as $N \to \infty$ and also $\sup_{i,N} |\lambda_i(N)/N| < \infty$. Then for every k as $N \to \infty$ we have

$$\frac{\Upsilon^k_{\lambda(N)} - NE(f)}{\sqrt{NS(f)}} \to \mathbb{GUE}_k$$

in the sense of weak convergence, where

$$E(f) = \int_0^1 f(t)dt, \quad S(f) = \int_0^1 f(t)^2 dt - E(f)^2 + \int_0^1 f(t)(1-2t)dt.$$

Corollary 1.8 (Corollary 5.2)**.** *Under the same assumptions as in Theorem 1.7 the (rescaled) joint distribution of $k(k+1)/2$ horizontal lozenges on the left k lines weakly converges to the joint distribution of the eigenvalues of the k top-left corners of a $k \times k$ matrix from a GUE.*

Note that, in principle, our domains may approximate a non–polygonal limit domain as $N \to \infty$, thus, the results of [KO] describing the limit shape in terms of algebraic curves are not applicable here and not much is known about the exact shape of the frozen boundary. In particular, even the explicit expression for the coordinate of the point where the frozen boundary touches the left boundary (which we get as a side result of Theorem 1.7) seems not to be present in the literature.

Our approach to the proof of Theorem 1.7 is the following: We express the expectations of certain observables of uniformly random lozenge tilings through normalized Schur polynomials S_λ and investigate the asymptotics of these polynomials. In this case we prove and use the following asymptotic expansion (given in Proposition 4.3 and Proposition 5.8)

$$S_\lambda(e^{\frac{h_1}{\sqrt{N}}}, \dots, e^{\frac{h_k}{\sqrt{N}}}; N, 1) \\ = \exp\left(\sqrt{N} E(f)(h_1 + \dots + h_k) + \frac{1}{2} S(f)(h_1^2 + \dots + h_k^2) + o(1) \right).$$

We believe that our approach can be extended to a natural q–deformation of uniform measure, which assigns the weight q^{vol} to lozenge tiling with volume vol below the corresponding stepped surface; and also to lozenge tilings with axial symmetry, as in [FN], [BK]. In the latter case the Schur polynomials are replaced with characters of orthogonal or symplectic groups and the limit object also changes. We postpone the thorough study of these cases to a future publication.

We note that there might be another approach to the proof of Theorem 1.7. Recently there was progress in understanding random tilings of polygonal domains, Petrov found double integral representations for the correlation kernel describing the local structure of tilings of a wide class of polygonal domains, see [Pe1] (and also [Me] for a similar result in context of random matrices). Starting from these formulas, one could try to prove the GUE–corners asymptotics along the lines of [OR1].

1.5. **Application: six–vertex model and random ASMs.** An *Alternating Sign Matrix* of size N is a $N \times N$ matrix whose entries are either $\mathbf{0}$, $\mathbf{1}$ or $-\mathbf{1}$, such that the sum along every row and column is 1 and, moreover, along each row and each column the nonzero entries alternate in sign. Alternating Sign Matrices are in bijection with configurations of the six-vertex model with domain wall boundary conditions as shown at Figure 4, more details on this bijection are given in Section 5.2. A good review of the six–vertex model can be found e.g. in the book [Bax] by Baxter.

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

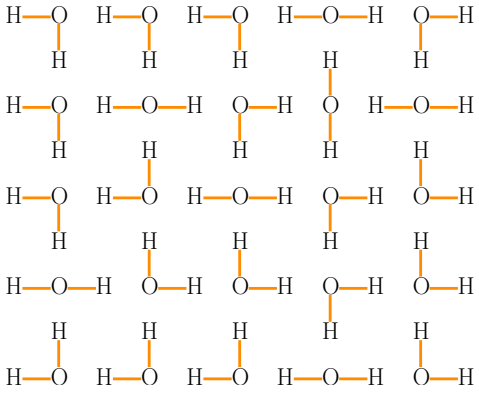

FIGURE 4. An alternating sign matrix of size 5 and the corresponding configuration of the six–vertex model ("square ice") with domain wall boundary condition. **1**s in ASM correspond to horizontal molecules H-O-H and $-\mathbf{1}$s to the vertical ones.

Interest in ASMs from combinatorial perspective emerged since their discovery in connection with Dodgson condensation algorithm for determinant evaluations. Initially, questions concerned enumeration problems, for instance, finding the total number of ASMs of given size n (this was the long-standing *ASM conjecture* proved by Zeilberger [Ze] and Kuperberg [Ku], the full story can be found in the Bressoud's book [Br]). Physicists' interest stems from the fact that ASMs are in one-to-one bijection with configurations of the six–vertex model. Many questions on ASMs still remain open. Examples of recent breakthroughs include the Razumov–Stroganov [RS] conjecture relating ASMs to yet another model of statistical mechanics (so-called O(1) loop model), which was finally proved very recently by Cantini and Sportiello [CS], and the still open question on a bijective proof of the fact that Totally Symmetric Self-Complementary Plane Partitions and ASMs are equinumerous. A brief up-to-date introduction to the subject can be found e.g. in [BFZ].

Our interest in ASMs and the six–vertex model is probabilistic. We would like to know how a *uniformly random* ASM of size n looks like when n is large. Conjecturally, the features of this model should be similar to those of lozenge tilings: we expect the formation of a limit shape and various connections with random matrices. The properties of the limit shape for ASMs were addressed by Colomo and Pronko [CP3], however their arguments are mostly not mathematical, but physical.

In the present article we prove a partial result toward the following conjecture.

Conjecture 1.9. *Fix any k. As $n \to \infty$ the probability that the number of $-\mathbf{1}$s in the first k rows of a uniformly random ASM of size n is maximal (i.e. there is one $-\mathbf{1}$*

in second row, two $-\mathbf{1}$s in third row, etc) tends to 1*, and, thus,* 1*s in first k rows are interlacing. After proper centering and rescaling, the distribution of the positions of* $\mathbf{1}$*s tends to the GUE–corners process as $n \to \infty$.*

Let $\Psi_k(n)$ denote the sum of coordinates of $\mathbf{1}$s minus the sum of coordinates of $-\mathbf{1}$s in the kth row of the uniformly random ASM of size n. We prove that the centered and rescaled random variables $\Psi_k(n)$ converge to the collection of i.i.d. Gaussian random variables as $n \to \infty$.

Theorem 1.10 (Theorem 5.9)**.** *For any fixed k the random variable $\frac{\Psi_k(n)-n/2}{\sqrt{n}}$ weakly converges to the normal random variable $N(0, \sqrt{3/8})$. Moreover, the joint distribution of any collection of such variables converges to the distribution of independent normal random variables $N(0, \sqrt{3/8})$.*

Remark. We also prove a bit stronger statement, see Theorem 5.9 for the details.

Note that Theorem 1.10 agrees with Conjecture 1.9. Indeed, if the latter holds, then $\Psi_k(n)$ converges to the difference of the sums of the eigenvalues of a $k\times k$ GUE–random matrix and of its $(k-1)\times(k-1)$ top left submatrix. But these sums are the same as the traces of the corresponding matrices, therefore, the difference of sums equals the bottom right matrix element of the $k \times k$ matrix, which is a Gaussian random variable by the definition of GUE.

Our proof of Theorem 1.10 has two components. First, a result of Okada [Ok], based on earlier work of Izergin and Korepin [I], [Kor], shows that sums of certain quantities over all ASMs can be expressed through Schur polynomials (in an equivalent form this was also shown by Stroganov [St]). Second, our method gives the asymptotic analysis of these polynomials.

In fact, we claim that Theorem 1.10 together with an additional probabilistic argument implies Conjecture 1.9. However, this argument is unrelated to the asymptotics of symmetric polynomials and, thus, is left out of the scope of the (already long) present paper; the proof of Conjecture 1.9 based on Theorem 1.10 is presented by one of the authors in the later article [G2].

In the literature one can find another probability measure on ASMs assigning the weight 2^{n_1} to the matrix with n_1 $\mathbf{1}$s. For this measure there are many rigorous mathematical results, due to the connection to the uniform measure on *domino tilings of the Aztec diamond*, see [EKLP], [FS]. The latter measure can be viewed as a *determinantal point process*, which gives tools for its analysis. An analogue of Conjecture 1.9 for the tilings of Aztec diamond was proved by Johansson and Nordenstam [JN].

In regard to the combinatorial questions on ASMs, we note that there has been interest in *refined* enumerations of Alternating Sign Matrices, i.e. counting the number of ASMs with fixed positions of 1s along the boundary. In particular, Colomo–Pronko [CP1], [CP2], Behrend [Be] and Ayyer–Romik [AR] found formulas relating k–refined enumerations to 1–refined enumerations for ASMs. Some of these formulas are closely related to particular cases of our multivariate formulas (Theorem 3.7) for staircase Young diagrams.

1.6. **Application: $O(n = 1)$–loop model.** Recently found parafermionic observables in the so-called completely packed $O(n = 1)$ dense loop model in a strip are also simply related to symmetric polynomials, see [GNP]. The $O(n = 1)$ dense loop model is one of the representations of the percolation model on the square lattice. For the critical percolation models similar observables and their asymptotic behavior were

studied (see e.g. [Sm]), however, the methods involved are usually completely different from ours.

A configuration of the $O(n = 1)$ loop model in a vertical strip consists of two parts: a tiling of the strip on a square grid of width L and infinite height with squares of two types shown in Figure 5 (left panel), and a choice of one of the two types of boundary conditions for each 1×2 segment along each of the vertical boundaries of the strip; the types appearing at the left boundary are shown in Figure 5 (right panel). Let $\urcorner_L$ denote the set of all configurations of the model in the strip of width L. An element of $\urcorner_6$ is shown in Figure 5. Note that the arcs drawn on squares and boundary segments form closed loops and paths joining the boundaries. Therefore, the elements of $\urcorner_L$ have an interpretation as collections of non-intersecting paths and closed loops.

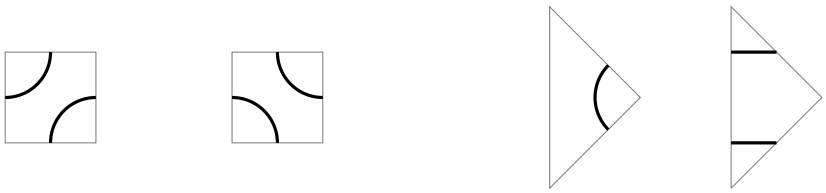

FIGURE 5. Left panel: the two types of squares. Right panel: the two types of boundary conditions.

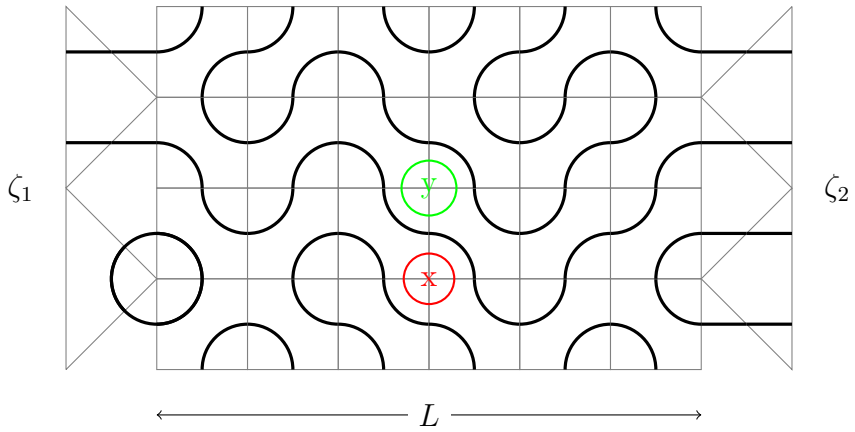

FIGURE 6. A particular configuration of the dense loop model showing a path passing between two vertically adjacent points x and y.

In the simplest homogeneous case a probability distribution on $\urcorner_L$ is defined by declaring the choice of one of the two types of squares to be an independent Bernoulli random variable for each square of the strip and for each segment of the boundary. I.e. for each square of the strip we flip an unbiased coin to choose one of the two types of squares (shown in Figure 5) and similarly for the boundary conditions. More generally, the type of a square is chosen using a (possibly signed or even complex) weight defined as a certain function of its horizontal coordinate and depending on L parameters $z_1, \dots, z_L$; two other parameters ζ_1, ζ_2 control the probabilities of the boundary conditions and, using a parameter q, the whole configuration is further weighted by its number of closed loops. We refer the reader to [GNP] and references therein for the exact dependence of weights on the parameters of the model and for the explanation of the choices of parameters.

Fix two points x and y and consider a configuration $\omega \in \urcorner_L$. There are finitely many paths passing between x and y. For each such path τ we define the current $c(\tau)$ as 0 if τ is a closed loop or joins points of the same boundary; 1 if τ joins the two boundaries and x lies above τ; -1 if τ joins the two boundaries and x lies below τ. The total current $C^{x,y}(\omega)$ is the sum of $c(\tau)$ over all paths passing between x and y. The *mean total current* $F^{x,y}$ is defined as the expectation of $C^{x,y}$.

Two important properties of $F^{x,y}$ are skew-symmetry

$$F^{x,y} = -F^{y,x}$$

and additivity

$$F^{x_1,x_3} = F^{x_1,x_2} + F^{x_2,x_3}.$$

These properties allow to express $F^{(x,y)}$ as a sum of several instances of the mean total current between two horizontally adjacent points

$$F^{(i,j),(i,j+1)}$$

and the mean total current between two vertically adjacent points

$$F^{(j,i),(j+1,i)}.$$

The authors of [GNP] present a formula for $F^{(i,j),(i,j+1)}$ and $F^{(j,i),(j+1,i)}$ which, based on certain assumptions, expresses them through the symplectic characters $\chi_{\lambda^L}(z_1^2, \dots, z_L^2, \zeta_1^2, \zeta_2^2)$ where $\lambda^L = (\lfloor \frac{L-1}{2} \rfloor, \lfloor \frac{L-2}{2} \rfloor, \dots, 1, 0, 0)$. The precise relationship is given in Section 5.3. Our approach allows us to compute the asymptotic behavior of the formulas of [GNP] as the lattice width $L \to \infty$, see Theorem 5.12. In particular, we prove that the leading term in the asymptotic expansion is independent of the boundary parameters ζ_1, ζ_2.

This problem was presented to the authors by de Gier [Gi], [GP] during the program "Random Spatial Processes" at MSRI, Berkeley.

1.7. **Application: matrix integrals.** Let A and B be two $N \times N$ Hermitian matrices with eigenvalues $a_1, \dots, a_N$ and $b_1, \dots, b_N$, respectively. The Harish–Chandra formula [H1], [H2] (sometimes known also as Itzykson–Zuber [IZ] formula in physics literature) is the following evaluation of the integral over the unitary group:

$$\int_{U(N)} \exp(\mathrm{Trace}(AUBU^{-1}))dU = \frac{\det_{i,j=1,\dots,N}\Big(\exp(a_i b_j)\Big)}{\prod_{i<j}(a_i - a_j)\prod_{i<j}(b_i - b_j)} \prod_{i<j}(j-i), \tag{1.11}$$

where the integration is with respect to the normalized Haar measure on the unitary group $U(N)$. Comparing (1.11) with the definition of Schur polynomials and using Weyl's dimension formula

$$s_\lambda(1, \dots, 1) = \prod_{i<j} \frac{(\lambda_i - i) - (\lambda_j - j)}{j - i},$$

we observe that when $b_j = \lambda_j + N - j$ the above matrix integral is the normalized Schur polynomial times explicit product, i.e.

$$\frac{s_\lambda(e^{a_1}, \dots, e^{a_n})}{s_\lambda(1, \dots, 1)} \cdot \prod_{i<j} \frac{e^{a_i} - e^{a_j}}{a_i - a_j}.$$

Guionnet and Maïda studied (after some previous results in the physics literature, see [GM] and references therein) the asymptotics of the above integral as $N \to \infty$ when the rank of A is finite and does not depend on N. This is precisely the asymptotics

of (1.1). Therefore, our methods (in particular, Propositions 4.1, 4.2, 4.7) give a new proof of some of the results of [GM]. In the context of random matrices the asymptotics of this integral in the case when rank of A grows as the size of A grows was also studied, see e.g. [GZ], [CS]. However, currently we are unable to use our methods for this case.

1.8. **Comparison with other approaches.** Since asymptotics of symmetric polynomials as the number of variables tends to infinity already appeared in various contexts in the literature, it makes sense to compare our approach to the ones used before.

In the context of asymptotic representation theory the known approach (see [VK], [OO], [OO2], [G1]) is to use the so-called *binomial formulas*. In the simplest case of Schur polynomials such formula reads as

$$S_\lambda(1+x_1,\dots,1+x_k;N,1)=\sum_\mu s_\mu(x_1,\dots,x_k)c(\mu,\lambda,N), \tag{1.12}$$

where the sum is taken over all Young diagrams μ with at most k rows, and $c(\mu,\lambda,N)$ are certain (explicit) coefficients. In the asymptotic regime of Theorem 1.5 the convergence of the left side of (1.12) implies the convergence of numbers $c(\mu,\lambda,N)$ to finite limits as $N\to\infty$. Studying the possible asymptotic behavior of these numbers one proves the limit theorems for normalized Schur polynomials.

Another approach uses the decomposition

$$S_\lambda(x_1,\dots,x_k;N,1)=\sum_\mu S_\mu(x_1,\dots,x_k;k,1)d(\mu,\lambda,N), \tag{1.13}$$

where the sum is taken over all signatures of length k. Recently in [BO] and [Pe2] $k\times k$ determinantal formulas were found for the coefficients $d(\mu,\lambda,N)$. Again, these formulas allow the asymptotic analysis which leads to the limit theorems for normalized Schur polynomials.

The asymptotic regime of Theorem 1.5 is distinguished by the fact that $\sum_i|\lambda_i(N)|/N$ is bounded as $N\to\infty$. This no longer holds when one studies asymptotics of lozenge tilings, ASMs, or $O(n=1)$ loop model. As far as authors know, in the latter limit regime neither formulas (1.12) nor (1.13) give simple ways to compute the asymptotics. The reason for that is the fact that for any fixed μ both $c(\mu,\lambda,N)$ and $d(\mu,\lambda,N)$ would converge to zero as $N\to\infty$ and more delicate analysis would be required to reconstruct the asymptotics of normalized Schur polynomials.

Yet another, but similar approach to the proof of Theorem 1.5 was used in [Bo2] but, as far as authors know, it also does not extend to the regime we need for other applications.

On the other hand the random–matrix asymptotic regime of [GM] is similar to the one we need for studying lozenge tilings, ASMs, or $O(n=1)$ loop model. The approach of [GM] is based on the matrix model and the proofs rely on *large deviations* for Gaussian random variables. However, it seems that the results of [GM] do not suffice to obtain our applications: for $k>1$ only the first order asymptotics (which is the limit of $\ln(S_\lambda(x_1,\dots,x_k;N,1))/N$) was obtained in [GM], while our applications require more delicate analysis. It also seems that the results of [GM] (even for $k=1$) cannot be applied in the framework of the representation theoretic regime of Theorem 1.5.

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2. Definitions and problem setup

In this section we set up notations and introduce the symmetric functions of our interest.

A *partition* (or a *Young diagram*) λ is a collection of non-negative numbers $\lambda_1 \geq \lambda_2 \geq \dots$, such that $\sum_i \lambda_i < \infty$. The numbers λ_i are *row lengths* of λ, and the numbers $\lambda'_i = |\{j : \lambda_j \geq i\}|$ are *column lengths* of λ.

More generally a *signature* λ of size N is an N–tuple of integers $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_N$. The set of all signatures of size N is denoted $\mathbb{GT}_N$. It is also convenient to introduce *strict signatures*, which are N–tuples satisfying strict inequalities $\lambda_1 > \lambda_2 > \dots > \lambda_N$; they from the set $\widehat{\mathbb{GT}}_N$. We are going to use the following identification between elements of $\mathbb{GT}_N$ and $\widehat{\mathbb{GT}}_N$:

$$\mathbb{GT}_N \ni \lambda \longleftrightarrow \lambda + \delta_N = \mu \in \widehat{\mathbb{GT}}_N, \quad \mu_i = \lambda_i + N - i,$$

where we set $\delta_N = (N-1, N-2, \dots, 1, 0)$. The subset of $\mathbb{GT}_N$ ($\widehat{\mathbb{GT}}_N$) of all signatures (strict signatures) with non-negative coordinates is denoted $\mathbb{GT}_N^+$ ($\widehat{\mathbb{GT}}_N^+$).

One of the main objects of study in this paper are the rational Schur functions, which originate as the characters of the irreducible representations of the unitary group $U(N)$ (equivalently, of irreducible rational representations of the general linear group $GL(N)$). Irreducible representations are parameterized by elements of $\mathbb{GT}_N$, which are identified with the *dominant weights*, see e.g. [W] or [Zh]. The value of the character of the irreducible representation V_λ indexed by $\lambda \in \mathbb{GT}_N$, on a unitary matrix with eigenvalues $u_1, \dots, u_N$ is given by the *Schur function*,

$$s_\lambda(u_1, \dots, u_N) = \frac{\det\left[u_i^{\lambda_j + N - j}\right]_{i,j=1}^N}{\prod_{i<j}(u_i - u_j)}, \tag{2.1}$$

which is a symmetric Laurent polynomial in $u_1, \dots, u_N$. The denominator in (2.1) is the Vandermonde determinant and we denote it through Δ:

$$\Delta(u_1, \dots, u_N) = \det\left[u_i^{N-j}\right]_{i,j=1}^N = \prod_{i<j}(u_i - u_j).$$

When the numbers u_i form a geometric progression, the determinant in (2.1) can be evaluated explicitly as

$$s_\lambda(1, q \dots, q^{N-1}) = \prod_{i<j} \frac{q^{\lambda_i + N - i} - q^{\lambda_j + N - j}}{q^{N-i} - q^{N-j}}. \tag{2.2}$$

In particular, sending $q \to 1$ we get

$$s_\lambda(1^N) = \prod_{1\le i<j\le N} \frac{(\lambda_i - i) - (\lambda_j - j)}{j-i}, \tag{2.3}$$

where we used the notation

$$1^N = (\underbrace{1, \dots, 1}_{N}).$$

The identity (2.3) gives the dimension of V_λ and is known as the Weyl's dimension formula.

In what follows we intensively use the normalized versions of Schur functions:

$$S_\lambda(x_1, \dots, x_k; N, q) = \frac{s_\lambda(x_1, \dots, x_k, 1, q, q^2, \dots, q^{N-1-k})}{s_\lambda(1, \dots, q^{N-1})},$$

in particular,

$$S_\lambda(x_1, \dots, x_k; N, 1) = \frac{s_\lambda(x_1, \dots, x_k, 1^{N-k})}{s_\lambda(1^N)}.$$

The Schur functions are characters of type A (according to the classification of root systems), their analogues for other types are related to the *multivariate Jacobi polynomials*.

For $a, b > -1$ and $m = 0, 1, 2 \dots$ let $\mathfrak{p}_m(x; a, b)$ denote the classical Jacobi polynomials orthogonal with respect to the weight $(1-x)^a(1+x)^b$ on the interval $[-1, 1]$, see e.g. [Er1], [KoSw]. We use the normalization of [Er1], thus, the polynomials can be related to the Gauss hypergeometric function ${}_2F_1$:

$$\mathfrak{p}_m(x; a, b) = \frac{\Gamma(m+a+1)}{\Gamma(m+1)\Gamma(a+1)}\ {}_2F_1\left(-m, m+a+b+1, a+1; \frac{1-x}{2}\right).$$

For any strict signature $\lambda \in \widehat{\mathbb{GT}}_N^+$ set

$$\mathfrak{P}_\lambda(x_1, \dots, x_N; a, b) = \frac{\det[\mathfrak{p}_{\lambda_i}(x_j; a, b)]_{i,j=1}^N}{\Delta(x_1, \dots, x_N)}.$$

and for any (non-strict) $\lambda \in \mathbb{GT}_N^+$ define

$$\mathfrak{J}_\lambda(z_1, \dots, z_N; a, b) = c_\lambda \mathfrak{P}_{\lambda+\delta}\left(\frac{z_1 + z_1^{-1}}{2}, \dots, \frac{z_N + z_N^{-1}}{2}; a, b\right), \tag{2.4}$$

where c_λ is a constant chosen so that the leading coefficient of $\mathfrak{J}_\lambda$ is 1. The polynomials $\mathfrak{J}_\lambda$ are (a particular case of) BC_N multivariate Jacobi polynomials, see e.g. [OO2] and also [HS], [M2], [Koo]. We also use their normalized versions

$$J_\lambda(z_1, \dots, z_k; N, a, b) = \frac{\mathfrak{J}_\lambda(z_1, \dots, z_k, 1^{N-k}; a, b)}{\mathfrak{J}_\lambda(1^N; a, b)}. \tag{2.5}$$

Again, there is an explicit formula for the denominator in (2.5) and also for its q–version. For special values of parameters a and b the functions J_λ can be identified with spherical functions of classical Riemannian symmetric spaces of compact type, in particular, with normalized characters of orthogonal and symplectic groups, see e.g. [OO2, Section 6].

Let us give more details on the latter case of the symplectic group $Sp(2N)$, as we need it for one of our applications. This case corresponds to $a = b = 1/2$ and here the formulas can be simplified.

The value of character of irreducible representation of $Sp(2N)$ parameterized by $\lambda \in \mathbb{GT}_N^+$ on symplectic matrix with eigenvalues $x_1, x_1^{-1}, \dots, x_N, x_N^{-1}$ is given by (see e.g. [W], [Zh])

$$\chi_\lambda(x_1, \dots, x_N) = \frac{\det\left[x_i^{\lambda_j+N+1-j} - x_i^{-(\lambda_j+N+1-j)}\right]_{i,j=1}^N}{\det\left[x_i^{N+1-j} - x_i^{-N-1+j}\right]_{i,j=1}^N}.$$

The denominator in the last formula can be evaluated explicitly and we denote it Δ_{s}

$$\Delta_{\mathrm{s}}(x_1, \dots, x_N) = \det\left[x_i^{N-j+1} - x_i^{-N+j-1}\right]_{i,j=1}^N \tag{2.6}$$
$$= \prod_i (x_i - x_i^{-1}) \prod_{i<j} (x_i + x_i^{-1} - (x_j + x_j^{-1})) = \frac{\prod_{i<j}(x_i - x_j)(x_i x_j - 1)\prod_i (x_i^2 - 1)}{(x_1 \cdots x_n)^n}.$$

The normalized symplectic character is then defined as

$$\mathfrak{X}_\lambda(x_1, \dots, x_k; N, q) = \frac{\chi_\lambda(x_1, \dots, x_k, q, \dots, q^{N-k})}{\chi_\lambda(q, q^2, \dots, q^N)},$$

in particular

$$\mathfrak{X}_\lambda(x_1, \dots, x_k; N, 1) = \frac{\chi_\lambda(x_1, \dots, x_k, 1^{N-k})}{\chi_\lambda(1^N)},$$

and both denominators again admit explicit formulas.

In most general terms, in the present article we study the symmetric functions S_λ, J_λ, X_λ, their asymptotics as $N \to \infty$ and its applications.

Some further notations.

We intensively use the q–algebra notations

$$[m]_q = \frac{q^m - 1}{q - 1}, \quad [a]_q! = \prod_{m=1}^a [m]_q,$$

and q-Pochhammer symbol

$$(a; q)_k = \prod_{i=0}^{k-1} (1 - aq^i).$$

Since there are lots of summations and products in the text where i plays the role of the index, we write $\mathbf{i}$ for the imaginary unit to avoid the confusion.

3. Integral and multivariate formulas

In this section we derive integral formulas for normalized characters of one variable and also express the multivariate normalized characters as determinants of differential (or, sometimes, difference) operators applied to the product of the single variable normalized characters.

We first exhibit some general formulas, which we later specialize to the cases of Schur functions, symplectic characters and multivariate Jacobi polynomials.

3.1. **General approach.**

Definition 3.1. *For a given sequence of numbers $\theta = (\theta_1, \theta_2, \dots)$, a collection of functions $\{A_\mu(x_1, \dots, x_N)\}$, $N = 1, 2, \dots$, $\mu \in \widehat{\mathbb{GT}}_N$ (or $\widehat{\mathbb{GT}}_N^+$) is called a* class of determinantal symmetric functions *with parameter θ, if there exist functions $\alpha(u)$, $\beta(u)$, $g(u, v)$, numbers c_N, and linear operator T such that for all N and μ we have*

(1)
$$\mathcal{A}_\mu(x_1, \dots, x_N) = \frac{\det[g(x_j; \mu_i)]_{i,j=1}^N}{\Delta(x_1, \dots, x_N)}$$

(2)
$$\mathcal{A}_\mu(\theta_1, \dots, \theta_N) = c_N \prod_{i=1}^N \beta(\mu_i) \prod_{i<j} (\alpha(\mu_i) - \alpha(\mu_j)),$$

(3) *$g(x; m)$ ($m \in \mathbb{Z}$ for the case of $\widehat{\mathbb{GT}}$ and $m \in \mathbb{Z}_{\geq 0}$ for the case $\widehat{\mathbb{GT}}^+$) are eigenfunctions of T acting on x with eigenvalues $\alpha(m)$, i.e.*
$$T(g(x, m)) = \alpha(m) g(x, m).$$

(4) *$\alpha'(m) \neq 0$ for all m as above.*

Proposition 3.2. *For $\mathcal{A}_\mu(x_1, \dots, x_N)$, as in Definition 3.1 we have the following formula*

$$\frac{\mathcal{A}_\mu(x_1, \dots, x_k, \theta_1, \dots, \theta_{N-k})}{\mathcal{A}_\mu(\theta_1, \dots, \theta_N)} = \frac{c_{N-k}}{c_N} \prod_{i=1}^k \prod_{j=1}^{N-k} \frac{1}{x_i - \theta_j} \times \frac{\det\left[T_i^{j-1}\right]_{i,j=1}^k}{\Delta(x_1, \dots, x_k)} \prod_{i=1}^k \left(\frac{\mathcal{A}_\mu(x_i, \theta_1, \dots, \theta_{N-1})}{\mathcal{A}_\mu(\theta_1, \dots, \theta_N)} \prod_{j=1}^{N-1} (x_i - \theta_j) \frac{c_N}{c_{N-1}} \right), \tag{3.1}$$

where T_i is operator T acting on variable x_i.

Remark. Since operators T_i commute, we have
$$\det\left[T_i^{j-1}\right]_{i,j=1}^k = \prod_{i<j} (T_i - T_j).$$

We also note that some of the denominators in (3.1) can be grouped in the compact form
$$\prod_{i=1}^k \prod_{j=1}^{N-k} (x_i - \theta_j) \Delta(x_1, \dots, x_k) = \frac{\Delta(x_1, \dots, x_k, \theta_1, \dots, \theta_{N-k})}{\Delta(\theta_1, \dots, \theta_{N-k})}.$$
Moreover, in our applications, the coefficients c_m will be inversely proportional to $\Delta(\theta_1, \dots, \theta_m)$, so we will be able to write alternative formulas where the prefactors are simple ratios of Vandermondes.

Proof of Proposition 3.2. We will compute the determinant from property (1) of $\mathcal{A}$ by summing over all $k \times k$ minors in the first k columns, where we set the convention that i is a row index and j is a column index. The rows in the corresponding minors will be indexed by $I = \{i_1 < i_2 < \dots < i_k\}$ and $\mu_I = (\mu_{i_1}, \dots, \mu_{i_k})$. I^c denotes the complement of I in $\{1, 2, \dots, n\}$. We have

$$\frac{\mathcal{A}_\mu(x_1, \dots, x_k, \theta_1, \dots, \theta_{N-k})}{\mathcal{A}_\mu(\theta_1, \dots, \theta_N)} = \frac{1}{\prod_{i=1}^k \prod_{j=1}^{N-k} (x_i - \theta_j)} \tag{3.2}$$

$$\times \sum_{I=\{i_1<i_2<\cdots<i_k\}} (-1)^{\sum_{\ell\in I}(\ell-1)} \mathcal{A}_{\mu_I}(x_1,\dots,x_k)\frac{\mathcal{A}_{\mu_{I^c}}(\theta_1,\dots,\theta_{N-k})}{\mathcal{A}_\mu(\theta_1,\dots,\theta_N)}$$

For each set I we have

$$\begin{aligned}(3.3)\quad \frac{\mathcal{A}_{\mu_{I^c}}(\theta_1,\dots,\theta_{N-k})}{\mathcal{A}_\mu(\theta_1,\dots,\theta_N)}\frac{c_N}{c_{N-k}} &= \frac{\prod_{i\in I^c}\beta(\mu_i)\prod_{i<j;i,j\in I^c}(\alpha(\mu_i)-\alpha(\mu_j))}{\prod_{i=1}^N\beta(\mu_i)\prod_{1\le i<j\le N}(\alpha(\mu_i)-\alpha(\mu_j))}\\ &= \left[\prod_{i\in I}\left(\frac{1}{\beta(\mu_i)}\prod_{j=i+1}^N\frac{1}{\alpha(\mu_i)-\alpha(\mu_j)}\right)\right]\prod_{i\notin I,\,j\in I,\,i<j}\frac{1}{\alpha(\mu_i)-\alpha(\mu_j)}\\ &= \left[\prod_{i\in I}\left(\frac{1}{\beta(\mu_i)}\prod_{j=i+1}^N\frac{1}{\alpha(\mu_i)-\alpha(\mu_j)}\right)\right]\frac{\prod_{i<j,\,i,j\in I}(\alpha(\mu_i)-\alpha(\mu_j))}{\prod_{r<s,\,r\in I,\,s\in[1..N]}-(\alpha(\mu_s)-\alpha(\mu_r))}\\ &= \prod_{i<j;i,j\in I}(\alpha(\mu_i)-\alpha(\mu_j))\cdot\prod_{r\in I}\frac{(-1)^{r-1}}{\beta(\mu_r)\prod_{s\ne r}(\alpha(\mu_r)-\alpha(\mu_s))}.\end{aligned}$$

We also have that

$$\begin{aligned}(3.4)\quad &\prod_{i<j;i,j\in I}(\alpha(\mu_i)-\alpha(\mu_j))A_{\mu_I}(x_1,\dots,x_k)\Delta(x_1,\dots,x_k)\\ &= \det\left[\alpha(\mu_{i_\ell})^{j-1}\right]_{\ell,j=1}^k\sum_{\sigma\in S_k}(-1)^\sigma\prod_{\ell=1}^k g(x_{\sigma_i};\mu_{i_\ell})\\ &= \sum_{\sigma\in S_k}(-1)^\sigma\det\left[\alpha(\mu_{i_\ell})^{j-1}g(x_{\sigma_j};\mu_{i_\ell})\right]_{\ell,j=1}^k = \sum_{\sigma\in S_k}(-1)^\sigma\det\left[T_{\sigma_j}^{j-1}g(x_{\sigma_j};\mu_{i_\ell})\right]_{\ell,j=1}^k\\ &= \det\left[T_i^{j-1}\right]_{i,j=1}^k\sum_{\sigma\in S_k}\prod_{\ell=1}^k g(x_{\sigma_i};\mu_{i_\ell}).\end{aligned}$$

Combining (3.2), (3.3) and (3.4) we get

$$\begin{aligned}(3.5)\quad &\frac{\mathcal{A}_\mu(x_1,\dots,x_k,\theta_1,\dots,\theta_{N-k})}{\mathcal{A}_\mu(\theta_1,\dots,\theta_N)}\prod_{i=1}^k\prod_{j=1}^{N-k}(x_i-\theta_j)\frac{c_N}{c_{N-k}}\\ &= \frac{\det\left[T_i^{j-1}\right]_{i,j=1}^k}{\Delta(x_1,\dots,x_k)}\sum_{I=\{i_1<i_2<\cdots<i_k\}}\sum_{\sigma\in S(k)}\prod_\ell\frac{g(x_\ell;\mu_{i_{\sigma_\ell}})}{\beta(\mu_{i_{\sigma_\ell}})\prod_{j\ne i_{\sigma_\ell}}(\alpha(\mu_{i_{\sigma_\ell}})-\alpha(\mu_j))}.\end{aligned}$$

Note that double summation in the last formula is a summation over all (ordered) collections of distinct numbers. We can also include into the sum the terms where some indices i_ℓ coincide, since application of the Vandermonde of linear operators annihilates such terms. Therefore, (3.5) equals

$$\frac{\det\left[T_i^{j-1}\right]_{i,j=1}^k}{\Delta(x_1,\dots,x_k)}\prod_{\ell=1}^k\sum_{i_\ell=1}^N\frac{g(x_\ell;\mu_{i_\ell})}{\beta(\mu_{i_\ell})\prod_{j\ne i_\ell}(\alpha(\mu_{i_\ell})-\alpha(\mu_j))}.$$

When $k=1$ the operators and the product over ℓ disappear, so we see that the remaining sum is exactly the univariate ratio $\dfrac{\mathcal{A}_\mu(x_\ell,\theta_1,\dots,\theta_{N-1})}{\mathcal{A}_\mu(\theta_1,\dots,\theta_N)}\prod_{j=1}^{N-1}(x_\ell-\theta_j)\dfrac{c_N}{c_{N-1}}$ and we obtain the desired formula. □

Proposition 3.3. *Under the assumptions of Definition 3.1 we have the following integral formula for the normalized univariate $\mathcal{A}_\mu$*

$$\frac{\mathcal{A}_\mu(x,\theta_1,\dots,\theta_{N-1})}{\mathcal{A}_\mu(\theta_1,\dots,\theta_N)}=\left(\frac{c_{N-1}}{c_N}\prod_{i=1}^{N-1}\frac{1}{x-\theta_i}\right)\frac{1}{2\pi\mathbf{i}}\oint_C\frac{g(x;z)\alpha'(z)}{\beta(z)\prod_{i=1}^N(\alpha(z)-\alpha(\mu_i))}dz. \tag{3.6}$$

Here the contour C includes only the poles of the integrand at $z=\mu_i$, $i=1,\dots,N$.

Proof. As a byproduct in the proof of Proposition 3.2 we obtained the following formula:

$$\frac{\mathcal{A}_\mu(x,\theta_1,\dots,\theta_{N-1})}{\mathcal{A}_\mu(\theta_1,\dots,\theta_N)}\prod_{j=1}^{N-1}(x-\theta_j)\frac{c_N}{c_{N-1}}=\sum_{i=1}^N\frac{g(x;\mu_i)}{\beta(\mu_i)\prod_{j\neq i}(\alpha(\mu_i)-\alpha(\mu_j))}. \tag{3.7}$$

Evaluating the integral in (3.6) as the sum of residues we arrive at the right side of (3.7). □

3.2. **Schur functions.** Here we specialize the formulas of Section 3.1 to the Schur functions.

Proposition 3.4. *Rational Schur functions $s_\lambda(x_1,\dots,x_N)$ (as above we identify $\lambda\in\mathbb{GT}_N$ with $\mu=\lambda+\delta\in\widehat{\mathbb{GT}_N}$) are class of determinantal functions with*

$$\theta_i=q^{i-1},\quad g(x;m)=x^m,\quad \alpha(x)=\frac{q^x-1}{q-1},\quad \beta(x)=1,$$

$$c_N=\prod_{1\le i<j\le N}\frac{q-1}{q^{j-1}-q^{i-1}}=\frac{1}{q^{\binom{N}{3}}}\prod_{j=1}^{N-1}\frac{1}{[j]_q!},\quad [Tf](x)=\frac{f(qx)-f(x)}{q-1}.$$

Proof. This immediately follows from the definition of Schur functions (2.1) and evaluation formula (2.2). □

Propositions 3.2 and 3.3 specialize to the following theorems.

Theorem 3.5. *For any signature $\lambda\in\mathbb{GT}_N$ and any $k\le N$ we have*

$$S_\lambda(x_1,\dots,x_k;N,q)=\frac{q^{\binom{k+1}{3}-(N-1)\binom{k}{2}}\prod_{i=1}^k[N-i]_q!}{\prod_{i=1}^k\prod_{j=1}^{N-k}(x_i-q^{j-1})}\times \frac{\det\left[D_{i,q}^{j-1}\right]_{i,j=1}^k}{\Delta(x_1,\dots,x_k)}\prod_{i=1}^k\frac{S_\lambda(x_i;N,q)\prod_{j=1}^{N-1}(x_i-q^{j-1})}{[N-1]_q!},$$

where $D_{i,q}$ is the difference operator acting on the function $f(x_i)$ by the formula

$$[D_{i,q}f](x_i)=\frac{f(qx_i)-f(x_i)}{q-1}.$$

Theorem 3.6. *For any signature $\lambda\in\mathbb{GT}_N$ and any $x\in\mathbb{C}$ other than 0 or q^i, $i\in\{0,\dots,N-2\}$ we have*

$$S_\lambda(x;N,q)=\frac{[N-1]_q!q^{\binom{N-1}{2}}(q-1)^{N-1}}{\prod_{i=1}^{N-1}(x-q^{i-1})}\cdot\frac{\ln(q)}{2\pi\mathbf{i}}\oint_C\frac{x^zq^z}{\prod_{i=1}^N(q^z-q^{\lambda_i+N-i})}dz,$$

where the contour C includes the poles at $\lambda_1+N-1,\dots,\lambda_N$ and no other poles of the integrand.

Remark. There is an alternative derivation of Theorem 3.6 suggested by A. Okounkov. Let $x = q^k$ with $k > N$. The definition of Schur polynomials implies the following symmetry for any $\mu, \lambda \in \mathbb{GT}_N$

$$\frac{s_\lambda(q^{\mu_1+N-1},\dots,q^{\mu_N})}{s_\lambda(1,\dots,q^{N-1})} = \frac{s_\mu(q^{\lambda_1+N-1},\dots,q^{\lambda_N})}{s_\mu(1,\dots,q^{N-1})}. \tag{3.8}$$

Using this symmetry,

$$S_\lambda(q^k;N,q) = \frac{h_{k+1-N}(q^{\lambda_1+N-1},\dots,q^{\lambda_N})}{h_{k+1-N}(1,\dots,q^{N-1})},$$

where $h_k = s_{(k,0,\dots)}$ is the complete homogeneous symmetric function. Integral representation for h_k can be obtained using their generating function (see e.g. [M, Chapter I, Section 2])

$$H(z) = \sum_{\ell=0}^{\infty} h_\ell(y_1,\dots,y_N)z^\ell = \prod_{i=1}^{N}\frac{1}{1-zy_i}.$$

Extracting h_ℓ as

$$h_\ell = \frac{1}{2\pi\mathbf{i}}\oint \frac{H(z)}{z^{\ell+1}}dz,$$

we arrive at the integral representation equivalent to Theorem 3.6. In fact the symmetry (3.8) holds in a greater generality, namely, one can replace Schur functions with *Macdonald polynomials*, which are their (q,t)–deformation, see [M, Chapter VI]. This means that, perhaps, Theorem 3.6 can be extended to the Macdonald polynomials. On the other hand, we do not know whether a simple analogue of Theorem 3.5 for Macdonald polynomials exists.

Sending $q \to 1$ in Theorems 3.5, 3.6 we get

Theorem 3.7. *For any signature $\lambda \in \mathbb{GT}_N$ and any $k \le N$ we have*

$$S_\lambda(x_1,\dots,x_k;N,1) = \prod_{i=1}^{k}\frac{(N-i)!}{(N-1)!(x_i-1)^{N-k}}\times \frac{\det\left[D_{i,1}^{j-1}\right]_{i,j=1}^{k}}{\Delta(x_1,\dots,x_k)}\prod_{j=1}^{k}S_\lambda(x_j;N,1)(x_j-1)^{N-1},$$

where $D_{i,1}$ is the differential operator $x_i\frac{\partial}{\partial x_i}$.

Theorem 3.8. *For any signature $\lambda \in \mathbb{GT}_N$ and any $x \in \mathbb{C}$ other than 0 or 1 we have*

$$S_\lambda(x;N,1) = \frac{(N-1)!}{(x-1)^{N-1}}\frac{1}{2\pi\mathbf{i}}\oint_C \frac{x^z}{\prod_{i=1}^{N}(z-(\lambda_i+N-i))}dz, \tag{3.9}$$

where the contour C includes all the poles of the integrand.

Note that this formula holds when $x \to 1$. Clearly, $\lim_{x\to 1} S_\lambda(x;N,1) = 1$. The convergence of the integral in (3.9) to 1 can be independently seen e.g. by application of L'Hospital's rule and evaluation of the resulting integral.

Let us state and prove several corollaries of Theorem 3.7.

For any integers j, ℓ, N, such that $0 \le \ell < j < N$, define the polynomial $P_{j,\ell,N}(x)$ as

$$P_{j,\ell,N}(x) = \binom{j-1}{\ell}\frac{N^\ell(N-j)!}{(N-1)!}(x-1)^{j-\ell-N}\left[\left(x\frac{\partial}{\partial x}\right)^{j-1-\ell}(x-1)^{N-1}\right], \tag{3.10}$$

it is easy to see (e.g. by induction on $j-\ell$) that $P_{j,\ell,N}$ is a polynomial in x of degree $j-\ell-1$ and its coefficients are bounded as $N\to\infty$. Also, $P_{j,0,N}(x)=x^{j-1}+O(1/N)$.

Proposition 3.9. *For any signature $\lambda\in\mathbb{GT}_N$ and any $k\le N$ we have*

$$S_\lambda(x_1,\dots,x_k;N,1)=\frac{1}{\Delta(x_1,\dots,x_k)} \times\det\left[\sum_{\ell=0}^{j-1}\frac{D_{i,1}^{\ell}[S_\lambda(x_i;N,1)]}{N^\ell}P_{j,\ell,N}(x_i)(x_i-1)^{\ell+k-j}\right]_{i,j=1}^{k}. \tag{3.11}$$

Proof. We apply Theorem 3.7 and, noting that

$$\left(x\frac{\partial}{\partial x}\right)^j[f(x)g(x)]=\sum_{\ell=0}^{j}\binom{j}{\ell}\left(x\frac{\partial}{\partial x}\right)^{\ell}[f(x)]\left(x\frac{\partial}{\partial x}\right)^{j-\ell}[g(x)]$$

for any f and g, we obtain

$$\begin{aligned}S_\lambda(x_1,\dots,x_k;N,1)&=\frac{1}{\Delta(x_1,\dots,x_k)}\det\left[\frac{(N-j)!}{(N-1)!}\frac{D_i^{j-1}(S_\lambda(x_i;N,1)(x_i-1)^{N-1})}{(x_i-1)^{N-k}}\right]_{i,j=1}^{k}\\&=\frac{\det\left[\sum_{\ell=0}^{j-1}D_{i,1}^{\ell}[S_\lambda(x_i;N,1)]\binom{j-1}{\ell}\frac{(N-j)!}{(N-1)!}\frac{D_{i,1}^{j-\ell-1}(x_i-1)^{N-1}}{(x_i-1)^{N-k}}\right]_{i,j=1}^{k}}{\Delta(x_1,\dots,x_k)}.\end{aligned} \tag{3.12}$$

□

Corollary 3.10. *Suppose that the sequence $\lambda(N)\in\mathbb{GT}_N$ is such that*

$$\lim_{N\to\infty}S_{\lambda(N)}(x;N,1)=\Phi(x)$$

uniformly on compact subsets of some region $M\subset\mathbb{C}$, then for any k

$$\lim_{N\to\infty}S_{\lambda(N)}(x_1,\dots,x_k;N,1)=\Phi(x_1)\cdots\Phi(x_k)$$

uniformly on compact subsets of M^k.

Proof. Since $S_{\lambda(N)}(x;N,1)$ is a polynomial, it is an analytic function. Therefore, the uniform convergence implies that the limit $\Phi(x)$ is analytic and all derivatives of $S_{\lambda(N)}(x)$ converge to the derivatives of $\Phi(x)$.

Now suppose that all x_i are distinct. Then we can use Proposition 3.9 and get as $N\to\infty$

$$\begin{aligned}S_{\lambda(N)}(x_1,\dots,x_k;N,1)&=\frac{\det\left[(x_i-1)^{k-j}S_{\lambda(N)}(x_i;N,1)P_{j,0,N}(x_i)+O(1/N)\right]_{i,j=1}^{k}}{\Delta(x_1,\dots,x_k)}\\&=\frac{\det\left[(x_i-1)^{k-j}S_{\lambda(N)}(x_i;N,1)x_i^{j-1}+O(1/N)\right]_{i,j=1}^{k}}{\Delta(x_1,\dots,x_k)}\\&=\prod_{i=1}^{k}S_{\lambda(N)}(x_i;N,1)\frac{\det\left[(x_i-1)^{k-j}x_i^{j-1}\right]_{i,j=1}^{k}+O(1/N)}{\Delta(x_1,\dots,x_k)}\end{aligned}$$

$$= \prod_{i=1}^{k} S_{\lambda(N)}(x_i; N, 1) \left(1 + \frac{O(1/N)}{\Delta(x_1, \dots, x_k)}\right),$$

where $O(1/N)$ is uniform over compact subsets of M^k. We conclude that

$$\lim_{N\to\infty} S_{\lambda(N)}(x_1, \dots, x_k; N, 1) = \Phi(x_1) \cdots \Phi(x_k) \tag{3.13}$$

uniformly on compact subsets of

$$M^k \setminus \bigcup_{i<j} \{x_i = x_j\}.$$

Since the left-hand side of (3.13) is analytic with only possible singularities at 0 for all N, the uniform convergence in (3.13) also holds when some of x_i coincide. □

Corollary 3.11. *Suppose that the sequence $\lambda(N) \in \mathbb{GT}_N$ is such that*

$$\lim_{N\to\infty} \frac{\ln\left(S_{\lambda(N)}(x; N, 1)\right)}{N} = \Psi(x)$$

uniformly on compact subsets of some region $M \subset \mathbb{C}$, in particular, there is a well defined branch of logarithm in M for large enough N. Then for any k

$$\lim_{N\to\infty} \frac{\ln\left(S_{\lambda(N)}(x_1, \dots, x_k; N, 1)\right)}{N} = \Psi(x_1) + \cdots + \Psi(x_k)$$

uniformly on compact subsets of M^k.

Proof. As in the proof of Corollary 3.10 we can first work with compact subsets of $M^k \setminus \bigcup_{i<j}\{x_i = x_j\}$ and then remove the restriction $x_i \neq x_j$ using the analyticity. Notice that

$$\frac{\left(\frac{\partial}{\partial x}\right)^j S_\lambda(x; N, 1)}{S_\lambda(x; N, 1)} \in \mathbb{Z}\left[\frac{\partial}{\partial x}\ln(S_\lambda(x; N, 1)), \dots, \frac{\partial^j}{\partial x^j}\ln(S_\lambda(x; N, 1))\right],$$

i.e. it is a polynomial in the derivatives of $\ln(S_\lambda(x; N, 1))$ of degree j and so

$$\frac{\left(x\frac{\partial}{\partial x}\right)^j S_\lambda(x; N, 1)}{S_\lambda(x; N, 1)} \in \mathbb{Z}\left[x, \frac{\partial}{\partial x}\ln(S_\lambda(x; N, 1)), \dots, \frac{\partial^j}{\partial x^j}\ln(S_\lambda(x; N, 1))\right].$$

Thus, when

$$\lim_{N\to\infty} \frac{\ln\left(S_{\lambda(N)}(x; N, 1)\right)}{N}$$

exists, then

$$\frac{\left(x\frac{\partial}{\partial x}\right)^j S_{\lambda(N)}(x; N, 1)}{N^j S_{\lambda(N)}(x; N, 1)}$$

converges and so does

$$\frac{\det\left[\sum_{\ell=0}^{j-1} \frac{D^\ell_{i,1}[S_{\lambda(N)}(x_i;N,1)]}{N^\ell} P_{j,\ell,N}(x_i)(x_i - 1)^{\ell+k-j}\right]_{i,j=1}^{k}}{\prod_{i=1}^{k} S_{\lambda(N)}(x_i; N, 1)\Delta(x_1, \dots, x_k)}.$$

Applying equation (3.11) to the last expression, we get that

$$\frac{S_{\lambda(N)}(x_1, \dots, x_k; N, 1)}{\prod_{i=1}^{k} S_{\lambda(N)}(x_i; N, 1)}$$

converges to a bounded function and so does its logarithm

$$\ln S_{\lambda(N)}(x_1,\dots,x_k;N,1)-\sum_{i=1}^{k}\ln S_{\lambda(N)}(x_i;N,1).$$

Dividing the last expression by N and letting $N\to\infty$ we get the statement. □

Corollary 3.12. *Suppose that for some number A*

$$S_{\lambda(N)}\left(e^{\frac{y}{\sqrt{N}}};N,1\right)e^{A\sqrt{N}y}\to G(y)$$

uniformly on compact subsets of domain $\mathbb{D}\subset\mathbb{C}$ as $N\to\infty$. Then

$$\lim_{N\to\infty}S_{\lambda(N)}\left(e^{\frac{y_1}{\sqrt{N}}},\dots,e^{\frac{y_k}{\sqrt{N}}};N,1\right)\exp\left(A\sqrt{N}(y_1+\dots+y_k)\right)=\prod_{i=1}^{k}G(y_i) \tag{3.14}$$

uniformly on compact subsets of $\mathbb{D}^k$.

Proof. Let $S_{\lambda(N)}(e^{y/\sqrt{N}};N,1)e^{A\sqrt{N}y}=G_N(y)$. Since $G_N(y)$ are entire functions, $G(y)$ is analytic on $\mathbb{D}$. Notice that

$$x\frac{\partial}{\partial x}f\left(\sqrt{N}\ln(x)\right)=\sqrt{N}f'\left(\sqrt{N}\ln(x)\right),$$

therefore

$$\begin{aligned}\left(x\frac{\partial}{\partial x}\right)^{\ell}S_{\lambda(N)}(x;N,1)&=N^{\ell/2}\left[\frac{\partial^{\ell}}{\partial y^{\ell}}\left(G_N(y)e^{-A\sqrt{N}y}\right)\right]_{y=\sqrt{N}\ln x}\\&=N^{\ell/2}\left[\sum_{r=0}^{\ell}\binom{l}{r}G_N^{(\ell-r)}(y)(-A)^rN^{r/2}e^{-A\sqrt{N}y}\right]_{y=\sqrt{N}\ln x}\\&=N^{\ell}(-A)^{\ell}\left[e^{-A\sqrt{N}y}G_N(y)\left(1+O(1/\sqrt{N})\right)\right]_{y=\sqrt{N}\ln x},\end{aligned}$$

since the derivatives of $G_N(y)$ are uniformly bounded on compact subsets of $\mathbb{D}$ as $N\to\infty$. Further,

$$(x-1)^{\ell}=N^{-\ell/2}y^{\ell}(1+O(1/\sqrt{N})),\quad x=e^{y/\sqrt{N}},$$

and $P_{j,\ell,N}\left(e^{y/\sqrt{N}}\right)=1+O(1/\sqrt{N})$ with $O(1/\sqrt{N})$ uniformly bounded on compact sets. Thus, setting $x_i=e^{y_i/\sqrt{N}}$ in Proposition 3.9, we get (for distinct y_i)

$$\begin{aligned}&S_{\lambda(N)}\left(e^{y_1/\sqrt{N}},\dots,e^{y_k/\sqrt{N}};N,1\right)e^{A\sqrt{N}(y_1+\dots+y_k)}\\&\quad=\frac{1}{\Delta(x_1,\dots,x_k)}\det\left[(x_i-1)^{k-j}G_N(y_i)\left(1+O(1/\sqrt{N})\right)\right]_{i,j=1}^{k}\\&\quad=G_N(y_1)\cdots G_N(y_k)\frac{\det\left[(x_i-1)^{k-j}\left(1+O(1/\sqrt{N})\right)\right]_{i,j=1}^{k}}{\Delta(x_1,\dots,x_k)}\\&\quad=G_N(y_1)\cdots G_N(y_k)\left(1+O(1/\sqrt{N})\right).\end{aligned}$$

Since the convergence is uniform, it also holds without the assumption that y_i are distinct. □

3.3. **Symplectic characters.** In this section we specialize the formulas of Section 3.1 to the characters χ_λ of the symplectic group.

For $\mu \in \widehat{\mathbb{GT}}_N^+$ let

$$\mathcal{A}^s_\mu(x_1,\dots,x_N) = \frac{\det\left[x_i^{\mu_j+1} - x_i^{-\mu_j-1}\right]_{i,j=1}^N}{\Delta(x_1,\dots,x_N)}.$$

Clearly, for $\lambda \in \mathbb{GT}_N^+$ we have

$$\mathcal{A}^s_{\lambda+\delta} = \chi_\lambda(x_1,\dots,x_N)\frac{\prod_{i<j}(x_ix_j-1)\prod_i(x_i^2-1)}{(x_1\cdots x_N)^N},$$

where χ_λ is a character of the symplectic group $Sp(2N)$.

Proposition 3.13. *Family $\mathcal{A}^s_\mu(x_1,\dots,x_N)$ forms a class of determinantal functions with*

$$\theta_i = q^i, \quad g(x;m) = x^{m+1} - x^{-m-1}, \quad \beta(x) = \frac{q^{x+1}-q^{-x-1}}{q-1},$$

$$\alpha(x) = \frac{q^{x+1}+q^{-x-1}}{(q-1)^2}, \quad [Tf](x) = \frac{f(qx)+f(q^{-1}x)}{(q-1)^2},$$

$$c_N = (q-1)^N \prod_{1\le i<j\le N}\frac{(q-1)^2}{q^j-q^i} = \frac{(q-1)^{N^2}}{(-1)^{\binom{N}{2}}\Delta(q,\dots,q^N)}.$$

Proof. Immediately follows from the definitions and identity

$$\mathcal{A}^s_\mu(q,\dots,q^n) = \frac{\prod_i(q^{\mu_i+1}-q^{-\mu_i-1})\prod_{i<j}(q^{\mu_i+1}+q^{-\mu_i-1}-(q^{\mu_j+1}+q^{-\mu_j-1}))}{(-1)^{\binom{N}{2}}\Delta(q,\dots,q^n)}.$$

□

Let us now specialize Proposition 3.2.

We have that

$$\begin{aligned}\mathfrak{X}_\lambda(x_1,\dots,x_k;N,q) &= \frac{\chi_\lambda(x_1,\dots,x_k,q,\dots,q^{N-k})}{\chi_\lambda(q,\dots,q^N)}\\ &= \frac{\Delta_{\mathrm{s}}(q,\dots,q^N)\Delta(x_1,\dots,x_k,q,\dots,q^{N-k})}{\Delta_{\mathrm{s}}(x_1,\dots,x_k,q\dots,q^{N-k})\Delta(q,\dots,q^N)}\frac{\mathcal{A}^s_\mu(x_1,\dots,x_k,q,\dots,q^{N-k})}{\mathcal{A}^s_\mu(q,\dots,q^N)},\end{aligned}$$

Theorem 3.14. *For any signature $\lambda \in \mathbb{GT}_N^+$ and any $k \le N$ we have*

$$\begin{aligned}\mathfrak{X}_\lambda(x_1,\dots,x_k;N,q) = \frac{\Delta_{\mathrm{s}}(q,\dots,q^N)(q-1)^{k^2-k}(-1)^{\binom{k}{2}}}{\Delta_{\mathrm{s}}(x_1,\dots,x_k,q\dots,q^{N-k})}\times\\ \det[(D^s_{q,i})^{j-1}]_{i,j=1}^k\prod_{i=1}^k\mathfrak{X}_\lambda(x_i;N,q)\frac{\Delta_s(x_i,q,\dots,q^{N-1})}{\Delta_s(q,\dots,q^N)}\end{aligned} \tag{3.15}$$

where $D^s_{q,i}$ is the difference operator

$$f(x) \to \frac{f(qx)+f(q^{-1}x)-2f(x)}{(q-1)^2}$$

acting on variable x_i.

Remark. Note that in Proposition 3.13 the difference operator differed by the shift $2/(q-1)^2$. This is still valid, since in either case the operator is equal to $\prod_{i<j}(D^s_{q,i} - D^s_{q,j})$ and the additional shifts cancel. However, the operator $D^s_{q,i}$ is well defined when $q \to 1$, which is used later.

Using Proposition 3.3 and computing the coefficient in front of the integral by straightforward algebraic manipulations we get the following.

Theorem 3.15. *For any signature $\lambda \in \mathbb{GT}^+_N$ and any $q \neq 1$ we have*

$$\mathfrak{X}_\lambda(x;N,q) = \frac{(-1)^{N-1}\ln(q)(q-1)^{2N-1}[2N]_q!}{(xq;q)_{N-1}(x^{-1}q;q)_{N-1}(x-x^{-1})[N]_q} \times \frac{1}{2\pi\mathbf{i}}\oint \frac{(x^{z+1}-x^{-z-1})}{\prod_{i=1}^N (q^{z+1}+q^{-z-1}-q^{-\lambda_i+N-i-1}-q^{\lambda_i+N-i+1})}dz \tag{3.16}$$

with contour C enclosing the singularities of the integrand at $z = \lambda_1+N-1,\dots,\lambda_N$.

Theorem 3.15 looks very similar to the integral representation for Schur polynomials, this is summarized in the following statement.

Proposition 3.16. *Let $\lambda \in \mathbb{GT}^+_N$, we have*

$$\mathfrak{X}_\lambda(x;N,q) = \frac{(1+q^N)}{x+1} S_\nu(xq^{N-1};2N,q),$$

where $\nu \in \mathbb{GT}_{2N}$ is a signature of size $2N$ given by $\nu_i = \lambda_i + 1$ for $i = 1,\dots,N$ and $\nu_i = -\lambda_{2N-i+1}$ for $i = N+1,\dots,2N$.

Proof. First notice that for any $\mu \in \widehat{\mathbb{GT}}^+_N$, we have

$$\oint_C \frac{(x^z-x^{-z})}{\prod_i (q^z+q^{-z}-q^{-\mu_i-1}-q^{\mu_i+1})}dz = \oint_{C'} \frac{x^z}{\prod_i (q^z+q^{-z}-q^{-\mu_i-1}-q^{\mu_i+1})}dz,$$

where C encloses the singularities of the integrand at $z = \lambda_1+N-1,\dots,\lambda_N$ and C' encloses all the singularities. Indeed, to prove this just write both integrals as the sums or residues. Further,

$$q^z+q^{-z}-q^{-\mu_i-1}-q^{\mu_i+1} = (q^z-q^{\mu_i+1})(q^z-q^{-\mu_i-1})q^{-z}.$$

Therefore, the integrand in (3.16)transforms into

$$\frac{x^z q^{Nz}}{\prod_i (q^z-q^{\lambda_i+N-i+1})(q^z-q^{-(\lambda_i+N-i)-1})} = \frac{\left(xq^{N-1}\right)^{z'} q^{z'}x^{-N}q^{N^2}}{\prod_i (q^{z'}-q^{\lambda_i+1+2N-i})(q^{z'}-q^{-(\lambda_i+1-i)})} \tag{3.17}$$

where $z' = z+N$. The contour integral of (3.17) is readily identified with that of Theorem 3.6 for $S_\nu(xq^{N-1};2N,q)$. It remains only to match the prefactors. □

Next, sending $q \to 1$ we arrive at the following 3 statements.

Define

$$\Delta^1_s(x_1,\dots,x_k,1^{N-k}) = \lim_{q\to 1} \frac{\Delta_{\mathrm{s}}(x_1,\dots,x_k,q,\dots,q^{N-k})}{(q-1)^{\binom{N-k+1}{2}}} = \Delta_{\mathrm{s}}(x_1,\dots,x_k)\prod_i \frac{(x_i-1)^{2(N-k)}}{x_i^{N-k}} \prod_{1\le i<j\le N-k}(i^2-j^2)2^{N-k}(N-k)! \tag{3.18}$$

Theorem 3.17. *For any signature $\lambda \in \mathbb{GT}_N^+$ and any $k \le N$ we have*

$$\mathfrak{X}_\lambda(x_1,\dots,x_k;N,1) = \frac{\Delta_s^1(1^N)}{\Delta_s^1(x_1,\dots,x_k,1^{N-k})}\times \tag{3.19}$$
$$(-1)^{\binom{k}{2}} \det\left[\left(x_i\frac{\partial}{\partial x_i}\right)^{2(j-1)}\right]_{i,j=1}^{k} \prod_{i=1}^{k} \mathfrak{X}_\lambda(x_i;N,1)\frac{(x_i-x_i^{-1})(2-x_i-x_i^{-1})^{N-1}}{2(2N-1)!}.$$

Remark. The statement of Theorem 3.17 was also proved by De Gier and Ponsaing, see [GP].

Theorem 3.18. *For any signature $\lambda \in \mathbb{GT}_N^+$ we have*

$$\mathfrak{X}_\lambda(x;N,1) = \frac{2(2N-1)!}{(x-x^{-1})(x+x^{-1}-2)^{N-1}} \times \frac{1}{2\pi\mathbf{i}}\oint_C \frac{(x^z-x^{-z})}{\prod_{i=1}^N (z-(\lambda_i+N-i+1))(z+\lambda_i+N-i+1)}dz,$$

where the contour includes only the poles at $\lambda_i+N-i+1$ for $i=1,\dots,N$.

Proposition 3.19. *For any signature $\lambda \in \mathbb{GT}_N^+$ we have*

$$\mathfrak{X}_\lambda(x;N,1) = \frac{2}{x+1}S_\nu(x;2N,1), \tag{3.20}$$

where $\nu \in \mathbb{GT}_{2N}$ is a signature of size $2N$ given by $\nu_i = \lambda_i + 1$ for $i = 1,\dots,N$ and $\nu_i = -\lambda_{2N-i+1}$ for $i = N+1,\dots,2N$.

Remark. We believe that the statement of Proposition 3.19 should be known, but we are unable to locate it in the literature.

Analogously to the treatment of the multivariate Schur case we can also derive the same statements as in Proposition 3.9 and Corollaries 3.10, 3.11, 3.12 for the multivariate normalized symplectic characters.

3.4. **Jacobi polynomials.** Here we specialize the formulas of Section 3.1 to the multivariate Jacobi polynomials. We do not present the formula for the q–version of (2.5), although it can be obtained in a similar way.

Recall that for $\lambda \in \mathbb{GT}_N^+$

$$J_\lambda(z_1,\dots,z_k;N,a,b) = \frac{\mathfrak{J}_\lambda(z_1,\dots,z_k,1^{N-k};a,b)}{\mathfrak{J}_\lambda(1^N;a,b)}$$

We produce the formulas in terms of polynomials $\mathfrak{P}_\mu$, $\mu \in \widehat{\mathbb{GT}}_N^+$ and, thus, introduce their normalizations as

$$P_\mu(x_1,\dots,x_k;N,a,b) = \frac{\mathfrak{P}_\mu\left(x_1,\dots,x_k,1^{N-k};a,b\right)}{\mathfrak{P}_\mu(1^N;a,b)}.$$

These normalized polynomials are related to the normalized Jacobi via

$$J_\lambda(z_1,\dots,z_k;N,a,b) = P_\mu\left(\frac{z_1+z_1^{-1}}{2},\dots,\frac{z_k+z_k^{-1}}{2};N,a,b\right),$$

where as usual $\lambda_i+N-i=\mu_i$ for $i=1,\dots,N$.

Proposition 3.20. *The polynomials $P_\mu(x_1,\dots,x_N)$, $\mu\in\widehat{\mathbb{GT}}_N^+$ are a class of determinantal functions with*

$$\theta_i=1,\quad g(x;m)=\mathfrak{p}_m(x;a,b),\quad \alpha(x)=x(x+a+b+1),\quad \beta(x)=\frac{\Gamma(x+a+1)}{\Gamma(x+1)\Gamma(a)},$$

$$c_N=\prod_{r=1}^N\frac{\Gamma(r)\Gamma(a)}{\Gamma(r+a)}\prod_{1\le i<j\le N}\frac{1}{(j-i)(2N-i-j+a+b+1)}$$

$$T=(x^2-1)\frac{\partial^2}{\partial x^2}+((a+b+2)x+a-b))\frac{\partial}{\partial x}$$

Proof. We have (see e.g. [OO2, Section 2C] and references therein)

$$P_\mu(1^n;a,b)=\prod_i\frac{\Gamma(\mu_i+a+1)}{\Gamma(\mu_i+1)}\times\prod_{i<j}(\mu_i-\mu_j)(\mu_i+\mu_j+a+b+1)\times\prod_{r=1}^n\frac{\Gamma(r)}{\Gamma(r+a)}\prod_{0\le i<j<n}\frac{1}{(j-i)(i+j+a+b+1)},\tag{3.21}$$

and also (see e.g. [Ed], [KoSw])

$$m(m+2\sigma)p_m(x;a,b)=\left[(x^2-1)\frac{\partial^2}{\partial x^2}+((a+b+2)x+a-b)\frac{\partial}{\partial x}\right]p_m(x;a,b),$$

Now the statement follows from the definition of polynomials P_μ. □

Specializing Proposition 3.2, using the fact that for $x=\frac{z+z^{-1}}{2}$ we have $\frac{\partial}{\partial x}=\frac{2}{1-z^{-2}}\frac{\partial}{\partial z}$ and $P_\mu(x)=J_\lambda(z)$, we obtain the following.

Theorem 3.21. *For any $\lambda\in\mathbb{GT}_N^+$ and any $k\le N$ we have*

$$J_\lambda(z_1,\dots,z_k;N,a,b)=\prod_{m=N-k+1}^N\frac{\Gamma(m+a)\Gamma(2m-1+a+b)}{\Gamma(m+a+b)}\cdot\frac{1}{\prod_{i=1}^k(z_i+z_i^{-1}-2)^{N-k}}\times\frac{\det[\mathcal{D}_{i,a,b}^{j-1}]_{i,j=1}^k}{2^{\binom{k}{2}}\Delta(z_1+z_1^{-1},\dots,z_k+z_k^{-1})}\prod_{i=1}^k J_\lambda(z_i;N,a,b)\frac{(z_i+z_i^{-1}-2)^{N-1}\Gamma(N+a+b)}{\Gamma(N+a)\Gamma(2N-1+a+b)}\tag{3.22}$$

where $\mathcal{D}_{i,a,b}$ is the differential operator

$$z_i^2\frac{\partial^2}{\partial z_i^2}+\frac{((a+b+2)(z_i+z_i^{-1})+2a-2b-2z_i^{-1})}{1-z_i^{-2}}\frac{\partial}{\partial z_i}.$$

Next, we specialize Proposition 3.3 to the case of multivariate Jacobi polynomials. Note that thanks to the symmetry under $\zeta+(a+b+1)/2\leftrightarrow-(\zeta+(a+b+1)/2)$ of the integrand we can extend the contour C to include all the poles.

Theorem 3.22. *For any $\lambda\in\mathbb{GT}_N^+$ we have*

$$J_\lambda(z;N,a,b)=\frac{\mathfrak{J}_\lambda(x,1^{N-1};a,b)}{\mathfrak{J}_\lambda(1^N;a,b)}=\frac{\Gamma(2N+a+b-1)}{\Gamma(n+a+b)\Gamma(a+1)}\frac{1}{\left(\frac{z+z^{-1}}{2}-1\right)^{N-1}}\tag{3.23}$$

$$\frac{1}{2\pi\mathbf{i}}\oint_C \frac{{}_2F_1\Big(-\zeta,\zeta+a+b+1;a+1;-\dfrac{(1-z)^2}{4z}\Big)(\zeta+(a+b+1)/2)}{\prod_i(\zeta-\mu_i)(\zeta+\mu_i+a+b+1)}d\zeta,$$

where the contour includes the poles of the integrand at $\zeta=-(a+b+1)/2\pm(\mu_i+(a+b+1)/2)$ *and* $\mu_i=\lambda_i+N-i$ *for* $i=1,\dots,N$.

4. General asymptotic analysis

Here we derive the asymptotics for the single-variable normalized Schur functions $S_\lambda(x;N,1)$. In what follows O and o mean uniform estimates, not depending on any parameters and *const* stands for a positive constant which might be different from line to line.

4.1. **Steepest descent.** Suppose that we are given a sequence of signatures $\lambda(N)\in\mathbb{GT}_N$ (or, even, more generally, $\lambda(N_k)\in\mathbb{GT}_{N_k}$ with $N_1<N_2<N_3<\dots$). We are going to study the asymptotic behavior of $S_{\lambda(N)}(x;N,1)$ as $N\to\infty$ under the assumption that there exists a function $f(t)$ for which as $N\to\infty$ the vector $(\lambda_1(N)/N,\dots,\lambda_N(N)/N)$ converges to $(f(1/N),\dots,f(N/N))$ in a certain sense which is explained below.

Let R_1,R_∞ denote the corresponding norms of the difference of the vectors $(\lambda_j(N)/N)$ and $f(j/N))$:

$$R_1(\lambda,f)=\sum_{j=1}^N\left|\frac{\lambda_j(N)}{N}-f(j/N)\right|,\quad R_\infty(\lambda,f)=\sup_{j=1\dots,N}\left|\frac{\lambda_j(N)}{N}-f(j/N)\right|.$$

In order to keep the computations compact we also introduce a modified form $\widehat{f}(t)$ of the function $f(t)$ via:

$$\widehat{f}(t)=f(t)+1-t.$$

As in the previous sections, let $\mu(N)=\lambda(N)+\delta_N$, so $\widehat{f}$ is the limit of $\mu(N)/N$. In order to state our results we introduce w, defined for any $y\in\mathbb{C}$ by the equation

$$\int_0^1\frac{dt}{w-\widehat{f}(t)}=y. \tag{4.1}$$

We remark that a solution to (4.1) can be interpreted as an *inverse Hilbert transform*. We also introduce the function $\mathcal{F}(w;f)$

$$\mathcal{F}(w;f)=\int_0^1\ln(w-\widehat{f}(t))dt,\qquad w\in\mathbb{C}\setminus\left\{\widehat{f}(t)\mid t\in[0,1]\right\}. \tag{4.2}$$

Note that we need to specify which branch of the logarithm we choose in (4.2). This choice is not very important at the moment, but it should be consistent in all the formulas which follow.

Observe that (4.1) can be rewritten as $\mathcal{F}'(w;f)=y$.

Proposition 4.1. *For* $y\in\mathbb{R}\setminus\{0\}$, *suppose that* $f(t)$ *is piecewise-continuous,* $R_\infty(\lambda(N),f)$ *is bounded,* $R_1(\lambda(N),f)/N$ *tends to zero as* $N\to\infty$, *and* $w_0=w_0(y)$ *is the (unique) real root of* (4.1). *Further, let* $y\in\mathbb{R}\setminus\{0\}$ *be such that* w_0 *is outside the interval* $[\frac{\lambda_N(N)}{N},\frac{\lambda_1(N)}{N}+1]$ *for all* N *large enough. Then*

$$\lim_{N\to\infty}\frac{\ln S_{\lambda(N)}(e^y;N,1)}{N}=yw_0-\mathcal{F}(w_0)-1-\ln(e^y-1). \tag{4.3}$$

Remark 1. When y is positive, we can choose the branch of the logarithm which has real values at positive real points both in (4.2) and in $\ln(e^y-1)$ inside (4.3). For negative ys we can choose the branch which has the values with imaginary part π.

Remark 2. Note that piecewise-continuity of $f(t)$ is a reasonable assumption since f is monotonic.

Remark 3. A somehow similar statement was proven by Guionnet and Maïda, see [GM, Theorem 1.2].

When an accurate asymptotics of $\lambda(N)$ is known, Proposition 4.1 can be further refined. For $w\in\mathbb{C}$ denote (as before $\mu_j(N)=\lambda_j(N)+N-j$)

$$\mathcal{Q}(w;\lambda(N),f)=\frac{\exp\bigl(N\mathcal{F}(w;f)\bigr)}{\prod\limits_{j=1}^{N}\Bigl(w-\frac{\mu_j(N)}{N}\Bigr)}. \tag{4.4}$$

Proposition 4.2. *Let $y\in\mathbb{R}\setminus\{0\}$ be such that $w_0=w_0(y)$ (which is the (unique) real root of (4.1)) is outside the interval $[\frac{\lambda_N(N)}{N},\frac{\lambda_1(N)}{N}+1]$ for all large enough N. Suppose that for a function $f(t)$*

$$\lim_{N\to\infty}\mathcal{Q}(w;\lambda(N),f)=g(w) \tag{4.5}$$

uniformly on an open $\mathcal{M}$ set in $\mathbb{C}$, containing w_0. Assume also that $g(w_0)\neq 0$ and $\mathcal{F}''(w_0;f)\neq 0$. Then as $N\to\infty$

$$S_{\lambda(N)}(e^y;N,1)=\frac{g(w_0)}{\sqrt{-\mathcal{F}''(w_0;f)}}\cdot\frac{\exp\bigl(N(yw_0-\mathcal{F}(w_0;f))\bigr)}{e^N\bigl(e^y-1\bigr)^{N-1}}\cdot\Bigl(1+o(1)\Bigr).$$

The remainder $o(1)$ is uniform over y belonging to compact subsets of $\mathbb{R}\setminus\{0\}$ and such that $w_0=w_0(y)\in\mathcal{M}$.

Remark. If the complete asymptotic expansion of $\mathcal{Q}(w;\lambda(N),f)$ as $N\to\infty$ is known, then, with some further work, we can obtain the expansion of $S_{\lambda(N)}(e^y;N,1)$ up to arbitrary precision. In such expansion, $o(1)$ in Proposition 4.2 is replaced by a power series in $N^{-1/2}$ with coefficients being the analytic functions of y. The general procedure is as follows: we use the expansion of $\mathcal{Q}(w;\lambda(N),f)$ (instead of only the first term) everywhere in the below proof and further obtain the asymptotic expansion for each term independently through the steepest descent method. This level of details is enough for our applications and we will not discuss it any further; all the technical details can be found in any of the classical treatments of the steepest descent method, see e.g. [Co], [Er2].

Proposition 4.3. *Suppose that $f(t)$ is piecewise-differentiable, $R_\infty(\lambda(N),f)=O(1)$ (i.e. it is bounded), and $R_1(\lambda(N),f)/\sqrt{N}$ goes to 0 as $N\to\infty$. Then for any fixed $h\in\mathbb{R}$*

$$S_{\lambda(N)}(e^{h/\sqrt{N}};N,1)=\exp\left(\sqrt{N}E(f)h+\frac{1}{2}S(f)h^2+o(1)\right)$$

as $N\to\infty$, where

$$E(f)=\int_0^1 f(t)dt,\quad S(f)=\int_0^1 f(t)^2dt-E(f)^2+\int_0^1 f(t)(1-2t)dt.$$

Moreover, the remainder $o(1)$ is uniform over h belonging to compact subsets of $\mathbb{R}\setminus 0$.

We prove the above three propositions simultaneously.

We start investigating the asymptotic behavior of the integral in the right side of the integral representation of Theorem 3.8

$$S_\lambda(e^y;N,1)=\frac{(N-1)!}{(e^y-1)^{N-1}}\frac{1}{2\pi\mathbf{i}}\oint_C\frac{e^{yz}}{\prod_{j=1}^N(z-\mu_j(N))}dz \tag{4.6}$$

Changing the the variables $z=Nw$ transforms (4.6) into

$$\frac{(N-1)!}{(e^y-1)^{N-1}}N^{1-N}\frac{1}{2\pi\mathbf{i}}\oint_{\mathcal{C}}\frac{\exp(Nyw)}{\prod_{j=1}^N(w-\frac{\mu_j(N)}{N})}dw \tag{4.7}$$

From now on we study the integral

$$\oint_{\mathcal{C}}\frac{\exp(Nyw)}{\prod_{j=1}^N\left(w-\frac{\mu_j(N)}{N}\right)}dw=\oint_{\mathcal{C}}\exp\big(N(yw-\mathcal{F}(w;f))\big)\cdot\mathcal{Q}(w;\lambda(N),f)dw, \tag{4.8}$$

where the contour $\mathcal{C}$ encloses all the poles of the integrand.

Note that $Re\,(\mathcal{F}(w;f))$ is a continuous function in w, while $Im\,(\mathcal{F}(w;f))$ has discontinuities along the real axis (if we choose the principal branch of logarithm with a cut along the negative real axis), both these functions are harmonic outside the real axis.

In fact, the factor $\mathcal{Q}(w;\lambda(N),f)$ in (4.8) has subexponential growth. Indeed, under the assumptions of Proposition 4.2 this is automatically true, while for other cases we use the following two lemmas whose proofs are presented at the end of this section.

Lemma 4.4. *Let A be the smallest interval in $\mathbb{R}$ containing all the points $\{\widehat{f}(t)\mid 0\le t\le 1\}$ and $\{\frac{\mu_j(N)}{N}\mid j=1,\dots,N\}$. Under the assumptions of Proposition 4.1 as $N\to\infty$*

$$\ln|Q(w;\lambda(N),f)|\le o(N)\left(1+\sup_{a\in A}|\ln(w-a)|+\sup_{a\in A}\left|\frac{1}{w-a}\right|\right),$$

where $o(N)$ is uniform in w outside A.

Lemma 4.5. *Let A be the smallest interval in $\mathbb{R}$ containing all the points $\{\widehat{f}(t)\mid 0\le t\le 1\}$ and $\{\frac{\mu_j(N)}{N}\mid j=1,\dots,N\}$. Under the assumptions of Proposition 4.3 as $N\to\infty$*

$$\begin{aligned}\ln|Q(w;\lambda(N),f)|\le o(\sqrt{N})\sup_{a\in A}\left|\frac{1}{w-a}\right|\\ +O(1)\sup_{|t-s|\le 1/N}\left|\ln\left(\frac{w-\widehat{f}(t)}{w-\widehat{f}(s)}\right)\right|+\sup_{0\le t\le 1}\left|\frac{\hat{f}'(t)}{w-\widehat{f}(t)}\right|\end{aligned}$$

where $o(\sqrt{N})$ and $O(1)$ are uniform in w outside A, and the last sup *is taken only over such t in which $\widehat{f}$ is differentiable.*

The asymptotic analysis of the integrals of the kind (4.8) is usually performed using the so-called steepest descent method (see e.g. [Co], [Er2].). We will deform the contour to pass through the critical point of $yw-\mathcal{F}(w;f)$. This point satisfies the equation

$$0=(yw-\mathcal{F}(w;f))'=y-\int_0^1\frac{dt}{w-\widehat{f}(t)}. \tag{4.9}$$

In general, equation (4.9) (which is the same as (4.1)) may have several roots and one has to be careful to choose the needed one.

Lemma 4.6. *Suppose that $y \in \mathbb{R} \setminus \{0\}$. If $y > 0$ then (4.9) has a unique real root $w_0(y) > \widehat{f}(0)$. If $y < 0$ then (4.9) has a unique real root $w_0(y) < \widehat{f}(1)$. Further, $w_0(y) \to \infty$ as $y \to 0$.*

Proof. For $y > 0$ the statement follows from the fact that the integral in (4.9) is a monotonic function of $w > \widehat{f}(0)$ changing from $+\infty$ down to zero (when $w \to +\infty$). Similarly, for $y < 0$ we use the that the integral in (4.9) is a monotonic function of $w < \widehat{f}(1)$ changing from zero (when $w \to -\infty$) down to $-\infty$. □

In what follows, without loss of generality, we assume that $y > 0$ and use $w_0 = w_0(y)$ of Lemma 4.6.

Next, we want to prove that one can deform the contour $\mathcal{C}$ into $\mathcal{C}'$ which passes through w_0 in such a way that $Re(yw - \mathcal{F}(w; f))$ has maximum at w_0. The fact that y is real, simplifies the choice of the contour.

Let $\mathcal{C}'$ be the vertical line passing through w_0. We claim that the contour $\mathcal{C}$ in (4.8) can be deformed into $\mathcal{C}'$ without changing the value of integral. Indeed, observe that the integrand in (4.8) decays like $|w|^{-N}$ as $|w| \to \infty$ in such way that $Re(w)$ stays bounded from above. Therefore, for $N \geq 2$ we can deform the contour as desired.

We will now study the integral over $w \in \mathcal{C}'$. The definitions immediately imply that

$$Re(yw - \mathcal{F}(w; f)) < Re(yw_0 - \mathcal{F}(w_0; f)), \quad w \in \mathcal{C}', \quad w \neq w_0.$$

Now the integrand is exponentially small in N (compared to its value at w_0) everywhere on the contour $\mathcal{C}'$ outside arbitrary neighborhood of w_0. Inside a small ε–neighborhood of w_0 we can do the Taylor expansion for $yw - \mathcal{F}(w; f)$:

$$yw - \mathcal{F}(w; f) = yw_0 - \mathcal{F}(w_0; f) - \frac{(w - w_0)^2}{2} \cdot \mathcal{F}''(w_0; f) + (w - w_0)^3 \cdot \delta,$$

where the absolute value of the remainder δ is bounded by the maximum of $|\mathcal{F}'''(w; f)|$ in the ε–neighborhood.

Note that $\mathcal{F}''(w_0; f) < 0$ and denote $u = -\mathbf{i}\sqrt{-\mathcal{F}''(w_0; f)}$. Setting $w = w_0 + s/(u\sqrt{N})$, and choosing a small $\varepsilon > 0$, whose exact value will be specified later, (4.8) is approximated by

$$\begin{aligned}
(4.10)\quad & \exp\Big(N(yw_0 - \mathcal{F}(w_0; f))\Big) \\
& \times \int_{w_0 - i\varepsilon}^{w_0 + i\varepsilon} \exp\Big(- N\mathcal{F}''(w_0; f)(w - w_0)^2/2 + N\delta(w - w_0)^3 \Big) \mathcal{Q}(w; \lambda(N), f) dw \\
& = \frac{\exp\Big(N(yw_0 - \mathcal{F}(w_0; f))\Big)}{u\sqrt{N}} \\
& \times \int_{-\sqrt{N}\varepsilon|u|}^{+\sqrt{N}\varepsilon|u|} \exp\Big(-s^2/2 + s^3\tilde{\delta}/\sqrt{N}\Big) \mathcal{Q}\Big(w_0 + s/(u\sqrt{N}); \lambda(N), f\Big) ds \\
& \approx \sqrt{2\pi}\frac{1}{u\sqrt{N}} \mathcal{Q}(w_0; \lambda(N), f) \exp\big(N(yw_0 - \mathcal{F}(w_0; f))\big),
\end{aligned}$$

where

$$|\tilde{\delta}| \leq |u|^{-3} \sup_{w \in [w_0 - i\varepsilon, w_0 + i\varepsilon]} \mathcal{F}'''(w; f)$$

When we approximate the integral over vertical line by the integral over the ε-neighborhood (reduction of (4.8) to the first line in (4.10)) the relative error can be bounded as

$$const\times\exp(NRe(\mathcal{F}(w_0+i\varepsilon;f)-\mathcal{F}(w_0;f))\approx const\times\exp(-N\varepsilon^2|\mathcal{F}''(w_0;f)|/2) \tag{4.11}$$

Next, we estimate the relative error in the approximation in (4.10) (i.e. the sign $\approx$ in (4.10)). Suppose that $\varepsilon<|u/\delta|/2$ and divide the integration segment into a smaller subsegment $|s|<N^{-1/10}\sqrt[3]{\sqrt{N}/|\tilde{\delta}|}$ and its complement. When we omit the s^3 term in the exponent we get the relative error at most $const\times N^{-3/10}$ when integrating over the smaller subsegment (which comes from the factor $\exp\left(s^3\tilde{\delta}/\sqrt{N}\right)$ itself) and $const\times\exp\left(-N^{-2/15}|\tilde{\delta}|^{-2/3}/4\right)$ when integrating over its complement (which comes from the estimate of the integral on this complement).

When we replace the integral over $[-\sqrt{N}\varepsilon|u|,+\sqrt{N}\varepsilon|u|]$ by the integral over $(-\infty,+\infty)$ in (4.10) we get the error

$$const\exp(-N\varepsilon^2|u^2|/2),$$

Finally, there is an error of

$$const\sup_{w\in[w_0-i\varepsilon,w_0+i\varepsilon]}|\mathcal{Q}(w;\lambda(N),f)-\mathcal{Q}(w_0;\lambda(N),f)|$$

coming from the factor $\mathcal{Q}\left(w_0+s/(u\sqrt{N});\lambda(N),f\right)$. Summing up, the total relative error in the approximation in (4.10) is at most constant times

$$N^{-3/10}+\exp\left(-N^{-2/15}|\tilde{\delta}|^{-2/3}/4\right)+\exp(-N\varepsilon^2|u^2|/2)+\sup_{w\in[w_0-i\varepsilon,w_0+i\varepsilon]}|\mathcal{Q}(w;\lambda(N),f)-\mathcal{Q}(w_0;\lambda(N),f)|. \tag{4.12}$$

Combining (4.7) and (4.10) we get

$$\frac{s_\lambda(e^y,1^{N-1})}{s_\lambda(1^N)}$$
$$\approx\frac{1}{\sqrt{2\pi}}\frac{(N-1)!}{(e^y-1)^{N-1}}N^{1-N}\frac{1}{\sqrt{-\mathcal{F}''(w_0;f)}\sqrt{N}}\mathcal{Q}(w_0;\lambda(N),f)\exp(N(yw_0-\mathcal{F}(w_0;f))).$$

Using Stirling's formula we arrive at

$$\frac{s_\lambda(e^y,1^{N-1})}{s_\lambda(1^N)}\approx\frac{1}{e^N(e^y-1)^{N-1}}\frac{\mathcal{Q}(w_0;\lambda(N),f)}{\sqrt{-\mathcal{F}''(w_0;f)}}\exp(N(yw_0-\mathcal{F}(w_0;f))). \tag{4.13}$$

With the relative error in (4.13) being the sum of (4.11), (4.12) and $O(1/N)$ coming from Stirling's approximation, and ε satisfying $\varepsilon<|u/\delta|/2$.

Now we are ready to prove the three statements describing the asymptotic behavior of normalized Schur polynomials.

Proof of Proposition 4.1. Use (4.13), Lemma 4.4 and note that after taking logarithms and dividing by N the relative error in (4.13) vanishes. □

Proof of Proposition 4.2. Again this follows from (4.13). It remains to check that the error term in (4.13) is negligible. Indeed, all the derivatives of $\mathcal{F}$, as well as $|u|$, $|\delta|$, $|\tilde{\delta}|$ are bounded in this limit regime. Thus, choosing $\varepsilon = N^{-1/10}$ we conclude that all the error terms vanish. □

Proof of Proposition 4.3. The equation (4.9) for w_0 reads

$$h/\sqrt{N} - \int_0^1 \frac{dt}{w_0 - \widehat{f}(t)} = 0.$$

Clearly, as $N \to \infty$ we have $w_0 \approx \sqrt{N}/h \to \infty$. Thus, we can write

$$\int_0^1 \frac{dt}{w_0 - \widehat{f}(t)} = \frac{1}{w_0} \int_0^1 \left(1 + \frac{\widehat{f}(t)}{w_0} + \left(\frac{\widehat{f}(t)}{w_0}\right)^2 + O\left(\frac{1}{w_0^3}\right)\right) dt.$$

Denote

$$A = \int_0^1 \widehat{f}(t)dt, \quad B = \int_0^1 \left(\widehat{f}(t)\right)^2 dt$$

and rewrite (4.9) as

$$w_0^2 - w_0 \frac{\sqrt{N}}{h} - \frac{A\sqrt{N}}{h} = O(1),$$

If follows that as $N \to \infty$ we have

$$\begin{aligned} &w_0 = \frac{\sqrt{N}}{2h} + \frac{1}{2}\sqrt{\frac{N}{h^2} + 4\frac{A\sqrt{N}}{h}} + O(1/\sqrt{N}) = \frac{\sqrt{N}}{h} + A + O\left(1/\sqrt{N}\right) \\ &\text{and alternatively} \qquad \frac{1}{w_0} = \frac{h}{\sqrt{N}} - \frac{Ah^2}{N} + O\left(N^{-3/2}\right). \end{aligned} \tag{4.14}$$

Next, let us show that the error in (4.13) is negligible. For that, choose ε in (4.10) to be $N^{1/10}$. Note that $|\mathcal{F}''(w_0; f)|$ is of order N^{-1}, and $|\mathcal{F}'''(w; f)|$ (and, thus, also $|\delta|$) is of order $N^{-3/2}$ on the integration contour and $|u|$ is of order $N^{-1/2}$. The inequality $\varepsilon < |u/\delta|/2$ is satisfied. The term coming from (4.11) is bounded by $\exp(-const \times N^{1/5})$ and is negligible. As for (4.12) the first term in it is negligible, the second one is bounded by $\exp(-const \times N^{2/15})$ and negligible, the third one is bounded by $\exp(-const \times N^{1/5})$ which is again negligible. Turning to the fourth term, Lemma 4.5 and asymptotic expansion (4.14) imply that both $\mathcal{Q}(w; \lambda(N), f)$ and $\mathcal{Q}(w_0; \lambda(N), f)$ can be approximated as $1 + o(1)$ as $N \to \infty$ and we are done.

Note that

$$\frac{e^{h/\sqrt{N}} - 1}{\sqrt{-\mathcal{F}''(w_0; f)}} = 1 + o(1)$$

as $N \to \infty$. Now (4.13) yields that

$$\frac{s_\lambda(e^{h/\sqrt{N}}, 1^{N-1})}{s_\lambda(1^N)} = \exp\left(N\left(-1 - \ln(e^{h/\sqrt{N}} - 1) + hw_0/\sqrt{N} - \mathcal{F}(w_0; f)\right)\right)(1 + o(1)) \tag{4.15}$$

As $N \to \infty$ using the Taylor expansion of the logarithm, we have

$$\mathcal{F}(w_0; f) = \int_0^1 \ln(w_0 - \widehat{f}(t))dt = \ln(w_0) + \int_0^1 \left(-\frac{\widehat{f}(t)}{w_0} - \frac{(\widehat{f}(t))^2}{2w_0^2} + O\left(\frac{1}{w_0^3}\right)\right) dt$$

$$= \ln(w_0) - \frac{A}{w_0} - \frac{B}{2w_0^2} + O\left(\frac{1}{w_0^3}\right)$$

and using (4.9) together with (4.14)

$$\frac{hw_0}{\sqrt{N}} = 1 + \frac{A}{w_0} + \frac{B}{w_0^2} + O\left(\frac{1}{w_0^3}\right) = 1 + \frac{Ah}{\sqrt{N}} - \frac{A^2h^2}{N} + \frac{Bh^2}{N} + O(N^{-3/2}).$$

Thus,

$$\begin{aligned}
&-1 - \ln(e^{h/\sqrt{N}} - 1) + hw_0/\sqrt{N} - \mathcal{F}(w_0; f) \\
&\qquad = -\ln(w_0(e^{h/\sqrt{N}} - 1)) + \frac{A}{w_0} + \frac{B}{w_0^2} + \frac{A}{w_0} + \frac{B}{2w_0^2} + O(N^{-3/2}) \\
&\qquad = -\ln\left(\frac{w_0 h}{\sqrt{N}}\left(1 + \frac{h}{2\sqrt{N}} + \frac{h^2}{6N} + O(N^{-3/2})\right)\right) + \frac{2A}{w_0} + \frac{3B}{2w_0^2} + O(N^{-3/2}) \\
&= -\ln\left(1 + \frac{Ah}{\sqrt{N}} + \frac{(B - A^2)h^2}{N}\right) - \ln\left(1 + \frac{h}{2\sqrt{N}} + \frac{h^2}{6N}\right) + \frac{2Ah}{\sqrt{N}} + \frac{(\frac{3}{2}B - 2A^2)h^2}{N} + O(N^{-3/2}) \\
&\qquad = \frac{Ah}{\sqrt{N}} + \frac{B - A^2}{2} \cdot \frac{h^2}{N} - \frac{h}{2\sqrt{N}} - \frac{h^2}{24N} + O(N^{-3/2})
\end{aligned}$$

To finish the proof observe that

$$A = E(f) + 1/2, \quad B = \int_0^1 f^2(t)dt + 2\int_0^1 f(t)(1-t)dt + 1/3,$$

thus, (4.15) transforms into

$$\exp(E(f)h\sqrt{N} + S(f)/2)(1 + o(1)). \quad \square$$

Now we prove Lemmas 4.4 and 4.5.

Proof of Lemma 4.4. We take the logarithm of $\mathcal{Q}(w; \lambda(N), f)$ and aim to prove that the result is small. For that observe the following estimate

$$\begin{aligned}
(4.16) \quad &\left|\sum_{j=1}^N \ln\left(w - \frac{\mu_j(N)}{N}\right) - \sum_{j=1}^N \ln(w - \widehat{f}(j/N))\right| \leq \\
&\qquad \sum_{j=1}^N \left|\int_{\frac{\mu_j(N)}{N}}^{\widehat{f}(j/N)} \frac{dx}{w - x}\right| \leq \frac{1}{N} \cdot \sup_{a \in A}\left|\frac{1}{w - a}\right| \cdot \sum_{j=1}^N |\lambda_j(N) - f(j/N)|.
\end{aligned}$$

Further, using a usual second-order approximation of the integral (trapezoid formula) we can write

$$\begin{aligned}
(4.17) \quad &\sum_{j=1}^N \ln(w - \hat{f}(j/N)) = N\left(\sum_{j=1}^N \frac{\ln(w - \hat{f}(j/N)}{N}\right) \\
&\qquad = N\int_0^1 \ln(w - \hat{f}(t))dt + \frac{\ln(w - \hat{f}(1)) - \ln(w - \hat{f}(0))}{2} \\
&\qquad + T(w, f, N) = N\mathcal{F}(w; f) + \frac{\ln(w - \hat{f}(1)) - \ln(w - \hat{f}(0))}{2} + T(w, f, N).
\end{aligned}$$

Under the conditions of Proposition 4.1 the function $\widehat{f}(t)$ is piecewise–continuous and the remainder $T(w,f,N)$ can be bounded via

$$|T(w,f,N)| \le N\sum_{j=1}^{N}\sup_{\frac{j-1}{N}\le t,s\le j/N}\frac{\left|\ln(w-\widehat{f}(t))-\ln(w-\widehat{f}(s))\right|}{N} \le o(N)(1+\sup_{a\in A}|\ln(w-a)|). \tag{4.18}$$

On the other hand, the right-hand side of (4.16) is bounded from above by $o(N)\sup_{a\in A}\left|\frac{1}{w-a}\right|$. Combining these two bounds we arrive at the desired estimate for $\mathcal{Q}(w;\lambda(N),f)$. □

Proof of Lemma 4.5. We proceed in the same way as in the proof of Lemma 4.4. This time, the right-hand side of (4.16) is bounded from above by $o(\sqrt{N})\sup_{a\in A}\left|\frac{1}{w-a}\right|$. We also have

$$|T(w,f,N)| \le N\sum_{j=1}^{N}\sup_{\frac{j-1}{N}\le t,s\le j/N}\frac{\left|\ln(w-\widehat{f}(t))-\ln(w-\widehat{f}(s))\right|}{N} \le O(1)\sup_{|t-s|\le 1/N}\left|\ln\left(\frac{w-\widehat{f}(t)}{w-\widehat{f}(s)}\right)\right| + \sup_{0\le t\le 1}\left|\frac{\widehat{f}'(t)}{w-\widehat{f}(t)}\right|, \tag{4.19}$$

where the last sup is taken only over those points where $\widehat{f}$ is differentiable and the term with prefactor $O(1)$ arises because of the possible discontinuities of $\widehat{f}$. □

Remark. Note that the restriction of Propositions 4.1 and 4.3 that $f(t)$ should have finitely many points of discontinuity is used only in the proofs of the above two lemmas. It is very plausible that this restriction can be removed if one uses more delicate estimates in these proofs.

4.2. **Values at complex points.** The propositions of the previous section deal with $S_\lambda(e^y;N,1)$ when y is *real*. In this section we show that under mild assumptions the results extend to complex ys.

In the notations of the previous section, suppose that we are given a weakly-decreasing non-negative function $f(t)$, the complex function $\mathcal{F}(w;f)$ is defined through (4.2), y is an arbitrary complex number and w_0 is a critical point of $yw-\mathcal{F}(w;f)$, i.e. a solution of equation (4.9).

We call a simple piecewise-smooth contour $\gamma(s)$ in $\mathbb{C}$ a *steepest descent contour* for the above data if the following conditions are satisfied.

(1) $\gamma(0)=w_0$,
(2) The vector $(\mathcal{F}''(w_0;f))^{-1/2}$ is tangent to γ at point 0,
(3) $Re(y\gamma(s)-\mathcal{F}(\gamma(s);f))$ has a global maximum at $s=0$,
(4) The following integral is finite

$$\int_{-\infty}^{\infty}\exp\Big(Re(y\gamma(s)-\mathcal{F}(\gamma(s);f)\Big)|\gamma'(t)|dt<\infty.$$

Remark. Often the steepest descent contour can be found as a level line $Im(yw-\mathcal{F}(w;f))=Im(yw_0-\mathcal{F}(w_0;f))$.

Example 1. Suppose that $f(t) = 0$. Then

$$\mathcal{F}(w; f) = \int_0^1 \ln(w - 1 + t)dt = w \ln(w) - (w - 1) \ln(w - 1) - 1$$

and

$$\mathcal{F}'(w; 0) = \ln(w) - \ln(w - 1) = -\ln(1 - 1/w). \tag{4.20}$$

And for any y such that $e^y \neq 1$, the critical point is

$$w_0 = w_0(y) = \frac{1}{1 - e^{-y}}.$$

Let us assume that e^{-y} is not a negative real number. This implies that w_0 does not belong to the segment $[0, 1]$.

Figure 7 sketches the level lines $Re(yw - \mathcal{F}(w; 0)) = Re(yw_0 - \mathcal{F}(w_0; 0))$ for one particular value of y. Let es explain the qualitative features of these level lines.

Taylor expanding $yw - \mathcal{F}(w; 0)$ near w_0 we observe that there are 4 level lines going out of w_0. Note that the level lines can not cross. Indeed, any intersection of the level lines is a critical point of $yw - \mathcal{F}(w; 0)$, but the only critical point is at w_0. When $|w| \gg 1$, we have $Re(yw - \mathcal{F}(w; 0) \approx Re(yw) - \ln |w|$, therefore level lines intersect a circle of big radius $R \gg 1$ in 2 points and the level lines picture should have two infinite branches which are close to the rays of the line $Re(yw) = const$ and one loop. We claim that this loop should enclose some points of the segment $[0, 1]$. Indeed, due to the maximum principle a non-constant harmonic function can not have closed level line; on the other hand, the only points where $Re(yw - \mathcal{F}(w; 0)$ is not harmonic lie in the segment $[0, 1]$.

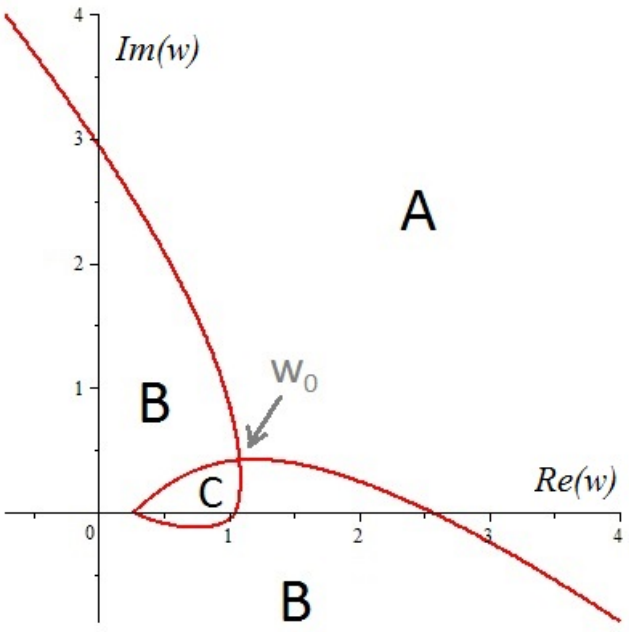

FIGURE 7. Sketch of the level lines $Re(yw - \mathcal{F}(w; 0)) = Re(yw_0 - \mathcal{F}(w_0; 0))$ for $y = 1 - \mathbf{i}$.

Now the plane is divided into three regions A, B and C as shown in Figure 7. $Re(yw - \mathcal{F}(w; 0) > Re(yw_0 - \mathcal{F}(w_0; 0)$ in A, C, and $Re(yw - \mathcal{F}(w; 0) < Re(yw_0 -$

$\mathcal{F}(w_0;0)$ in B. One way to see this fact is by analyzing $Re(yw-\mathcal{F}(w;0)$ for very large $|w|$.

There are two smooth curves $Im(yw-\mathcal{F}(w;0) = Im(yw_0-\mathcal{F}(w_0;0)$ passing through w_0. Taylor expanding $yw-\mathcal{F}(w;0)$ near w_0 we observe that one of them has a tangent vector parallel to $\sqrt{\mathcal{F}''(w_0;0)}$ and another one has a tangent vector parallel to $\mathbf{i}\sqrt{\mathcal{F}''(w_0;0)}$. We conclude that the former one lies inside the region B. In the neighborhood of w_0 this curve is our steepest descent contour. The only property which still might not hold is the property number 4. But in this case, we can modify the contour outside a small neighborhood of w_0, so that $Re(yw-\mathcal{F}(w;0)$ rapidly decays along it. This is always possible because for $|w| \gg 1$, we have $Re(yw-\mathcal{F}(w;0)) \approx Re(yw) - \ln|w|$.

Example 2. More generally let $f(t) = \alpha(1-t)$, then

$$\mathcal{F}(w;\alpha(1-t)) = \int_0^1 \ln(w+(\alpha+1)(t-1))dt$$
$$= \frac{w\ln(w) - (w-(\alpha+1))\ln(w-(\alpha+1))}{\alpha+1} - 1$$

and

$$\mathcal{F}'(w;\alpha(1-t)) = \frac{\ln(1-(\alpha+1)/w)}{\alpha+1}.$$

For any y such that $e^y \neq 1$, the critical point is

$$w_0 = w_0(y) = (\alpha+1)/(1-e^{-y(\alpha+1)}).$$

Note that if we set $w = u(\alpha+1)$, then

$$\mathcal{F}(w;\alpha(1-t)) = u\ln(u) - (u-1)\ln(u-1) + \ln(\alpha+1) - 1,$$

which is a constant plus $\mathcal{F}(u;0)$ from Example 1. Therefore, the linear transformation of the steepest descent contour of Example 1 gives a steepest descent contour for Example 2.

Proposition 4.7. *Suppose that $f(t)$, y and w_0 are such that there exists a steepest descent contour γ and, moreover, the contour of integration in* (4.6) *can be deformed to γ without changing the value of the integral. Then Propositions 4.1 and 4.2 hold for this $f(t)$, y and w_0.*

Proof. The proof of Propositions 4.1 and 4.2 remains almost the same. The only changes are in formula (4.10) and subsequent estimates of errors. Note that condition 4 in the definition of steepest descent contour guarantees that the integral over γ outside arbitrary neighborhood of w_0 is still negligible as $N \to \infty$.

Observe that the integration in (4.10) now goes not over the segment $[w_0-i\varepsilon, w_0+i\varepsilon]$ but over the neighborhood of w_0 on the curve γ_0. This means that in the relative error calculation a new term appears, which is a difference of the integral

$$\int e^{-s^2/2}ds$$

over the interval $[-\sqrt{N}\varepsilon|u|, \sqrt{N}\varepsilon|u|]$ of real line and over the part of rescaled curve $\frac{\gamma(t)-\gamma(0)}{\sqrt{N}u}$ inside circle of radius $\sqrt{N}\varepsilon|u|$ around the origin. The difference of the two integrals equals to the integral of $\exp(-s^2/2)$ over the lines connecting their endpoints. But since $1/u = -(\mathcal{F}''(w_0;f))^{-1/2}$ is tangent to γ at 0, it follows that for small ε the error is the integral of $\exp(-s^2/2)$ over segment joining $\sqrt{N}\varepsilon|u|$ and $\sqrt{N}\varepsilon|u| + Q_1$

plus the integral of $\exp(-s^2/2)$ joining $-\sqrt{N}\varepsilon|u|$ and $-\sqrt{N}\varepsilon|u| + Q_2$ with $|Q_1| < (\sqrt{N}\varepsilon|u|)/100$ and similarly for Q_2. Clearly, these integrals exponentially decay as $N \to \infty$ and we are done. $\square$

It turns out that in the context of Proposition 4.3 the required contour always exists.

Proposition 4.8. *Proposition 4.3 is valid for any $h \in \mathbb{C}$.*

Proof. Recall that in the context of Proposition 4.3 $y = h/\sqrt{N}$ and goes to 0 as $N \to \infty$, while $w_0 \approx 1/y$ goes to infinity. In what follows without loss of generality we assume that h is not an element of $\mathbb{R}_{\leq 0}$ and choose in all arguments the principal branch of logarithms with cut along negative real axis. (In order to work with $h \in \mathbb{R}_{<0}$ we should choose other branches.)

Let us construct the right steepest descent contour passing through the point w_0. Choose positive number r such that $r > |\widehat{f}(t)|$ for all $0 \leq t \leq 1$. Set Ψ to be the minimal strip (which is a region between two parallel lines) in complex plane parallel to the vector $\mathbf{i}/h$ and containing the disk of radius r around the origin .

Since w_0 is a saddle point of $yw - \mathcal{F}(w; f)$, in the neighborhood of w_0 there are two smooth curves $Im(yw - \mathcal{F}(w; f)) = Im(yw_0 - \mathcal{F}(w_0; f))$ intersecting at w_0. Along one of them $Re(yw - \mathcal{F}(w; f))$ has maximum at w_0, along another one it has minimum; we need the former one. Define the contour γ to be the smooth curve $Im(yw - \mathcal{F}(w; f)) = Im(yw_0 \mathcal{F}(w_0; f))$ until it leaves Ψ and the curve (straight line) $Re(yw) = const$ outside Ψ.

Let us prove that $Re(yw - \mathcal{F}(w; f))$ has no local extremum on γ except for w_0, which would imply that w_0 is its global maximum on γ. First note, that outside Ψ we have

$$Re(yw - \mathcal{F}(w; f)) = Re(yw) - \int_0^1 \ln|w - \widehat{f}(t)|dt,$$

with the first term here being a constant, while the second being monotone along the contour. Therefore, outside Ψ we can not have local extremum. Next, straightforward computation shows that if N is large enough, then one can always choose two independent of N constants $1/2 > G_1 > 0$ and $G_2 > 0$ such that $Re(yw - \mathcal{F}(w; f)) > Re(yw_0 - \mathcal{F}(w_0; f))$ for w in Ψ satisfying $|w| = G_1|w_0|$ or $|w| = G_2|w_0|$. It follows, that is $Re(yw - \mathcal{F}(w; f))$ had a local extremum, then such extremum would exist at some point $w_1 \in \Psi$ satisfying $G_1|w_0| < |w_1| < G_2|w_0|$. But since $Im(yw - \mathcal{F}(w; f))$ is constant on the contour inside Ψ, we conclude that w_1 is also a critical point of $yw - \mathcal{F}(w; f)$. However, there are no critical points other than w_0 in this region.

Now we use the contour γ and repeat the argument of Proposition 4.3 using it. Note that the deformation of the original contour of (4.6) into γ does not change the value of the integral. The only part of proof of Proposition 4.3 which we should modify is the estimate for the relative error in (4.13). Here we closely follow the argument of Proposition 4.7. The only change is that the bound on Q_1 and Q_2 is now based on the following observation: The straight line defined by $Re(yw) = Re(yw_0)$ (which is the main part of the contour γ) is parallel to the vector $\mathbf{i}/y$. On the other hand,

$$\sqrt{F''(w_0)} = \mathbf{i}/y(1 + O(1/\sqrt{N}) \approx \mathbf{i}/y. \quad \square$$

Remark. In the proof of Proposition 4.8 we have shown, in particular, that the steepest descent contour exists and, thus, asymptotic theorem is valid for all complex y, which are close enough to 1. This is somehow similar to the results of Guionnet and Maïda, cf. [GM, Theorem 1.4].

5. Statistical mechanics applications

5.1. **GUE in random tilings models.** Consider a tiling of a domain drawn on the regular triangular lattice of the kind shown at Figure 1 with rhombi of 3 types which are usually called *lozenges*. The configuration of the domain is encoded by the number N which is its width and N integers $\mu_1 > \mu_2 > \cdots > \mu_N$ which are the positions of *horizontal lozenges* sticking out of the right boundary. If we write $\mu_i = \lambda_i + N - i$, then λ is a signature of size N, see left panel of Figure 1. Due to combinatorial constraints the tilings of such domain are in correspondence with tilings of a certain polygonal domain, as shown on the right panel of Figure 1.

Let Ω_λ denote the domain encoded by $\lambda \in \mathbb{GT}_N$ and define Υ_λ to be a *uniformly random* lozenge tiling of Ω_λ. We are interested in the asymptotic properties of Υ_λ as $N \to \infty$ and λ changes in a certain regular way.

Given Υ_λ let $\nu_1 > \nu_2 > \cdots > \nu_k$ be positions of the horizontal lozenges at the kth vertical line from the left. (Horizontal lozenges are shown in blue in the left panel of Figure 1.) We again set $\nu_i = \kappa_i + k - i$ and denote the resulting random signature κ of size k by Υ^k_λ.

Recall that the Gaussian Unitary Ensemble is a probability measure on the set of $k \times k$ Hermitian random matrices with density proportional to $\exp(-Trace(X^2)/2)$. Let $\mathbb{GUE}_k$ denote the distribution of k (ordered) eigenvalues of such random matrices.

In this section we prove the following theorem.

Theorem 5.1. *Let $\lambda(N) \in \mathbb{GT}_N$, $N = 1, 2, \dots$ be a sequence of signatures. Suppose that there exists a non-constant piecewise-differentiable weakly decreasing function $f(t)$ such that*

$$\sum_{i=1}^{N} \left| \frac{\lambda_i(N)}{N} - f(i/N) \right| = o(\sqrt{N})$$

as $N \to \infty$ and also $\sup_{i,N} |\lambda_i(N)/N| < \infty$. Then for every k as $N \to \infty$ we have

$$\frac{\Upsilon^k_{\lambda(N)} - NE(f)}{\sqrt{NS(f)}} \to \mathbb{GUE}_k$$

in the sense of weak convergence, where

$$E(f) = \int_0^1 f(t)dt, \quad S(f) = \int_0^1 f(t)^2 dt - E(f)^2 + \int_0^1 f(t)(1-2t)dt.$$

Remark. For any non-constant weakly decreasing $f(t)$ we have $S(f) > 0$.

Corollary 5.2. *Under the same assumptions as in Theorem 5.1 the (rescaled) joint distribution of $k(k+1)/2$ horizontal lozenges on the left k lines weakly converges to the joint distribution of the eigenvalues of the k top-left corners of a $k \times k$ matrix from GUE.*

Proof. Indeed, conditionally on Υ^k_λ the distribution of the remaining $k(k-1)/2$ lozenges is uniform subject to interlacing conditions and the same property holds for the eigenvalues of the corners of GUE random matrix, see [Bar] for more details. □

Let us start the proof of Theorem 5.1.

Proposition 5.3. *The distribution of Υ^k_λ is given by:*

$$\mathrm{Prob}\{\Upsilon^k_\lambda = \eta\} = \frac{s_\eta(1^k) s_{\lambda/\eta}(1^{N-k})}{s_\lambda(1^N)},$$

where $s_{\lambda/\eta}$ is the skew Schur polynomial.

Proof. Let $\kappa \in \mathbb{GT}_M$ and $\mu \in \mathbb{GT}_{M-1}$. We say that κ and μ interlace and write $\mu \prec \kappa$, if

$$\kappa_1 \geq \mu_2 \geq \kappa_2 \geq \cdots \geq \mu_{M-1} \geq \kappa_M.$$

We also agree that $\mathbb{GT}_0$ consists of a single point point, *empty* signature $\emptyset$ and $\emptyset \prec \kappa$ for all $\kappa \in \mathbb{GT}_1$.

For $\kappa \in \mathbb{GT}_K$ and $\mu \in \mathbb{GT}_L$ with $K > L$ let $\mathrm{Dim}(\mu, \kappa)$ denote the number of sequences $\zeta^L \prec \zeta^{L+1} \prec \cdots \prec \zeta^K$ such that $\zeta^i \in \mathbb{GT}_i$, $\zeta^L = \kappa$ and $\zeta^K = \mu$. Note that through the identification of each ζ^i with configuration of horizontal lozenges on a vertical line, each such sequence corresponds to a lozenge tiling of a certain domain encoded by κ and μ, so that, in particular the tiling on the left panel of Figure 1 corresponds to the sequence

$$\emptyset \prec (2) \prec (3,0) \prec (3,1,0) \prec (3,3,0,0) \prec (4,3,3,0,0).$$

It follows that

$$\mathrm{Prob}\{\Upsilon^k_\lambda = \eta\} = \frac{\mathrm{Dim}(\emptyset, \eta)\mathrm{Dim}(\eta, \lambda)}{\mathrm{Dim}(\emptyset, \lambda)}.$$

On the other hand the *combinatorial formula* for (skew) Schur polynomials (see e.g. [M, Chapter I, Section 5]) yields that for $\kappa \in \mathbb{GT}_K$ and $\mu \in \mathbb{GT}_L$ with $K > L$ we have

$$\mathrm{Dim}(\mu, \kappa) = s_{\kappa/\mu}(1^{K-L}), \quad \mathrm{Dim}(\emptyset, \mu) = s_\mu(1^L).$$

□

Introduce multivariate normalized Bessel function $B_k(x; y)$, $x = (x_1, \ldots, x_k)$, $y = (y_1, \ldots, y_k)$ through

$$B_k(x; y) = \frac{\det_{i,j=1,\ldots,k}\Big(\exp(x_i y_j)\Big)}{\prod_{i<j}(x_i - x_j)\prod_{i<j}(y_i - y_j)} \prod_{i<j}(j-i).$$

The functions $B_k(x; y)$ appear naturally as a result of computation of Harish-Chandra-Itzykson-Zuber matrix integral (1.11). Their relation to Schur polynomials is explained in the following statement.

Proposition 5.4. *For $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \in \mathbb{GT}_k$ we have*

$$\frac{s_\lambda(e^{x_1}, \ldots, e^{x_k})}{s_\lambda(1^k)} \prod_{i<j} \frac{e^{x_i} - e^{x_j}}{x_i - x_j} = B_k(x_1, \ldots, x_k; \lambda_1 + k - 1, \lambda_2 + k - 2 \ldots, \lambda_k)$$

Proof. Immediately follows from the definition of Schur polynomials and the evaluation of $s_\lambda(1^k)$ given in (2.3). □

We study Υ^k_λ for $\lambda \in \mathbb{GT}_N$ through its moment generating functions $\mathbb{E}B_k(x; \Upsilon^k_\lambda + \delta_k)$, where $x = (x_1, \ldots, x_k)$, $\delta_k = (k-1, k-2, \ldots, 0)$ as above and $\mathbb{E}$ stays for the expectation. Note that for $k = 1$ the function $\mathbb{E}B_k(x; \Upsilon^k_\lambda + \delta_k)$ is nothing else but usual one-dimensional moment generating function $\mathbb{E}\exp(x\Upsilon^1_\lambda)$.

Proposition 5.5. *We have*

$$\mathbb{E}B_k(x; \Upsilon^k_\lambda + \delta_k) = \frac{s_\lambda(e^{x_1}, \ldots, e^{x_k}, 1^{N-k})}{s_\lambda(1^N)} \prod_{1 \leq i < j \leq k} \frac{e^{x_i} - e^{x_j}}{x_i - x_j}.$$

Proof. Let $Z = (z_1, \dots, z_m)$ and $Y = (y_1, \dots, y_n)$ and let $\mu \in \mathbb{GT}_{m+n}$, then (see e.g. [M, Chapter I, Section 5])

$$\sum_{\kappa \in \mathbb{GT}_M} s_\kappa(Z) s_{\mu/\kappa}(Y) = s_\mu(Z, Y).$$

Therefore, Propositions 5.3 and 5.4 yield

$$\begin{aligned}\left(\mathbb{E}B_k(x; \Upsilon^k_\lambda + \delta_k)\right) \prod_{i<j} \frac{x_i - x_j}{e^{x_i} - e^{x_j}} &= \sum_{\eta \in \mathbb{GT}_k} \frac{s_\eta(e^{x_1}, \dots, e^{x_k})}{s_\eta(1^k)} \cdot \frac{s_\eta(1^k) s_{\lambda/\eta}(1^{N-k})}{s_\lambda(1^N)} \\ &= \frac{\sum_{\eta \in \mathbb{GT}_k} s_\eta(e^{x_1}, \dots, e^{x_k}) s_{\lambda/\eta}(1^{N-k})}{s_\lambda(1^N)} = \frac{s_\lambda(e^{x_1}, \dots, e^{x_k}, 1^{N-k})}{s_\lambda(1^N)}.\end{aligned}$$

□

The counterpart of Proposition 5.5 for $\mathbb{GUE}_k$ distribution is the following.

Proposition 5.6. *We have*

$$\mathbb{E}B_k(x; \mathbb{GUE}_k) = \exp\left(\frac{1}{2}(x_1^2 + \dots + x_k^2)\right). \tag{5.1}$$

Proof. Let X be a (fixed) diagonal $k \times k$ matrix with eigenvalues $x_1, \dots, x_k$ and let A be random $k \times k$ Hermitian matrix from $\mathbb{GUE}$. Let us compute

$$\mathbb{E} \exp\left(Trace(XA)\right). \tag{5.2}$$

From one hand, standard integral evaluation shows that (5.2) is equal to the right side of (5.1). On the other hand, we can rewrite (5.2) as

$$\int_{y_1 \geq y_2 \geq \dots \geq y_k} P_{\mathbb{GUE}_k}(dy) \int_{u \in U(k)} P_{Haar}(du) \exp\left(Trace(YuXu^{-1})\right), \tag{5.3}$$

where $P_{\mathbb{GUE}_k}$ is probability distribution of $\mathbb{GUE}_k$, P_{Haar} is normalized Haar measure on the unitary group $U(k)$ and Y is Hermitian matrix (e.g. diagonal) with eigenvalues $y_1, \dots, y_k$. The evaluation of the integral over unitary group in (5.3) is well-known, see [H1], [H2], [IZ], [OV] and the answer is precisely $B_k(y_1, \dots, y_k; x_1, \dots, x_k)$. Thus, (5.3) transforms into the left side of (5.1). □

In what follows we need the following technical proposition.

Proposition 5.7. *Let $\phi^N = (\phi^N_1 \geq \phi^N_2 \geq \dots \geq \phi^N_k)$, $N = 1, 2, \dots$ be a sequence of k-dimensional random variables. Suppose that there exists a random variable ϕ^∞ such that for every $x = (x_1, \dots, x_k)$ in a neighborhood of $(0, \dots, 0)$ we have*

$$\lim_{N \to \infty} \mathbb{E}B_k(x; \phi^N) = \mathbb{E}B_k(x; \phi^\infty).$$

Then $\phi^N \to \phi$ in the sense of weak convergence of random variables.

Proof. For $k = 1$ this is a classical statement, see e.g. [Bi, Section 30]. For general k this statement is, perhaps, less known, but it can be proven by the same standard techniques as for $k = 1$. □

Next, note that the definition implies the following property for the moment generating function of k-dimensional random variable ϕ:

$$\mathbb{E}B_k(x_1, \dots, x_k; a\phi + b) = \exp(b(x_1 + \dots + x_k)) \mathbb{E}B_k(ax_1, \dots, ax_k; \phi).$$

Also observe that for any non-constant weakly decreasing $f(t)$ we have $S(f) > 0$. The following statement, together with Propositions 5.5, gives the moment generating function for the shifted and normalized $\Upsilon^k_{\lambda(N)}$ as $N \to \infty$.

Proposition 5.8. *In the assumptions of Theorem 5.1 for any k reals $h_1, \dots, h_k$ we have:*

$$\lim_{N\to\infty} \frac{s_{\lambda(N)}\left(e^{\frac{h_1}{\sqrt{NS(f)}}}, \dots, e^{\frac{h_k}{\sqrt{NS(f)}}}, 1^{N-k}\right)}{s_{\lambda(N)}(1^N)} \exp\left(-\sqrt{N}\frac{E(f)}{\sqrt{S(f)}}(h_1+\dots+h_k)\right) = \exp\left(\frac{1}{2}(h_1^2+\dots+h_k^2)\right).$$

Proof. For $k = 1$ this is precisely the statement of Proposition 4.3. For general k we combine Proposition 4.8 and Corollary 3.12. □

Proof of Theorem 5.1. Propositions 5.8 and 5.5, and the observation that $(e^{x_i} - e^{x_j})/(x_i - x_j)$ tends to 1 when $x_i, x_j \to 0$ show that as $N \to \infty$ the moment generating function for the shifted and normalized $\Upsilon^k_{\lambda(N)}$ converges to the corresponding moment generating function for the $\mathbb{GUE}_k$ as given in Proposition 5.6. Now Proposition 5.7 implies the weak convergence

$$(\Upsilon^k_{\lambda(N)} - NE(f))/\sqrt{NS(f)} \to \mathbb{GUE}_k$$

and Theorem 5.1 then follows. □

5.2. **Asymptotics of the six vertex model.** Recall that an *Alternating Sign Matrix* of size N is a $N \times N$ matrix filled with **0**s **1**s and $-$**1**s in such a way that the sum along every row and column is **1** and, moreover, along each row and each column **1**s and $-$**1**s are alternating, possibly separated by an arbitrary number of **0**s. Alternating Sign Matrices are in bijection with configurations of the six-vertex ("square ice") model with domain wall boundary conditions. The configurations of the 6–vertex model are assignments of one of 6 types of H_2O molecules shown in Figure 8 to the vertices of $N \times N$ square grid in such a way that the O atoms are at the vertices of the grid. To each O atom there are two H atoms attached, so that they are at angles 90° or 180° to each other, along the grid lines, and between any two adjacent O atoms there is exactly one H. We also impose the so-called domain wall boundary conditions as shown in Figure 4 in the introduction. In order to get an ASM we replace the vertex of each type with **0**, **1** or $-$**1**, as shown in Figure 8, see e.g. [Ku] and references therein for more details. Figure 4 gives one example of ASM and corresponding configuration of the 6–vertex model.

Let $\beth_N$ denote the set of all Alternating Sign Matrices of size N or, equivalently, all configurations of six-vertex model with domain wall boundary condition. Equip $\beth_N$ with *uniform* probability measure and let ω_N be a random element of $\beth_N$. We are going study the asymptotic properties of ω_N as $N \to \infty$.

For $\vartheta \in \beth_N$ let $a_i(\vartheta)$, $b_i(\vartheta)$, $c_i(\vartheta)$ denote the number of vertices in horizontal line i of types a, b and c, respectively (the types are shown in Figure 8). Likewise, let $\widehat{a}_j(\vartheta)$, $\widehat{b}_j(\vartheta)$ and $\widehat{c}_j(\vartheta)$ be the same quantities in vertical line j. Also let $a^{ij}(\vartheta)$, $b^{ij}(\vartheta)$ and $c^{ij}(\vartheta)$ be $0-1$ functions equal to the number of vertices of types a, b and c, respectively, at the intersection of vertical line j and horizontal line i. To simplify the notations we view a_i, b_i and c_i as random variables and omit their dependence on ϑ.

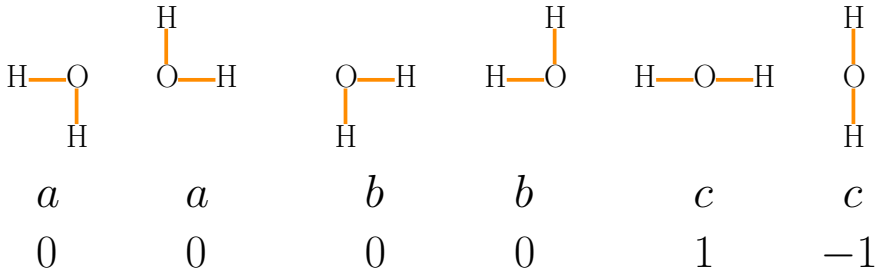

FIGURE 8. Types of vertices in the six vertex model divided by groups and their correspondence to numbers in ASM

Theorem 5.9. *For any fixed j the random variable $\frac{a_j - N/2}{\sqrt{N}}$ weakly converges to the normal random variable $N(0, \sqrt{3/8})$. The same is true for a_{N-j}, $\widehat{a}_j$ and $\widehat{a}_{N-j}$. Moreover, the joint distribution of any collection of such variables converges to the distribution of independent normal random variables $N(0, \sqrt{3/8})$.*

Inspecting the bijection between ASMs and the configurations of the six–vertex model one readily sees that Theorem 5.9 implies Theorem 1.10. The rest of this section is devoted to the proof of Theorem 5.9.

The 6 types of vertices in six vertex model are divided into 3 groups, as shown in Figure 8. Define a weight depending on the position (i, j) (i is the vertical coordinate) of the vertex and its type as follows:

$$a:\ q^{-1}u_i^2 - qv_j^2, \quad b:\ q^{-1}v_j^2 - qu_i^2, \quad c:\ (q^{-1}-q)u_iv_j$$

where $v_1, \dots, v_N$, $u_1, \dots, u_N$ are parameters and from now and till the end of the section we set $q = \exp(\pi \mathbf{i}/3)$ (notice that this implies $q^{-1} + q = 1; q - q^{-1} = \mathbf{i}\sqrt{3}$.)

Let the weight W of a configuration be equal to the product of weights of vertices. The partition function of the model can be explicitly evaluated in terms of Schur polynomials.

Proposition 5.10. *We have*

$$\sum_{\vartheta \in \beth_N} W(\vartheta) = (-1)^{N(N-1)/2}(q^{-1}-q)^N \prod_{i=1}^{N} (v_iu_i)^{-1} s_{\lambda(N)}(u_1^2, \dots, u_N^2, v_1^2, \dots, v_N^2),$$

where $\lambda(N) = (N-1, N-1, N-2, N-2, \dots, 1, 1, 0, 0) \in \mathbb{GT}_{2N}$.

Proof. See [Ok], [St], [FZ]. □

The following proposition is a straightforward corollary of Proposition 5.10.

Proposition 5.11. *Fix any n distinct vertical lines $i_1, \dots, i_n$ and m distinct horizontal lines $j_1, \dots, j_m$ and any set of complex numbers $u_1, \dots, u_n$, $v_1, \dots, v_m$. We have*

$$\mathbb{E}_N \prod_{k=1}^{n} \left[\left(\frac{q^{-1}u_k^2 - q}{q^{-1}-q} \right)^{a_{i_k}} \left(\frac{q^{-1} - qu_k^2}{q^{-1}-q} \right)^{b_{i_k}} (u_k)^{c_{i_k}} \right] = \left(\prod_{k=1}^{n} u_k^{-1} \right) \frac{s_{\lambda(N)}(u_1, \dots, u_n, 1^{2N-n})}{s_{\lambda(N)}(1^{2N})}, \tag{5.4}$$

$$\mathbb{E}_N \prod_{\ell=1}^{m} \left[\left(\frac{q^{-1}-qv_\ell^2}{q^{-1}-q} \right)^{\widehat{a}_{j_\ell}} \left(\frac{q^{-1}v_\ell^2-q}{q^{-1}-q} \right)^{\widehat{b}_{j_\ell}} (v_\ell)^{\widehat{c}_{j_\ell}} \right] = \left(\prod_{\ell=1}^{n} v_\ell^{-1} \right) \frac{s_{\lambda(N)}(v_1,\dots,v_m,1^{2N-m})}{s_{\lambda(N)}(1^{2N})} \tag{5.5}$$

and, more generally

$$\begin{aligned} &\mathbb{E}_N \left(\prod_{k=1}^{n} \left[\left(\frac{q^{-1}u_k^2-q}{q^{-1}-q} \right)^{a_{i_k}} \left(\frac{q^{-1}-qu_k^2}{q^{-1}-q} \right)^{b_{i_k}} (u_k)^{c_{i_k}} \right] \right. \\ &\qquad \times \prod_{\ell=1}^{m} \left[\left(\frac{q^{-1}-qv_\ell^2}{q^{-1}-q} \right)^{\widehat{a}_{j_\ell}} \left(\frac{q^{-1}v_\ell^2-q}{q^{-1}-q} \right)^{\widehat{b}_{j_\ell}} (v_\ell)^{\widehat{c}_{j_\ell}} \right] \\ &\left. \times \prod_{k=1}^{n} \prod_{\ell=1}^{m} \left[\left(\frac{(q^{-1}u_k^2-qv_\ell^2)(q^{-1}-q)}{(q^{-1}u_k^2-q)(q^{-1}-qv_\ell^2)} \right)^{a^{i_k,j_\ell}} \left(\frac{(q^{-1}v_\ell^2-qu_k^2)(q^{-1}-q)}{(q^{-1}-qu_k^2)(q^{-1}v_\ell^2-q)} \right)^{b^{i_k,j_\ell}} \right] \right) \\ &\qquad = \left(\prod_{\ell=1}^{m} v_\ell^{-1} \prod_{k=1}^{n} u_k^{-1} \right) \frac{s_{\lambda(N)}(u_1,\dots,u_n,v_1,\dots,v_m,1^{2N-n-m})}{s_{\lambda(N)}(1^{2N})}, \end{aligned} \tag{5.6}$$

where all the above expectations $\mathbb{E}_N$ are taken with respect to the uniform measure on $\rfloor_N$.

We want to study $N \to \infty$ limits of observables of Proposition 5.11. Suppose that $n = 1$, $m = 0$. Then we have two parameters $u_1 = u$ and $i_1 = i$. Suppose that as $N \to \infty$ we have

$$u = u(N) = \exp(y/\sqrt{N}) \tag{5.7}$$

and i remains fixed. Then we can use Proposition 4.3 to understand the asymptotics of the right-hand side of (5.4).

As for the left-hand side of (5.4), note that c_i is uniformly bounded, in fact $c_i < 2i$ because of the combinatorics of the model. Therefore, the factors involving c_i in the observable become negligible as $N \to \infty$. Also note that $a_i + b_i + c_i = N$, therefore the observable can be rewritten as

$$\left(\frac{q^{-1}-qe^{2y/\sqrt{N}}}{q^{-1}-q} \right)^{N} \left(\frac{q^{-1}e^{2y/\sqrt{N}}-q}{q^{-1}-qe^{2y/\sqrt{N}}} \right)^{a_i} G(y),$$

with G satisfying the estimate $|\ln G(y)| < Cy/\sqrt{N}$ with some constant C (independent of all other parameters).

Now let z be an auxiliary variable and choose $y = y(z, N)$ such that

$$\exp(z/\sqrt{N}) = \frac{q^{-1}e^{2y/\sqrt{N}}-q}{q^{-1}-qe^{2y/\sqrt{N}}} \tag{5.8}$$

Now the observable (as a function of z) turns into $\left(\frac{q^{-1}-qe^{2y/\sqrt{N}}}{q^{-1}-q} \right)^{N}$ times $\exp(za_i/\sqrt{N})$. Therefore, the expectation in (5.4) is identified with the exponential moment generating function for $a_i/\sqrt{N}$.

In order to obtain the asymptotics we should better understand the function $y(z,N)$. Rewrite (5.8) as

$$e^{2y/\sqrt{N}} = \frac{\exp(z/\sqrt{N})q^{-1}+q}{q^{-1}+q\exp(z/\sqrt{N})} = \frac{1+(\exp(z/\sqrt{N})-1)\frac{q^{-1}}{q^{-1}+q}}{1+(\exp(z/\sqrt{N})-1)\frac{q}{q^{-1}+q}}$$

Recall that $q^{-1}+q=1$, therefore

$$\begin{aligned} 2y &= \sqrt{N}\left(\ln(1+q^{-1}(\exp(z/\sqrt{N})-1)) - \ln(1+q(\exp(z/\sqrt{N})-1))\right) \\ &= -(q-q^{-1})z - \frac{q-q^{-1}}{2}z^2/\sqrt{N} + \frac{q^2-q^{-2}}{2}z^2/\sqrt{N} + O(z^3/N) \end{aligned}$$

Note that the last two terms cancel out and we get

$$y = -z\mathbf{i}\frac{\sqrt{3}}{2} + O(z^3/N). \tag{5.9}$$

Now we compute

$$\begin{aligned} \left(\frac{q^{-1}-qe^{2y/\sqrt{N}}}{q^{-1}-q}\right)^N &= \exp\left[N\ln\left(1-\frac{q}{q^{-1}-q}(e^{-\mathbf{i}\sqrt{3}z/\sqrt{N}+O(z^3N^{-3/2})}-1)\right)\right] \\ &= \exp\left[-\sqrt{N}qz + q\mathbf{i}\sqrt{3}z^2/2 - q^2z^2/2 + o(1)\right] = \exp\left[-\sqrt{N}qz - z^2/2 + o(1)\right] \end{aligned}$$

Summing up, the observable of (5.4) is now rewritten as

$$\exp\left[-\sqrt{N}z\mathbf{i}\frac{\sqrt{3}}{2} - z^2/2 + o(1)\right]\exp\left[\frac{a_i - N/2}{\sqrt{N}}z\right] \tag{5.10}$$

Now combining (5.4) with Propositions 4.3, 4.8 (note that parameter N in these two propositions differs by the factor 2 from that of (5.4)) we conclude that (for any complex z) the expectation of (5.10) is asymptotically

$$\exp\left[4\sqrt{N}yE(f) + 4S(f)y^2 + o(1)\right],$$

where f is the function $\frac{1-x}{2}$. Using (5.9) and computing

$$E(f) = 1/4, \quad S(f) = 5/48$$

we get

$$\exp\left[-\sqrt{N}z\mathbf{i}\frac{\sqrt{3}}{2} - \frac{5}{16}z^2 + o(1)\right]. \tag{5.11}$$

Now we are ready to prove Theorem 5.9.

Proof of Theorem 5.9. Choose z_k and z'_ℓ to be related to u_{i_k} and v_{j_ℓ}, respectively, in the same way as z was related to u (through (5.7) and (5.8)). Then, combining the asymptotics (5.11) with Corollary 3.12 we conclude that the righthand side of (5.6) as $N\to\infty$ is

(5.12)

$$\prod_{k=1}^{n}\exp\left[-\sqrt{N}z_k\mathbf{i}\frac{\sqrt{3}}{2} - \frac{5}{16}z_k^2 + o(1)\right]\prod_{\ell=1}^{m}\exp\left[-\sqrt{N}z'_k\mathbf{i}\frac{\sqrt{3}}{2} - \frac{5}{16}(z'_k)^2 + o(1)\right].$$

Now it is convenient to choose z_i (z'_i) to be purely imaginary $z_i = s_i\mathbf{i}$ ($z'_i = s'_i\mathbf{i}$).

Summing up the above discussion, observing that the case $n=0$, $m=1$ is almost the same as $n=1$, $m=0$, (only the sign of a_i changes), and that the observable (5.6) has a multiplicative structure, and the third (double) product in (5.6) is negligible as $N\to\infty$, we conclude that as $N\to\infty$ for all real s_i, s_i'

$$\lim_{N\to\infty}\mathbb{E}_N\exp\left[\sum_{k=1}^{n}\frac{a_{i_k}-N/2}{\sqrt{N}}s_k\mathbf{i}+\sum_{\ell=1}^{m}\frac{\widehat{a}_{j_\ell}-N/2}{\sqrt{N}}s'_\ell\mathbf{i}+o(1)\right] = \exp\left[-\frac{3}{16}\left(\sum_{k=1}^{n}s_k^2+\sum_{\ell=1}^{n}(s'_\ell)^2\right)\right]. \tag{5.13}$$

The remainder $o(1)$ in the left side of (5.13) is uniform in a_{i_k}, $\widehat{a}_{i_\ell}$ and, therefore, it can be omitted. Indeed, this follows from

$$\left|\mathbb{E}_N\exp\left[\frac{a_i-N/2}{\sqrt{N}}s\mathbf{i}+o(1)\right]-\mathbb{E}_N\exp\left[\frac{a_i-N/2}{\sqrt{N}}s\mathbf{i}\right]\right| \le \mathbb{E}_N\left|\exp\left[\frac{a_i-N/2}{\sqrt{N}}s\mathbf{i}\right]\right|o(1)=o(1).$$

Hence, (5.13) yields that the characteristic function of the random vector

$$\left(\frac{a_{i_1}-N/2}{\sqrt{N}},\dots,\frac{a_{i_n}-N/2}{\sqrt{N}},\frac{\widehat{a}_{j_1}-N/2}{\sqrt{N}},\dots,\frac{\widehat{a}_{j_m}-N/2}{\sqrt{N}}\right)$$

converges as $N\to\infty$ to

$$\exp\left[-\frac{3}{16}\left(\sum_{k=1}^{n}s_k^2+\sum_{\ell=1}^{n}(s'_\ell)^2\right)\right]$$

Since convergence of characteristic functions implies weak convergence of distributions (see e.g. [Bi, Section 26]) the proof of Theorem 5.9 is finished. □

5.3. **Towards dense loop model.** In [GNP] de Gier, Nienhuis and Ponsaing study the completely packed $O(n=1)$ dense loop model and introduce the following quantities related to the symplectic characters.

Following the notation from [GNP] we set

$$\tau_L(z_1,\dots,z_L)=\chi_{\lambda^L}(z_1^2,\dots,z_L^2)$$

where $\lambda^L\in\mathbb{GT}_L^+$ is given by $\lambda_i^L=\lfloor\frac{L-i}{2}\rfloor$ for $i=1,\dots,L$. Further, set

$$u_L(\zeta_1,\zeta_2;z_1,\dots,z_L)=(-1)^L\mathbf{i}\frac{\sqrt{3}}{2}\ln\left[\frac{\tau_{L+1}(\zeta_1,z_1,\dots,z_L)\tau_{L+1}(\zeta_2,z_1,\dots,z_L)}{\tau_L(z_1,\dots,z_L)\tau_{L+2}(\zeta_1,\zeta_2,z_1,\dots,z_L)}\right] \tag{5.14}$$

Define

$$X_L^{(j)}=z_j\frac{\partial}{\partial z_j}u_L(\zeta_1,\zeta_2;z_1,\dots,z_L)$$

and

$$Y_L=w\frac{\partial}{\partial w}u_{L+2}(\zeta_1,\zeta_2;z_1,\dots,z_L,vq^{-1},w)|_{w=v},$$

in particular, $X_L^{(j)}$ is a function of $z_1,\dots,z_L$ and ζ_1,ζ_2, while Y_L also depends on additional parameters v and q.

De Gier, Nienhuis and Ponsaing showed that $X_L^{(j)}$ and Y_L are related to the mean total current in the $O(n=1)$ dense loop model, which was presented in Section 1.6.

More precisely, they prove that under certain factorization assumption and with an appropriate choice of weights of configurations of the model, $X_L^{(j)}$ is the mean total current between two horizontally adjacent points in the strip of width L:

$$X_L^{(j)} = F^{(i,j),(i,j+1)},$$

and Y is the mean total current between two vertically adjacent points in the strip of width L:

$$Y_L = F^{(j,i),(j-1,i)},$$

see [GNP] for the details.

This connection motivated the question of the limit behavior of $X_L^{(j)}$ and $Y_L^{(j)}$ as the width L tends to infinity, this was asked in [Gi], [GP]. In the present paper we compute the asymptotic behavior of these two quantities in the homogeneous case, i.e. when $z_i = 1$, $i = 1, \dots, L$.

Theorem 5.12. *As $L \to \infty$ we have*

$$X_L^{(j)}\Big|_{z_j=z;\, z_i=1,\ i\neq j} = \frac{\mathbf{i}\sqrt{3}}{4L}(z^3 - z^{-3}) + o\left(\frac{1}{L}\right)$$

and

$$Y_L\Big|_{z_i=1,\ i=1,\dots,L} = \frac{\mathbf{i}\sqrt{3}}{4L}(w^3 - w^{-3}) + o\left(\frac{1}{L}\right)$$

Remark 1. When $z = 1$, $X_L^{(j)}$ is identical zero and so is our asymptotics.

Remark 2. The fully homogeneous case corresponds to $w = e^{-\pi\mathbf{i}/6}$, $q = e^{2\pi\mathbf{i}/3}$. In this case

$$Y_L = \frac{\sqrt{3}}{2L} + o\left(\frac{1}{L}\right).$$

Remark 3. The leading asymptotics terms do not depend on the boundary parameters ζ_1 and ζ_2.

The rest of this section is devoted to the proof of Theorem 5.12.

Proposition 5.13. *The normalized symplectic character for $\lambda^L = (\lfloor\frac{L-1}{2}\rfloor, \lfloor\frac{L-2}{2}\rfloor, \dots, 1, 0, 0)$ is asymptotically given for even L by*

$$\mathfrak{X}_{\lambda^L}(e^y; L) = \frac{3e^{-\frac{9}{4}y}(e^y - 1)}{(e^{3/2y} - 1)(e^y + 1)} \left(\frac{4}{9}\frac{(e^{3/2y} - 1)^2}{e^{y/2}(e^y - 1)^2}\right)^L \left(1 + \frac{t_1(y)}{L^{1/2}} + \frac{t_2(y)}{L^{2/2}} + \dots\right),$$

and for odd L by

$$\mathfrak{X}_{\lambda^L}(e^y; L) = \frac{3e^{-\frac{9}{4}y}(e^y - 1)}{(e^{3/2y} - 1)(e^y + 1)} \left(\frac{4}{9}\frac{(e^{3/2y} - 1)^2}{e^{y/2}(e^y - 1)^2}\right)^L \left(1 + \frac{t'_1(y)}{L^{1/2}} + \frac{t'_2(y)}{L^{2/2}} + \dots\right),$$

for some analytic functions t_1, t_2, ... and $t'_1, t'_2, \dots$ such that $t_1 = t'_1$ and

$$t'_2 = t_2 + \frac{1}{12}(e^{3/2y} - 1)^2 e^{-3/2y}$$

Proof. We will apply the formula from Proposition 3.19 to express the normalized symplectic character as a normalized Schur function. The corresponding ν is given by $\nu_i = \lfloor\frac{L-i}{2}\rfloor + 1$ for $i = 1, \dots, L$ and $\nu_i = -\lfloor\frac{i-L-1}{2}\rfloor$ for $i = L+1, \dots, 2L$, which is equivalent to $\nu_i = \lfloor\frac{L-i}{2}\rfloor + 1$ for all $i = 1, \dots, 2L$. We will apply Proposition 4.2 to directly derive the asymptotics for $S_\nu(e^y; 2L, 1)$. For the specific signature we find that $f(t) = \frac{1}{4} - \frac{1}{2}t$ and

$$\mathcal{F}(w;f)=\int_0^1 \ln(w-f(t)-1+t)dt$$
$$=\frac{1}{6}\left(-6+(5-4w)\ln\left[-\frac{5}{4}+w\right]+(1+4w)\ln\left[\frac{1}{4}+w\right]\right)$$

In particular, we have

$$\mathcal{F}'(w;f)=-\frac{2}{3}\left(\ln\left[-\frac{5}{4}+w\right]-\ln\left[\frac{1}{4}+w\right]\right),$$
$$\mathcal{F}''(w;f)=-\frac{1}{(w+\frac{1}{4})(w-\frac{5}{4})}.$$

The root of $\mathcal{F}'(w;f)=y$, referred to as the critical point, is given by

$$w_0=w_0(y)=\frac{1+5e^{3/2y}}{4(-1+e^{3/2y})}.$$

Example 2 of Section 4.2 shows that a steepest descent contour exists for any complex values of y for which $w_0\notin[-1/4,5/4]$, i.e. if $e^{3/2y}$ is not a negative real number. The values at w_0 are

$$yw_0-\mathcal{F}(w_0;f)=-\frac{1}{4}y+\ln(e^{3/2y}-1)+1-\ln\frac{3}{2}$$

and

$$\mathcal{F}''(w_0;f)=-\frac{4}{9}\frac{(e^{3/2y}-1)^2}{e^{3/2y}}$$

In order to apply Proposition 4.2 we need to ensure the convergence of $\mathcal{Q}(w;\nu,f)$, defined as in Section 4.1 via

$$\ln\mathcal{Q}(w;\nu,f)=(2L)\mathcal{F}(w;f)-\sum_{j=1}^{2L}\ln\left(w-\frac{\nu_j+2L-j}{2L}\right) \tag{5.15}$$
$$=\underbrace{\left(2L\mathcal{F}(w;f)-\sum_{j=1}^{2L}\ln\left(w-\widehat{f}\left(\frac{j}{2L}\right)\right)\right)}_{P_1(w;\nu,f)}-\underbrace{\sum_{j=1}^{2L}\ln\left(1+\frac{f\left(\frac{j}{2L}\right)-\frac{\nu_j}{2L}}{w-f\left(\frac{j}{2L}\right)-1+\frac{j}{2L}}\right)}_{P_2(w;\nu,f)}.$$

As in (4.17), we can write

$$P_1(w;\nu,f)=\frac{\ln(w-\hat{f}(0))-\ln(w-\hat{f}(1))}{2}+\frac{b(w)}{L}+o(1/L),$$

where the exact value of $b(w)$ does not depend on the parity of L and, thus, will not affect the differences t_1-t_1' and t_2-t_2' in the statement.

We now estimate $P_2(w;\nu,f)$. We substitute the values for ν and expand the logarithms as $\ln(1+x)\approx x-x^2/2$. Let

$$A(w;L):=-L\sum_{i=1}^{2L}\left(\frac{-\frac{\nu_i}{2L}+f(\frac{i}{2L})}{w-f(\frac{i}{2L})-1+\frac{i}{2L}}\right)^2 \tag{5.16}$$

be the second order term in this expansion, so that

$$P_2(w;\nu,f)=\sum_{i=1}^{2L}\frac{-\frac{\nu_i}{2L}+f(\frac{i}{2L})}{w-f(\frac{i}{2L})-1+\frac{i}{2L}}+\frac{A(w;L)}{2L}+O(1/L^2)$$

Approximating the last sum by integrals we have

(5.17)

$$\sum_{i=1,\, i\equiv L(\mathrm{mod}\ 2)}^{2L} \frac{-\frac{(L-i)/2+1}{2L}+\frac{1}{4}-\frac{1}{2}\cdot\frac{i}{2L}}{w-\frac{1}{4}+\frac{i}{4L}-1+\frac{i}{2L}} + \sum_{i=1,\, i\equiv L+1(\mathrm{mod}\ 2)}^{2L} \frac{-\frac{(L-i)/2+1/2}{2L}+\frac{1}{4}-\frac{1}{2}\cdot\frac{i}{2L}}{w-\frac{1}{4}+\frac{i}{4L}-1+\frac{i}{2L}}$$

$$= \sum_{i=1,\, i\equiv L(\mathrm{mod}\ 2)}^{2L} \frac{-\frac{1}{2L}}{w-\frac{5}{4}+\frac{3i}{4L}} + \sum_{i=1,\, i\equiv L+1(\mathrm{mod}\ 2)}^{2L} \frac{-\frac{1}{4L}}{w-\frac{5}{4}+\frac{3i}{4L}}$$

$$= \int_0^1 \frac{-\frac{1}{2}}{w-\frac{5}{4}+\frac{3}{2}\eta}d\eta + \int_0^1 \frac{-\frac{1}{4}}{w-\frac{5}{4}+\frac{3}{2}\eta}d\eta + \frac{B(w;L)}{L}$$

$$= \frac{1}{2}\ln\left(\frac{w-\frac{5}{4}}{w+\frac{1}{4}}\right) + \frac{B(w;L)}{L},$$

where $B(w;L)$ is the error term in the approximation of the Riemann sums by integrals. While both functions $A(w;L)$ and $B(w;L)$ are bounded in w and L, they could depend on the parity of L. The sum in (5.16) can be again approximated by an integral similarly to (5.17); therefore for both odd and even L we have

$$A(w;L) = \hat{A}(w) + O(1/L).$$

Next, $B(w;L)$ appears when we approximate the integrals by their Riemann sums. Using that the trapezoid formula for the integral gives $O(1/L^2)$ approximation, and denoting $v(x) = -\frac{1}{4(w-\frac{5}{4}+\frac{3}{2}x)}$, we have for even L

$$B(w;L) = -v(0) + v\left(\frac{2L}{2L}\right) + O(1/L) = v(1) - v(0) + O(1/L)$$

and for odd L

$$B(w;L) = -v(0)/2 + v\left(\frac{2L}{2L}\right)/2 + O(1/L) = v(1)/2 - v(0)/2 + O(1/L)$$

Therefore, we have

$$A(w,L) + B(w,L) = \hat{C}(w) + (-1)^{L+1}\frac{1}{16}\left(\frac{1}{w-\frac{5}{4}} - \frac{1}{w+\frac{1}{4}}\right) + O(1/L)$$

and hence we obtain as $L\to\infty$

$$\exp(\mathcal{Q}(w;\nu,f)) = \left(\frac{w-\frac{5}{4}}{w+\frac{1}{4}}\right)^{\frac{1}{2}}\left(1 + (-1)^{L+1}\frac{1}{16L}\left(\frac{1}{w-\frac{5}{4}} - \frac{1}{w+\frac{1}{4}}\right) + O(1/L^2)\right),$$

and

$$\exp\left(\mathcal{Q}(w_0;\nu,f)\right) = \exp\left(-\frac{3}{4}y\right)\left(1 + (-1)^{L+1}\frac{1}{24L}\left((e^{3/2y}-1)^2e^{-3/2y}\right) + O(1/L^2)\right),$$

Now combining Proposition 4.2 and remark after it with the expansion of $\mathcal{Q}$ and explicit values found above we obtain

$$S_\nu(e^y;2L,1) = \sqrt{-\frac{w_0 - f(0) - 1}{\mathcal{F}''(w_0)(w_0 - f(1))}}\left(\frac{w_0-\frac{5}{4}}{w_0+\frac{1}{4}}\right)^{\frac{1}{2}}\frac{\exp 2L(yw_0 - \mathcal{F}(w_0))}{e^{2L}(e^y-1)^{2L-1}}$$
$$\times\left(1 + (-1)^{L+1}\frac{1}{16L}\left(\frac{1}{w_0-\frac{5}{4}} - \frac{1}{w_0+\frac{1}{4}}\right) + \dots\right)(1+\dots) \tag{5.18}$$

$$= \frac{3e^{-\frac{9}{4}y}(e^y-1)}{2(e^{3/2y}-1)}\left(\frac{4}{9}\frac{(e^{3/2y}-1)^2}{e^{y/2}(e^y-1)^2}\right)^L$$
$$\times\left(1+\hat{t}_1L^{-1/2}+\left(\hat{t}_2+(-1)^{L+1}\frac{(e^{3/2y}-1)^2e^{-3/2y}}{12}\right)L^{-1}+\dots\right)$$

Proposition 3.19 then immediately gives $\mathfrak{X}_{\lambda^L}(e^y;L,1)$ as $\frac{2}{e^y+1}S_\nu(e^y;2L,1)$. □

We will now proceed to derive the multivariate formulas needed to compute u_L. First of all, set $h(x)=\frac{4}{9}x^{-3/2}(x^{3/2}-1)^2$ and define $\alpha_L(x)$ through

$$\mathfrak{X}_{\lambda^L}(x;L)=\alpha_L(x)\frac{x-1}{x+1}h(x)^L\left(2-x-x^{-1}\right)^{-L},$$

with λ^L as in Proposition 5.13.

Define

$$\widetilde{\tau}_L(z_1,\dots,z_k)=\frac{\chi_{\lambda^L}(z_1^2,\dots,z_k^2,1^{L-k})}{\chi_{\lambda^L}(1^L)}=\mathfrak{X}_{\lambda^L}(z_1^2,\dots,z_k^2;L,1)$$

$$\widetilde{u}_L(\zeta_1,\zeta_2;z_1,\dots,z_k)=(-1)^L\mathbf{i}\frac{\sqrt{3}}{2}\ln\left[\frac{\widetilde{\tau}_{L+1}(\zeta_1,z_1,\dots,z_k)\widetilde{\tau}_{L+1}(\zeta_2,z_1,\dots,z_k)}{\widetilde{\tau}_L(z_1,\dots,z_k)\widetilde{\tau}_{L+2}(\zeta_1,\zeta_2,z_1,\dots,z_k)}\right].$$

Then $\widetilde{u}_L(\zeta_1,\zeta_2;z_1,\dots,z_k)-u_L(\zeta_1,\zeta_2,z_1,\dots,z_k)$ is a constant and thus we have

$$z_j\frac{\partial}{\partial z_j}\widetilde{u}(\zeta_1,\zeta_2;z_1)=X_L^{(j)},$$

$$w\frac{\partial}{\partial w}\widetilde{u}_{L+2}(\zeta_1,\zeta_2;vq^{-1},w)\Big|_{v=w}=Y_L.$$

Therefore, we can work with $\mathfrak{X}_{\lambda^L}$ instead of χ_{λ^L} and with $\widetilde{u}$ instead of u.

For any function ξ and variables $v_1,\dots,v_m$ we define

$$B(v_1,\dots,v_m;\xi):=\frac{\sum_{i=1}^m\xi(v_i)v_i\frac{\partial}{\partial v_i}\Delta(\xi(v_1)^2,\dots,\xi(v_m)^2)}{\Delta(\xi(v_1)^2,\dots,\xi(v_m)^2)}.$$

Proposition 5.14. *Suppose that signature λ depends on a large parameter L in such a way that*

$$\mathfrak{X}_\lambda(x;L,1)=\alpha_L(x)h(x)^L\frac{x-1}{x+1}(x+x^{-1}-2)^{-L},$$

where

$$\alpha_L(x)=a(x)(1+b_1(x)L^{-1/2}+b_2(x)L^{-1}+\dots)\ \text{ for even } L,$$
$$\alpha_L(x)=a(x)(1+b_1(x)L^{-1/2}+\widehat{b}_2(x)L^{-1}+\dots)\ \text{ for odd } L$$

and $a(x),b_1(x),b_2(x),\widehat{b}_2(x),h(x)$ are some analytic functions of x. Let $\xi(x)=x\frac{\partial}{\partial x}\ln(h(x))$. Then for any k we have

$$\ln\left[\frac{\mathfrak{X}_\lambda(x_0,\dots,x_k;L+1)\mathfrak{X}_\lambda(x_1,\dots,x_{k+1};L+1)}{\mathfrak{X}_\lambda(x_1,\dots,x_k;L)\mathfrak{X}_\lambda(x_0,\dots,x_{k+1};L+2)}\right]$$
$$=c_1(x_0,x_{k+1};L)+\sum_{i=1}^k2\big(\widehat{b}_2(x_i)-b_2(x_i)\big)\frac{(-1)^L}{L}+$$
$$\ln\left[(\xi(x_{k+1})^2-\xi(x_0)^2)+\frac{2}{L}\big(B(x_0,..,x_k;\xi)-B(x_1,..,x_{k+1};\xi)+c_2(x_0,x_{k+1})\big)\right]$$
$$+o(L^{-1}),$$

where c_0 and c_1 are analytic functions not depending on $x_1,\dots,x_k$.

Proof. We use Theorem 3.17 to express the multivariate normalized character in terms of $\alpha_L(x_i)$ and $h(x_i)$ as follows

$$\frac{\mathfrak{X}_\lambda(x_1,\dots,x_m;N)}{\prod \mathfrak{X}_\lambda(x_i;N)} = \prod_{j=0}^{m-1} \frac{(2N-2j-1)!N^{2j}}{(2N-1)!} \frac{\prod_{i=1}^m (x_i-1)^{2m-1}(x_i+1)x_i^{-m}}{\Delta_s(x_1,\dots,x_m)} M_N(x_1,\dots,x_m), \tag{5.19}$$

which is applied with $N = L, L+1, L+2$, $m = k, k+1, k+2$ and define for any N and m

$$M_N(x_1,\dots,x_m) := \det\left[\frac{D_i^{2j-2}\left[\alpha_N(x_i)h(x_i)^N\right]}{N^{2j-2}\alpha_N(x_i)h(x_i)^N}\right]_{i,j=1}^m = \frac{\Delta\left(\frac{D_1^2}{N^2},\dots,\frac{D_m^2}{N^2}\right)\prod_{i=1}^m \alpha_N(x_i)h(x_i)^N}{\prod_{i=1}^m \alpha_N(x_i)h(x_i)^N}, \tag{5.20}$$

where, as above, $D_i = x_i\frac{\partial}{\partial x_i}$. The second form in (5.20) will be useful later.

We can then rewrite the expression of interest as

$$\begin{aligned}\ln\left[\frac{\mathfrak{X}_\lambda(x_0,\dots,x_k;L+1)\mathfrak{X}_\lambda(x_1,\dots,x_{k+1};L+1)}{\mathfrak{X}_\lambda(x_1,\dots,x_k;L)\mathfrak{X}_\lambda(x_0,\dots,x_{k+1};L+2)}\right] &= const_1(L) + \ln\left[\frac{\mathfrak{X}_\lambda(x_0;L+1)\mathfrak{X}_\lambda(x_{k+1};L+1)}{\mathfrak{X}_\lambda(x_0;L+2)\mathfrak{X}_\lambda(x_{k+1};L+2)}\right] \\ &+ \ln\left[\prod_{i=1}^k \frac{\mathfrak{X}_\lambda(x_i;L+1)^2}{\mathfrak{X}_\lambda(x_i;L)\mathfrak{X}_\lambda(x_i;L+2)}\right] - \ln\left[\frac{(x_0-1)^2x_0^{-1}(x_{k+1}-1)^2x_{k+1}^{-1}}{x_0+x_0^{-1}-(x_{k+1}+x_{k+1}^{-1})}\right] \\ &+ \ln\frac{M_{L+1}(x_0,x_1,\dots,x_k)M_{L+1}(x_1,\dots,x_{k+1})}{M_L(x_1,\dots,x_k)M_{L+2}(x_0,\dots,x_{k+1})},\end{aligned} \tag{5.21}$$

where $const_1(L)$ will be part of $c_1(x_0, x_{k+1}; L)$. We investigate each of the other terms separately. First, we have that

$$\begin{aligned}\ln\left[\frac{\mathfrak{X}_\lambda(x_0;L+1)\mathfrak{X}_\lambda(x_{k+1};L+1)}{\mathfrak{X}_\lambda(x_0;L+2)\mathfrak{X}_\lambda(x_{k+1};L+2)}\right] + \ln\left[\prod_{i=1}^k \frac{\mathfrak{X}_\lambda(x_i;L+1)^2}{\mathfrak{X}_\lambda(x_i;L)\mathfrak{X}_\lambda(x_i;L+2)}\right] &= \sum_{i=1}^k \ln\left(\frac{\alpha_{L+1}(x_i)^2}{\alpha_L(x_i)\alpha_{L+2}(x_i)}\right) + \ln\left(\frac{\alpha_{L+1}(x_0)\alpha_{L+1}(x_{k+1})}{\alpha_{L+2}(x_0)\alpha_{L+2}(x_{k+1})}\right) \\ &+ \ln\left[\frac{(x_0+x_0^{-1}-2)(x_{k+1}+x_{k+1}^{-1}-2)}{h(x_0)h(x_{k+1})}\right]\end{aligned}$$

where the terms involving x_0 and x_{k+1} are absorbed in c_1 and we notice that

$$\ln\left(\frac{\alpha_{L+1}(x)^2}{\alpha_L(x)\alpha_{l+2}(x)}\right) = 2\left(\widehat{b}_2(x) - b_2(x)\right)\frac{(-1)^L}{L} + O\left(\frac{1}{L^2}\right).$$

Next we observe that for any ℓ and N

$$\frac{\left(x\frac{\partial}{\partial x}\right)^\ell\left[\alpha_N(x)h(x)^N\right]}{N^\ell\alpha_N(x)h(x)^N} = \xi(x)^\ell + \left(\binom{\ell}{2}q_1 - \binom{\ell}{2}\xi(x)^\ell + \ell r_1\xi(x)^{\ell-1}\right)\frac{1}{N} + O\left(N^{-3/2}\right), \tag{5.22}$$

where $q_1 = \xi(x)(x\frac{\partial}{\partial x}\xi(x) + \xi(x)^2)$ and $r_1(x) = x\frac{\partial}{\partial x}\log(a(x))$. In particular, since M_N is a polynomial in the left-hand side of (5.22), it is of the form

$$M_N(x_1,\dots,x_m) = \Delta(\xi^2(x_1),\dots,\xi^2(x_m)) + p_1(x_1,\dots,x_m)\frac{1}{N} + O\left(N^{-3/2}\right) \tag{5.23}$$

for some function p_1 which depends only on ξ and a. That is, the second order asymptotics of M_N does not depend on the second order asymptotics of α_L. Further, we have

$$\frac{M_N}{M_{N+1}} = 1 + O\left(N^{-3/2}\right)$$

for any N, so in formula (5.21) we can replace M_{L+1} and M_{L+2} by M_L without affecting the second order asymptotics. Evaluation of M directly will not lead to an easily analyzable formula, therefore we will do some simplifications and approximations beforehand.

We will use Lewis Carroll's identity (Dodgson condensation), which states that for any square matrix A we have

$$(\det A)(\det A_{1,2;1,2}) = (\det A_{1;1})(\det A_{2;2}) - (\det A_{1;2})(\det A_{2;1}),$$

where $A_{I;J}$ denotes the submatrix of A obtained by removing the rows whose indices are in I and columns whose indices are in J. Applying this identity to the matrix

$$A = \left[\frac{D_i^{2j-2}\left[\alpha_L(x_i)h(x_i)^L\right]}{L^{2j}\alpha_L(x_i)h(x_i)^L}\right]_{i,j=0}^{k+1}$$

we obtain

$$\begin{aligned} &M_L(x_1,\dots,x_k)M_L(x_0,x_1,\dots,x_k,x_{k+1}) \\ &\quad= \det\left[\frac{D_i^{2j}(\alpha_L(x_i)h^L(x_i))}{L^{2j}\alpha_L(x_i)h(x_i)^L}\right]_{i=[1:k+1]}^{j=[0:k-1,k+1]} \det\left[\frac{D_i^{2j}(\alpha_L(x_i)h^L(x_i))}{L^{2j}\alpha_L(x_i)h(x_i)^L}\right]_{i,j=0}^{k} \\ &\qquad- \det\left[\frac{D_i^{2j}(\alpha_L(x_i)h^L(x_i))}{L^{2j}\alpha_L(x_i)h(x_i)^N}\right]_{i=[0:k]}^{j=[0:k-1,k+1]} \det\left[\frac{D_i^{2j}(\alpha_L(x_i)h^L(x_i))}{L^{2j}\alpha_L(x_i)h(x_i)^L}\right]_{i,j+1=1}^{k+1}, \end{aligned} \tag{5.24}$$

where $[0:k-1,k+1] = \{0,1,\dots,k-1,k+1\}$. The second factors in the two products on the right-hand side above are just M_L evaluated at the corresponding sets of variables. For the first factors, applying the alternate formula for M_L from (5.20) and using the fact that

$$\Delta(v_1,\dots,v_m)\sum_{i=1}^m v_i = \det\left[v_i^j\right]_{i=[1:m]}^{j=[0:m-2,m]},$$

we obtain

$$\begin{aligned} &\det\left[\frac{D_i^{2j}(\alpha_L(x_i)h(x_i)^L)}{L^{2j}\alpha_L(x_i)h(x_i)^L}\right]_{i=[1:k+1]}^{j=[0:k-1,k+1]} \\ &\qquad= \frac{1}{\prod_{i=1}^{k+1}\alpha_L(x_i)h(x_i)^L}\det\left[(D_i^2/L^2)^j\right]_{i=[1:k+1]}^{j=[0:k-1,k+1]}\prod_{i=1}^{k+1}\alpha_L(x_i)h(x_i)^L \end{aligned}$$

$$= \frac{1}{\prod_{i=1}^{k+1} \alpha_L(x_i)h(x_i)^L} \left(\sum_{i=1}^{k+1} D_i^2/L^2\right) \Delta(D_1^2/L^2, \dots, D_{k+1}^2/L^2) \prod_{i=1}^{k+1} \alpha_L(x_i)h(x_i)^L$$

$$= \frac{1}{\prod_{i=1}^{k+1} \alpha_L(x_i)h(x_i)^L} \left(\sum_{i=1}^{k+1} D_i^2/L^2\right) \left[\left(\prod_{i=1}^{k+1} \alpha_L(x_i)h(x_i)^L\right) M_L(x_1, \dots, x_{k+1})\right],$$

Substituting these computations into (5.24) we get

$$\begin{aligned}
(5.25)\quad & \frac{M_L(x_0, x_1, \dots, x_k)M_L(x_1, \dots, x_k, x_{k+1})}{M_L(x_1, \dots, x_k)M_L(x_0, x_1, \dots, x_k, x_{k+1})} \\
&= \frac{(\sum_{i=1}^{k+1} \frac{D_i^2}{L^2}) \left[(\prod_{i=1}^{k+1} \alpha_L(x_i)h(x_i)^L)M_L(x_1, \dots, x_{k+1})\right]}{\prod_{i=1}^{k+1} \alpha_L(x_i)h(x_i)^L M_L(x_1, \dots, x_{k+1})} \\
&\quad - \frac{(\sum_{i=0}^{k} \frac{D_i^2}{L^2}) \left[(\prod_{i=0}^{k} \alpha_L(x_i)h(x_i)^L)M_L(x_0, \dots, x_k)\right]}{\prod_{i=0}^{k} \alpha_L(x_i)h(x_i)^L M_L(x_0, \dots, x_k)} \\
&= \frac{D_{k+1}^2 \alpha_L(x_{k+1})h(x_{k+1})^L}{L^2 \alpha_L(x_{k+1})h(x_{k+1})^L} - \frac{D_0^2 \alpha_L(x_0)h(x_0)^L}{L^2 \alpha_L(x_0)h(x_0)^L} \\
&\quad + \frac{(\sum_{i=1}^{k+1} D_i^2)[M_L(x_1, \dots, x_{k+1})]}{L^2 M_L(x_1, \dots, x_{k+1})} - \frac{(\sum_{i=0}^{k} D_i^2)[M_L(x_0, \dots, x_k)]}{L^2 M_L(x_0, \dots, x_k)} \\
&+2\left(\sum_{i=1}^{k+1} \frac{D_i[\alpha_L(x_i)h(x_i)^L]}{L\alpha_L(x_i)h(x_i)^L} \frac{D_i M_L(x_1, \dots, x_{k+1})}{L M_L(x_1, \dots, x_{k+1})} - \sum_{i=0}^{k} \frac{D_i[\alpha_L(x_i)h(x_i)^L]}{L\alpha_L(x_i)h(x_i)^L} \frac{D_i M_L(x_0, \dots, x_k)}{L M_L(x_0, \dots, x_k)}\right)
\end{aligned}$$

Using the expansion for M_L from equation (5.23) and the expansion from (5.22) we see that the only terms contributing to the first two orders of approximation in (5.25) above are

$$\begin{aligned}
(5.26)\quad & \xi(x_{k+1})^2 - \xi(x_0)^2 + \frac{1}{L}(c_3(x_{k+1}) - c_3(x_0)) + \\
& \frac{2}{L}\left(\sum_{i=1}^{k+1} \xi(x_i) \frac{D_i \Delta(\xi(x_1)^2, \dots, \xi(x_{k+1})^2)}{\Delta(\xi(x_1)^2, \dots, \xi(x_{k+1})^2)} - \sum_{i=0}^{k} \xi(x_i) \frac{D_i \Delta(\xi(x_0)^2, \dots, \xi(x_k)^2)}{\Delta(\xi(x_0)^2, \dots, \xi(x_k)^2)}\right) \\
& + o\left(L^{-1}\right)
\end{aligned}$$

for some function c_3 not depending on L, so $c_2(x_0, x_{k+1}) = c_3(x_{k+1}) - c_3(x_0)$. Substituting this result into (5.21) we arrive at the desired formula.

□

Proof of Theorem 5.12. Proposition 5.14 with $x_0 = \zeta_2^2$, $x_{k+1} = \zeta_1^2$ and $x_i = z_i^2$ shows that

$$\begin{aligned}
(5.27)\quad & L\Bigg(\widetilde{u}_L(\zeta_1, \zeta_2, z_1, \dots, z_k) - c_1(\zeta_1^2, \zeta_2^2; L) - \sum_{i=1}^{k} 2\big(\widehat{b}_2(x_i) - b_2(x_i)\big) \frac{(-1)^L}{L} \\
& - \ln\left[(\xi(\zeta_1^2)^2 - \xi(\zeta_2^2)^2) + 2(B(x_0, .., x_k; \xi) - B(x_1, .., x_{k+1}; \xi) + c_2(\zeta_1^2, \zeta_2^2)) \frac{1}{L}\right]\Bigg)
\end{aligned}$$

converges uniformly to 0 and so its derivatives also converge to 0. Proposition 5.13 shows that in our case

$$h(x) = \frac{4}{9} x^{-3/2}(x^{3/2} - 1)^2$$

and thus $\xi(x)=\frac{3}{2}\cdot\frac{x^{3/2}+1}{x^{3/2}-1}$. Moreover, the function ξ satisfies the following equation

$$x\frac{\partial}{\partial x}\xi(x)=-\frac{9}{2}\frac{x^{3/2}}{(x^{3/2}-1)^2}=-\frac{9}{8}(\xi(x)^2-1)$$

and so we can simplify the function B as a sum as follows

$$\begin{aligned}(5.28)\quad B(v_1,\dots,v_m;\xi)&=\frac{\sum_i \xi(v_i)v_i\frac{\partial}{\partial v_i}\Delta(\xi(v_1)^2,\dots,\xi(v_m)^2)}{\Delta(\xi(v_1)^2,\dots,\xi(v_m)^2)}\\ &=\sum_i\sum_{j\neq i}\xi(v_i)v_i\frac{\frac{\partial}{\partial v_i}(\xi(v_i)^2-\xi(v_j)^2)}{\xi(v_i)^2-\xi(v_j)^2}=\sum_i\sum_{j\neq i}\frac{2\xi(v_i)^2v_i\frac{\partial}{\partial v_i}\xi(v_i)}{\xi(v_i)^2-\xi(v_j)^2}\\ =\sum_i\sum_{j\neq i}\frac{-\frac{9}{4}(\xi(v_i)^4-\xi(v_i)^2)}{\xi(v_i)^2-\xi(v_j)^2}&=\sum_{i<j}\frac{-\frac{9}{4}(\xi(v_i)^4-\xi(v_i)^2-\xi(v_j)^4+\xi(v_j)^2)}{\xi(v_i)^2-\xi(v_j)^2}\\ &=\sum_{i<j}-\frac{9}{4}(\xi(v_i)^2+\xi(v_j)^2-1)=-\frac{9}{4}(m-1)\left(\sum\xi(v_i)^2\right)+\frac{9}{4}\binom{m}{2}\end{aligned}$$

We thus have that

$$B(x_0,\dots,x_k;\xi)-B(x_1,\dots,x_{k+1};\xi)=-\frac{9}{4}k(\xi(x_{k+1})^2-\xi(x_0)^2),$$

which does not depend on $x_1,\dots,x_k$.

Differentiating (5.27) we obtain the asymptotics of $X_L^{(j)}$ as

$$\begin{aligned}X_L^{(j)}=\mathbf{i}\frac{\sqrt{3}}{2}(-1)^L z\frac{\partial}{\partial z}2\big(\widehat{b}_2(z^2)-b_2(z^2)\big)\frac{(-1)^L}{L}&=\mathbf{i}\frac{\sqrt{3}}{2}z\frac{\partial}{\partial z}\left[\frac{1}{6}(z^3-1)^2z^{-3}\right]\\ &=\mathbf{i}\frac{\sqrt{3}}{4}(z^3-z^{-3}).\end{aligned}$$

For $Y_L^{(j)}$ the computations is the same. □

6. Representation-theoretic applications

6.1. **Approximation of characters of $U(\infty)$.** In this section we give a new proof of Theorem 1.5 presented in the Introduction.

Recall that a character of $U(\infty)$ is given by the function $\chi(u_1,u_2,\dots)$, which is defined on sequences u_i such that $u_i=1$ for all large enough i. Also $\chi(1,1,\dots)=1$. By Theorem 1.3 extreme characters of $U(\infty)$ are parameterized by the points ω of the infinite-dimensional domain

$$\Omega\subset\mathbb{R}^{4\infty+2}=\mathbb{R}^\infty\times\mathbb{R}^\infty\times\mathbb{R}^\infty\times\mathbb{R}^\infty\times\mathbb{R}\times\mathbb{R},$$

where Ω is the set of sextuples

$$\omega=(\alpha^+,\alpha^-,\beta^+,\beta^-;\delta^+,\delta^-)$$

such that

$$\alpha^\pm=(\alpha_1^\pm\geq\alpha_2^\pm\geq\dots\geq 0)\in\mathbb{R}^\infty,\quad \beta^\pm=(\beta_1^\pm\geq\beta_2^\pm\geq\dots\geq 0)\in\mathbb{R}^\infty,$$

$$\sum_{i=1}^\infty(\alpha_i^\pm+\beta_i^\pm)\leq\delta^\pm,\quad \beta_1^++\beta_1^-\leq 1.$$

Let μ be a Young diagram with the length of main diagonal d. Recall that *modified Frobenius coordinates* are defined via

$$p_i = \mu_i - i + 1/2, \quad q_i = \mu'_i - i + 1/2, \quad i = 1, \dots, d.$$

Note that $\sum_{i=1}^{d} p_i + q_i = |\mu|$.

Now let $\lambda \in \mathbb{GT}_N$ be a signature, we associate two Young diagrams λ^+ and λ^- to it, corresponding to the positive and negative entries of λ, respectively: let $r = \max(i : \lambda_i \geq 0)$, then

$$\lambda^+ = (\lambda_1, \dots, \lambda_r) \qquad \text{and} \qquad \lambda^- = (-\lambda_N, -\lambda_{N-1}, \dots, -\lambda_{r+1}).$$

In this way we get two sets of modified Frobenius coordinates: p_i^+, q_i^+, $i = 1, \dots, d^+$ and p_i^-, q_i^-, $i = 1, \dots, d^-$.

Proposition 6.1. *Suppose that $\lambda(N) \in \mathbb{GT}_N$ is such a way that*

$$\frac{p_i^+}{N} \to \alpha_i^+, \quad \frac{q_i^+}{N} \to \beta_i^+, \quad \frac{p_i^-}{N} \to \alpha_i^-, \quad \frac{q_i^-}{N} \to \beta_i^-,$$

$$\frac{\sum_{i=1}^{d^+} p_i^+ + q_i^+}{N} \to \sum_{i=1}^{\infty} (\alpha_i^+ + \beta_i^+) + \gamma^+, \quad \frac{\sum_{i=1}^{d^-} p_i^- + q_i^-}{N} \to \sum_{i=1}^{\infty} (\alpha_i^- + \beta_i^-) + \gamma^-$$

then

$$\lim_{N\to\infty} S_{\lambda(N)}(x; N, 1) = \Phi_\infty\left(\alpha, \beta, \gamma; \frac{x}{x-1}\right),$$

where

$$\Phi_\infty\left(\alpha, \beta, \gamma; \frac{x}{x-1}\right) = \exp(\gamma^+(x-1) + \gamma^-(x^{-1}-1)) \times \prod_{i=1}^{\infty} \frac{1+\beta_i^+(x-1)}{1-\alpha_i^+(x-1)} \cdot \frac{1+(1-\beta_i^-)(x-1)}{1+(1+\alpha_i^-)(x-1)}.$$

The convergence is uniform over $1-\varepsilon < |x| < 1+\varepsilon$ for certain $\varepsilon > 0$.

Remark 1. Note that

$$\frac{1+(1-\beta_i^-)(x-1)}{1+(1+\alpha_i^-)(x-1)} = \frac{1+\beta_i^-(x^{-1}-1)}{1-\alpha_i^-(x^{-1}-1)},$$

which brings the function Φ_∞ into a more traditional form of Theorems 1.3, 1.5

Remark 2. Our methods, in principle, allow us to give a full description of the set on which the convergence holds.

Proof. The following combinatorial identity is known (see e.g. [BO, (5.15)] and references therein)

$$\prod_{i=1}^{N} \frac{s+i-\lambda_i}{s+i} = \prod_{i=1}^{d^+} \frac{s+1/2-p_i^+}{s+1/2+q_i^+} \prod_{i=1}^{d^-} \frac{s+1/2+N+p_i^+}{s+1/2+N-q_i^-}. \tag{6.1}$$

Introduce the following notation:

$$\Phi_N(\lambda(N); w) = \prod_{i=1}^{d^+} \frac{w-1+\frac{1/2+q_i^+}{N}}{w-1+\frac{1/2-p_i^+}{N}} \prod_{i=1}^{d^-} \frac{w-1+\frac{1/2+N-q_i^-}{N}}{w-1+\frac{1/2+N+p_i^+}{N}},$$

and observe that (6.1) implies that in the notation of Section 4.1 we have

$$\prod_j \frac{1}{(w-\mu_j(N)/N)} = \Phi_N(\lambda(N);w)\prod_i \frac{1}{w-\frac{N-i}{N}}. \tag{6.2}$$

Then the integral formula for the Schur function (Theorem 3.8) gives

$$S_{\lambda(N);N,1}(x) = \frac{(N-1)!}{(x-1)^{N-1}}\frac{1}{2\pi\mathbf{i}}\oint \frac{x^z}{\prod_{i=1}^N (z-(N-i))}\Phi_N(\lambda(N);z/N)\,dz.$$

We recognize in the integrand the setting of Proposition 4.2 with $f(t)=0$ for $t\in[0,1]$. Thus, following the notations of Proposition 4.2 we denote

$$\mathcal{Q}(w;\lambda(N),f) = \frac{\exp(\mathcal{F}(w;f))}{\prod_{i=1}^N (w-\frac{N-i}{N})}\Phi_N(\lambda(N);w).$$

As $N\to\infty$ we have

$$\Phi_N(\lambda(N);w)\to\Phi_\infty(\alpha,\beta,\gamma;w). \tag{6.3}$$

Further, we have that for $f\equiv 0$, $\mathcal{F}(w;0) = w\ln(w)-(w-1)\ln(w-1)-1$ and as $N\to\infty$

$$\frac{\exp(\mathcal{F}(w;f))}{\prod_{i=1}^N (w-\frac{N-i}{N})}\to 1. \tag{6.4}$$

Combining (6.3) and (6.4) we conclude that $\mathcal{Q}(w;\lambda(N),f)\to\Phi_\infty(\alpha,\beta,\gamma;w)$ as $N\to\infty$. Now we can use Propositions 4.2 and 4.7 with the steepest descent contours of Example 1 of Section 4.2. Recall that here $f(t)=0$, $x=e^y$, $\mathcal{F}(w;0)=w\ln(w)-(w-1)\ln(w-1)-1$ and $w_0=1/(1-e^{-y})$.

We conclude that as $N\to\infty$

$$S_{\lambda(N)}(e^y;N,1) = \frac{g(w_0)}{\sqrt{-\mathcal{F}''(w_0;f)}}\cdot\frac{\exp\big(N(yw_0-\mathcal{F}(w_0;f))\big)}{e^N\big(e^y-1\big)^{N-1}}\cdot\Big(1+o(1)\Big).$$

Substituting $\mathcal{F}$, w_0, $g(w_0)=\Phi_\infty(\alpha,\beta,\gamma;w_0)$ and simplifying we arrive at

$$S_{\lambda(N);N,1}(x)\to\Phi_\infty\left(\alpha,\beta,\gamma;\frac{x}{x-1}\right). \tag{6.5}$$

Note that the convergence in (6.3) is uniform (on compact subsets) outside the poles of $\Phi_\infty(\alpha,\beta,\gamma;w)$, while the convergence in (6.4) is uniform over outside the interval $[0,1]$. Therefore, the convergence in (6.5) is uniform over compact subsets of

$$\mathcal{D} = \{x=e^y\in\mathbb{C}\mid -\pi<Im(y)<\pi,\ -\varepsilon_2<Re(y)<\varepsilon_2,\ y\neq 0\}.$$

(Here the small parameter ε_2 shrinks to zero as $\alpha_1^\pm$ goes to infinity.)

It remains to prove that this implies uniform convergence over $1-\varepsilon<|x|<1+\varepsilon$.

Decompose

$$S_{\lambda(N)}(x;N,1) = \sum_{k=-\infty}^{\infty} c_k(N)x^k.$$

Since $S_{\lambda(N)}$ is a polynomial, only finitely many coefficients $c_k(N)$ are non-zero. The coefficients $c_k(N)$ are *non-negative*, see e.g. [M, Chapter I, Section 5], also $\sum_k c_k(N) = S_{\lambda(N)}(1;N,1)=1$.

Since $\Phi_\infty(\alpha,\beta,\gamma;\frac{x}{x-1})$ is analytic in the neighborhood of the unit circle we can similarly decompose

$$\Phi_\infty\left(\alpha,\beta,\gamma;\frac{x}{x-1}\right)=\sum_{k=-\infty}^{\infty}c_k(\infty)x^k.$$

We claim that $\lim_{N\to\infty}c_k(N)=c_k(\infty)$. Indeed this follows from the integral representations

$$c_k(N)=\frac{1}{2\pi\mathbf{i}}\oint_{|z|=1}S_{\lambda(N)}(z;N,1)z^{-k-1}dz, \tag{6.6}$$

and similarly for Φ_∞. Pointwise convergence for all but finitely many points of the unit circle and the fact that $|S_{\lambda(N)}(z;N,1)|\le 1$ for $|z|=1$ implies that we can send $N\to\infty$ in (6.6).

Now take two positive real numbers a and b, with $\exp(-\varepsilon_2)<a<1<b<\exp(\varepsilon_2)$ such that

$$\lim_{N\to\infty}S_{\lambda(N)}(a;N,1)=\Phi_\infty\left(\alpha,\beta,\gamma;\frac{a}{a-1}\right), \tag{6.7}$$

$$\lim_{N\to\infty}S_{\lambda(N)}(b;N,1)=\Phi_\infty\left(\alpha,\beta,\gamma;\frac{b}{b-1}\right). \tag{6.8}$$

For x satisfying $a\le|x|\le b$ and some positive integer M write

$$\begin{aligned}&\left|S_{\lambda(N)}(x)-\Phi_\infty\left(\alpha,\beta,\gamma;\frac{x}{x-1}\right)\right| \\ &\qquad=\left|\sum_k(c_k(N)-c_k(\infty))x^k\right|\le\sum_k|c_k(N)-c_k(\infty)|(a^k+b^k)\end{aligned} \tag{6.9}$$

$$\le\sum_{k=-M}^{M}|c_k(N)-c_k(\infty)|(a^k+b^k)+\sum_{|k|>M}c_k(N)(a^k+b^k)+\sum_{|k|>M}c_k(\infty)(a^k+b^k).$$

The third term goes zero as $M\to\infty$ because the series $\sum_k c_k(\infty)z^k$ converges for $z=a$ and $z=b$. The second term goes to zero as $M\to\infty$ because of (6.7),(6.8) and $c_k(N)\to c_k(\infty)$. Now for any δ we can choose M such that each of the last two terms in (6.9) are less than $\delta/3$. Since $c_k(N)\to c_k(\infty)$, the first term is a less than $\delta/3$ for large enough N. Therefore, all the expression (6.9) is less than δ and the proof is finished. □

Now applying Corollary 3.10 we arrive at the following theorem.

Theorem 6.2 (cf. Theorem 1.5)**.** *In the settings of Proposition 6.1 for any k we have*

$$\lim_{N\to\infty}S_{\lambda(N)}(x_1,\dots,x_k;N,1)=\prod_{\ell=1}^{k}\Phi_\infty\left(\alpha,\beta,\gamma;\frac{x_\ell}{x_\ell-1}\right),$$

The convergence is uniform over the set $1-\varepsilon<|x_\ell|<1+\varepsilon$, $\ell=1,\dots,k$ for certain $\varepsilon>0$.

Note that we can prove analogues of Theorem 1.5 for infinite-dimensional symplectic group $Sp(\infty)$ and orthogonal group $O(\infty)$ in exactly the same way as for $U(\infty)$. Even the computations remain almost the same. This should be compared to the analogy between the argument based on binomial formulas of [OO] for characters of $U(\infty)$

(and their Jack–deformation) and that of [OO2] for characters corresponding to other root series.

6.2. **Approximation of q–deformed characters of $U(\infty)$.** In [G1] a q–deformation for the characters of $U(\infty)$ related to the notion of quantum trace for quantum groups was proposed. One point of view on this deformation is that we *define* characters of $U(\infty)$ through Theorem 1.5, i.e. as all possible limits of functions S_λ, and then *deform* the function $S_{\lambda(N)}$ keeping the rest of the formulation the same. A "good" q–deformation of turns out to be (see [G1] for the details)

$$\frac{s_\lambda(x_1,\dots,x_k,q^{-k},\dots,q^{1-N})}{s_\lambda(1,q^{-1},\dots,q^{1-N})}.$$

Throughout this section we assume that q is a real number satisfying $0<q<1$. The next proposition should be viewed as q–analogue of Proposition 6.1.

Proposition 6.3. *Suppose that $\lambda(N)$ is such that $\lambda_{N-j+1}\to\nu_j$ for every j. Then*

$$\frac{s_\lambda(x,q^{-1},q^{-2},\dots,q^{1-N})}{s_\lambda(1,q^{-1},\dots,q^{1-N})}\to F_\nu(x),$$

$$F_\nu(x)=\prod_{j=0}^{\infty}\frac{(1-q^{j+1})}{(1-q^jx)}\frac{\ln(q)}{2\pi\mathbf{i}}\int_{\mathcal{C}'}\frac{x^z}{\prod_{j=1}^{\infty}(1-q^{-z}q^{\nu_j+j-1})}dz, \tag{6.10}$$

where the contour of integration $\mathcal{C}'$ consists of two infinite segments of $Im(z)=\pm\frac{\pi\mathbf{i}}{\ln(q)}$ going to the right and vertical segment $[-M(\mathcal{C}')-\frac{\pi\mathbf{i}}{\ln(q)},-M(\mathcal{C}')+\frac{\pi\mathbf{i}}{\ln(q)}]$ with arbitrary $M(\mathcal{C}')<\nu_1$. Convergence is uniform over x belonging to compact subsets of $\mathbb{C}\setminus\{0\}$.

Remark. Note that we can evaluate the integral in the definition of $F_\nu(x)$ as the sum of the residues:

$$F_\nu(x)=\prod_{j=0}^{\infty}\frac{(1-q^{j+1})}{(1-q^jx)}\sum_{k=1}^{\infty}\frac{x^{\nu_k+k-1}}{\prod_{j\neq k}(1-q^{-\nu_k-k+1}q^{\nu_j+j-1})}dz, \tag{6.11}$$

The sum in (6.11) is convergent for any x. Indeed, the product over $j>k$ can be bounded from above by $1/(q;q)_\infty$. The product over over $j<k$ is (up to the factor bounded by $(q;q)_\infty$)

$$\prod_{j=1}^{k-1}q^{\nu_k+k-\nu_j-j}.$$

Note that for any fixed m, if $k>k_0(m)$, then the last product is less than $q^{m(\nu_k+k-1)}$. We conclude that the absolute value of kth term in (6.11) is bounded by

$$|x|^{\nu_k+k-1}q^{m(\nu_k+k-1)}\frac{1}{((q;q)_\infty)^2}.$$

Choosing large enough m and $k>k_0(m)$ we conclude that (6.11) converges.

Proof of Proposition 6.3. We start from the formula of Theorem 3.6

$$\frac{s_\lambda(x,1,q^{-1},\dots,q^{2-N})}{s_\lambda(1,q^{-1},\dots,q^{1-N})}=\frac{-\ln(q)}{2\pi\mathbf{i}}\prod_{i=0}^{N-2}\frac{(q^{1-N}-q^{-i})}{(x-q^{-i})}\int_C\frac{(x/q)^z}{\prod_{j=1}^{N}(q^{-z}-q^{-\lambda_j-N+j})}dz, \tag{6.12}$$

where the contour contains only the real poles $z=\lambda_j+N-j$, e.g. C is the rectangle through $M+\frac{\pi\mathbf{i}}{\ln(q)},M-\frac{\pi\mathbf{i}}{\ln(q)},-M-\frac{\pi\mathbf{i}}{\ln(q)},-M+\frac{\pi\mathbf{i}}{\ln(q)}$ for a sufficiently large M.

Since

$$s_\lambda(x,1,q^{-1},\dots,q^{2-n}) = q^{|\lambda|}s_\lambda(q^{-1}x,q^{-1},q^{-2},\dots,q^{1-N}),$$

we may also write

(6.13)

$$\frac{s_\lambda(x,q^{-1},q^{-2},\dots,q^{1-N})}{s_\lambda(1,q^{-1},\dots,q^{1-N})} = -\prod_{i=0}^{N-2}\frac{(q^{1-N}-q^{-i})}{(x-q^{-i})}\frac{\ln(q)}{2\pi\mathbf{i}}q^{-|\lambda|}\int_C \frac{x^z}{\prod_{j=1}^N(q^{-z}-q^{-\lambda_j-N+j})}dz$$
$$= \prod_{i=0}^{N-2}\frac{(1-q^{i+1})}{(1-q^i x)}\frac{\ln(q)}{2\pi\mathbf{i}}\int_C \frac{x^z}{\prod_{j=1}^N(1-q^{-z}q^{\lambda_j+N-j})}dz.$$

Note that for large enough (compared to x) N the integrand rapidly decays as $Re(z)\to+\infty$. Therefore, we can deform the contour of integration to be $\mathcal{C}'$ which consists of two infinite segments of $Im(z)=\pm\frac{\pi\mathbf{i}}{\ln(q)}$ going to the right and vertical segment $[-M(\mathcal{C}')-\frac{\pi\mathbf{i}}{\ln(q)},-M(\mathcal{C}')+\frac{\pi\mathbf{i}}{\ln(q)}]$ with some $M(\mathcal{C}')$.

Note that prefactor in (6.13) converges as $N\to\infty$. Let us study the convergence of the integral. Clearly, the integrand converges to the same integrand in $F_\nu(x)$, thus, it remains only to check the contribution of infinite parts of contours. But note that for $z=s\pm\frac{\pi\mathbf{i}}{\ln(q)}$, $s\in\mathbb{R}$, we have

$$\frac{x^z}{\prod_{j=1}^N(1-q^{-z}q^{\lambda_j+N-j})} = \frac{x^z}{\prod_{j=1}^N(1+q^{-s}q^{\lambda_j+N-j})}.$$

Now the absolute value of each factor in denominator is greater than 1 and each factor rapidly grows to infinity as $s\to\infty$. We conclude that the integrand in (6.13) rapidly and uniformly in N decays as $s\to+\infty$.

It remains to deal with the singularities of the prefactors in (6.13) and (6.10) at $x=q^{-i}$. But note that pre-limit function is analytic in x (indeed it is a polynomial) and for the analytic functions uniform convergence on a contour implies the convergence everywhere inside. □

As a side-effect we have proved the following analytic statement

Corollary 6.4. *The integral in* (6.10) *and the sum in* (6.11) *vanish at* $x=q^{-i}$.

Theorem 6.5. *Suppose that* $\lambda(N)$ *is such that* $\lambda_{N-j+1}\to\nu_j$ *for every* j. *Then*

$$\frac{s_{\lambda(N)}(x_1,\dots,x_k,q^{-k},q^{-k-1},\dots,q^{1-N})}{s_{\lambda(N)}(1,q^{-1},\dots,q^{1-N})}\to F_\nu^{(k)}(x_1,\dots,x_k),$$

$$F_\nu^{(k)}(x_1,\dots,x_k) = \frac{(-1)^{\binom{k}{2}}q^{-2\binom{k}{3}}}{\Delta(x_1,\dots,x_k)\prod_i(x_iq^{k-1};q)_\infty}\times \det\left[D_{i,q^{-1}}^{j-1}\right]_{i,j=1}^k\prod_{i=1}^k F_\nu(x_iq^{k-1})(xq^{k-1};q)_\infty. \tag{6.14}$$

Convergence is uniform over each x_i *belonging to compact subsets of* $\mathbb{C}\setminus\{0\}$.

Remark. The formula (6.10) should be viewed as a q–analogue of the multiplicativity in the Voiculescu–Edrei theorem on characters of $U(\infty)$ (Theorem 1.3). There exist a natural linear transformation, which restores the multiplicitivity for q–characters, see [G1] for the details.

Proof of Theorem 6.5. Using Proposition 6.3 and Theorem 3.5 we get

$$F_\nu^{(k)}(x_1,\dots,x_k)=\lim_{N\to\infty} q^{-k|\lambda(N)|}S_\lambda(q^kx_1,\dots,q^kx_k;N,q^{-1})$$
$$=\frac{q^{-\binom{k+1}{3}+(N-1)\binom{k}{2}}\prod_{i=1}^k[N-i]_{q^{-1}}!}{\prod_{i=1}^k\prod_{j=1}^{N-k}(x_iq^k-q^{-j+1})}\times$$
$$\frac{(-1)^{\binom{k}{2}}\det\left[D_{i,q}^{j-1}\right]_{i,j=1}^k}{q^{k\binom{k}{2}}\Delta(x_1,\dots,x_k)}\prod_{i=1}^k\frac{S_\lambda(x_iq^k;N,q^{-1})\prod_{j=1}^{N-1}(x_iq^k-q^{-j+1})}{[N-1]_{q^{-1}}!}.$$

In order to simplify this expression we observe that

$$\frac{[N-i]_{q^{-1}}!}{[N-1]_{q^{-1}}!}\approx q^{N(i-1)-\binom{i-1}{2}},\qquad N\to\infty.$$

Also,

$$\prod_{j=1}^m(xq^k-q^{-j+1})=(-1)^mq^{-\binom{m}{2}}(xq^{k-1};q)_m.$$

Last, we have

$$\lim_{N\to\infty}q^{-|\lambda|}S_\lambda(xq^k;N,q^{-1})=F_\nu(q^{k-1}x).$$

Substituting all of these into the formula above, we obtain

$$F_\nu^{(k)}(x_1,\dots,x_k)=\lim_{N\to\infty}\frac{q^{-\binom{k+1}{3}+(N-1)\binom{k}{2}}\prod_{i=1}^k q^{N(i-1)-\binom{i-1}{2}}}{\prod_{i=1}^k(-1)^{N-k}q^{-\binom{N-k}{2}}(xq^{k-1};q)_{N-k}}\times$$
$$\frac{(-1)^{\binom{k}{2}}\det\left[D_{i,q^{-1}}^{j-1}\right]_{i,j=1}^k}{q^{k\binom{k}{2}}\Delta(x_1,\dots,x_k)}\prod_{i=1}^k S_\lambda(x_iq^k;N,q^{-1})(-1)^{N-1}q^{-\binom{N-1}{2}}(xq^{k-1};q)_{N-1}$$
$$=\frac{1}{q^{2\binom{k}{3}}\prod_i(x_iq^{k-1};q)_\infty}\frac{(-1)^{\binom{k}{2}}\det\left[D_{i,q^{-1}}^{j-1}\right]_{i,j=1}^k}{\Delta(x_1,\dots,x_k)}\prod_{i=1}^k F_\nu(x_iq^{k-1})(xq^{k-1};q)_\infty.\quad\square$$

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