Combinatorial applications of symmetric function theory to certain classes of permutations and truncated tableaux

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Abstract

The purpose of this dissertation is to study certain classes of permutations and plane partitions of truncated shapes. We establish some of their enumerative and combinatorial properties. The proofs develop methods and interpretations within various fields of algebraic combinatorics, most notably the theory of symmetric functions and their combinatorial properties. The first chapter of this thesis reviews the necessary background theory on Young tableaux, symmetric functions and permutations. In the second chapter I find a bijective proof of the enumerative formula of permutations starting with a longest increasing subsequence, thereby answering a question of A.Garsia and A.Goupil. The third chapter studies the shape under the Robinson-Schensted-Knuth correspondence of separable (3142 and 2413 avoiding) permutations and proves a conjecture of G.Warrington. In the last chapter I consider Young tableaux and plane partitions of truncated shapes, i.e., whose diagrams are obtained by erasing boxes in the north-east corner from straight or shifted Young diagrams. These objects were first introduced in a special case by R.Adin and Y.Roichman and appeared to possess product-type enumerative formulas, which is a rare property shared only by a few classes of tableaux, most notably the standard Young tableaux. I find formulas for the number of standard tableaux, whose shapes are shifted staircase truncated by one box and straight rectangles truncated by a shifted staircase. The proofs involve interpretations in terms of semi-standard Young tableaux, polytope volumes, Schur function identities and the Robinson-Schensted-Knuth correspondence.

Contents

To all my little things.

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CHAPTER 0

Introduction

In this thesis we study the combinatorial and enumerative properties of three different objects: permutations which start with a longest increasing subsequence, separable (i.e. $3142, 2413$-avoiding) permutations and plane partitions of certain truncated shapes. Our study of these objects uses tools that arise at the intersection of combinatorics and representation theory. The methods we develop rely on interpretations in terms of both algebraic and combinatorial aspects of the theory of symmetric functions: standard Young tableaux, the Robinson-Schensted-Knuth correspondence, Schur functions.

The first chapter of this thesis reviews the necessary background theory from algebraic combinatorics and follows the expositions in [**Sta99**, **Mac95**, **Ful97**]. We define the relevant objects, namely, partitions, Young tableaux and Schur functions. We state some of their combinatorial properties. We describe the Robinson-Schensted-Knuth (RSK) correspondence between matrices with nonnegative integer entries and pairs of same-shape semi-standard Young tableaux and explain several important properties of this bijection. In particular, the specialization of RSK to permutations translates some of their properties to standard Young tableaux (SYT); these will play a key role in the rest of this thesis. We also review a remarkable enumerative property of SYTs: their number is given by a hook-length product formula. SYTs are the linear extensions of the posets defined by their Young diagrams and only a few classes of posets possess nice formulas for the number of their linear extensions. The algebraic aspects of these objects are encompassed within the theory of symmetric functions, mainly the Schur functions, whose definition and properties are subsequently reviewed. These algebraic properties will play a major role in the last chapter of this thesis: the enumeration of standard Young tableaux of certain truncated shapes.

The second chapter studies a special class of permutations, $\Pi_{n,k}$. They are defined as permutations of the n numbers $1, \ldots, n$, whose first $n-k$ elements are increasing and which do not have an increasing subsequence longer than $n-k$. As A.Garsia and A.Goupil observed in [**GG**], the number of such permutations is given by a simple inclusion-exclusion type formula, namely

$$\#\Pi_{n,k} = \sum_{r=0}^{k} (-1)^{k-r} \binom{k}{r} \frac{n!}{(n-r)!}.$$

Their derivation of this fact was quite indirect and followed through computations of character polynomials. In this thesis we give a purely combinatorial, i.e. bijective, proof of these enumerative results. The bijection involves construction of SYT-like objects. It also proves directly a $q-$analogue of the formula, namely

$$\sum_{w \in \Pi_{n,k}} q^{\mathrm{maj}(w^{-1})} = \sum_{r=0}^{k} (-1)^{k-r} \binom{k}{r} [n]_q \cdots [n-r+1]_q,$$

where $\mathrm{maj}(\sigma) = \sum_{i|\sigma_i > \sigma_{i+1}} i$ denotes the major index of a permutation and $[n]_q = \frac{1-q^n}{1-q}$. We construct other bijective proofs of these formulas: a very direct one which gives a recurrence relation for the numbers $\#\Pi_{n,k}$, and a bijection using only permutations without passing through tableaux. This chapter is based on our work in [**Pan10a**].

The third chapter studies the properties under RSK of separable permutations, i.e. permutations which avoid the order patterns 3142 and 2413. A theorem of Greene states that the shape of the tableaux under RSK of any permutation can be determined by the maximal sets of disjoint increasing subsequences of this permutation. In particular, if the shape under RSK of a permutation w is $\lambda = (\lambda_1, \ldots)$, then for every k we have that $\lambda_1 + \cdots + \lambda_k$ is equal to the maximal possible total length of k disjoint increasing subsequences of w. However, these sequences are not necessarily the same for different ks. We show that in the case of separable permutation these sequences can, in fact, be chosen to be the same. Namely, if the shape of a separable permutation σ is $\lambda = (\lambda_1, \ldots, \lambda_k)$, we prove that σ has k disjoint increasing subsequences of lengths $\lambda_1, \ldots, \lambda_k$. As a corollary, we prove that if σ is a separable subsequence of a word w, then the shape of σ is contained in the shape of w as Young diagrams. This chapter is based on our joint work with A.Crites and G.Warrington [**CPW**].

The last chapter of this thesis studies new types of tableaux and plane partitions. These are tableaux and plane partitions whose diagrams are obtained from ordinary straight or shifted Young diagrams by removing boxes from their north-east corners. These objects originated in the work of Adin and Roichman, [**AR**], in the instance of a shifted staircase with the box in the NE corner removed. Computer calculations showed that the number of standard tableaux of this and some other shapes factor into only small primes, i.e. on the order of the number of squares of the tableaux. This suggested the existence of a product-form formula for the number of standard tableaux of such shapes, similar to the remarkable hook-length formula for the number of ordinary SYTs. In this chapter we find and prove formulas for the number of standard tableaux of shifted truncated shape $\delta_n \setminus (1)$, i.e. a shifted staircase without its NE corner box, and of truncated shape $n^m \setminus \delta_k$, i.e. rectangles without the boxes in a staircase at the NE corner. In particular, the number of shifted truncated tableaux of shape $\delta_n \setminus (1)$ is equal to

$$g_n \frac{C_n C_{n-2}}{2C_{2n-3}},$$

where $g_n = \frac{\binom{n+1}{2}!}{\prod_{0 \le i < j \le n}(i+j)}$ is the number of shifted staircase tableaux of shape $(n, n-1, \ldots, 1)$ and $C_m = \frac{1}{m+1}\binom{2m}{m}$ is the $m-$th Catalan number. The number of standard tableaux of truncated straight shape $\underbrace{(n, n, \ldots, n)}_{m} \setminus \delta_k$ (assume $n \le m$), is

$$(1) \quad (mn - \binom{k+1}{2})! \times \frac{f_{(n-k-1)^m}}{(m(n-k-1))!} \times \frac{g_{(m,m-1,\ldots,m-k)}}{((k+1)m - \binom{k+1}{2})!} \frac{E_1(k+1, m, n-k-1)}{E_1(k+1, m, 0)},$$

where

$$E_1(r,p,s) = \begin{cases} \displaystyle\prod_{r<l<2p-r+2} \frac{1}{(l+2s)^{r/2}} \prod_{2 \le l \le r} \frac{1}{((l+2s)(2p-l+2+2s))^{\lfloor l/2 \rfloor}}, & r \text{ even}, \\ \displaystyle\frac{((r-1)/2+s)!}{(p-(r-1)/2+s)!} E_1(r-1,p,s), & r \text{ odd}. \end{cases}$$

The proofs of these formulas go through several steps of combinatorial interpretations. We interpret the number of truncated SYTs as a polytope volume and evaluate it as a limit of a generating function. We form a bijection between these specific truncated tableaux and certain skew SSYTs with restrictions on their number of rows. We then evaluate the generating function as a sum of certain Schur functions and find formulas for these sums as complex integrals. In evaluating the limit we also use the RSK correspondence. This chapter is based on our work in [**Pan10b**].

CHAPTER 1

Background

In this chapter we state and explain the objects and theory which will serve as basis and toolkit for our results. The theorems we cite and their proofs can be found in [**Sta99**], [**Mac95**] and [**Ful97**].

1. Definitions

We will now define the basic combinatorial objects which the current research relies on.

Definition 1.1. An integer partition λ of $n \in \mathbb{N}$, denoted $\lambda \vdash n$, is a sequence of positive integers $\lambda = (\lambda_1, \lambda_2, \ldots)$, such that

- $\lambda_i \geq \lambda_{i+1}$ for every i,
- $\lambda_1 + \lambda_2 + \cdots = n$.

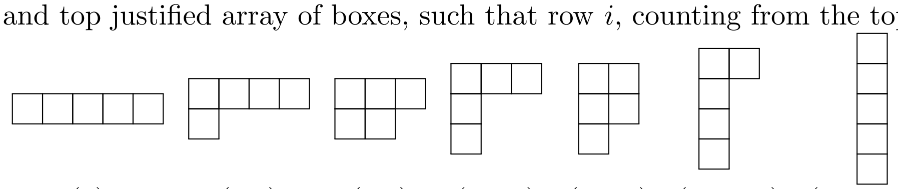
We refer to n as the size of λ, denoted by $|\lambda| = \lambda_1 + \cdots$. The number of nonzero parts in λ is called **the length** of λ and is denoted by $l(\lambda)$.

For example, the integer partitions of $n = 5$ are

$$(5), (4,1), (3,2), (3,1,1), (2,2,1), (2,1,1,1), (1,1,1,1,1) = (1^5)$$

and $l((2,1,1,1)) = 4$. We will use the following abbreviations for some special partitions: $(a^b) = (\underbrace{a, a, \ldots, a}_{b})$ and $\delta_n = (n, n-1, n-2, \ldots, 2, 1)$.

An integer partition can be represented graphically by its **Young diagram**, that is a left and top justified array of boxes, such that row i, counting from the top, has λ_i boxes.



(5) (4,1) (3,2) (3,1,1) (2,2,1) (2,1,1,1) (1,1,1,1,1)

Filling the boxes with nonnegative integers according to certain rules we obtain the central objects of algebraic combinatorics, namely Young tableaux and plane partitions.

Definition 1.2. A **semi-standard Young tableau (SSYT)** of shape λ is a filling of the Young diagram of λ with positive integers, such that the entries increase strictly down columns and weakly along rows left to right. For $\lambda \vdash n$ we define a **standard Young tableau (SYT)** as an SSYT of shape λ where each of the integers from 1 to n appears exactly once. A **plane partition (PP)** of shape λ is a filling of the Young diagram of λ with nonnegative integers, such that the entries decrease weakly down columns and along rows left to right.

The following figures represent, respectively, an SSYT, an SYT and a plane partition of shape $\lambda = (6, 5, 3, 2)$:

$$\begin{array}{cccccc} 1&1&2&3&3&5\\ 2&3&3&4&6&\\ 4&5&6&&&\\ 6&6&&&& \end{array} \qquad \begin{array}{cccccc} 1&3&5&6&11&15\\ 2&7&8&12&14&\\ 4&9&16&&&\\ 10&13&&&& \end{array} \qquad \begin{array}{cccccc} 6&6&5&3&1&0\\ 6&4&4&2&1&\\ 5&3&2&&&\\ 3&0&&&& \end{array}$$

A generalization of SSYTs are the **skew** SSYTs.

Definition 1.3. A **skew** Young diagram of shape λ/μ is the diagram complementary to the Young diagram of μ inside the Young diagram of λ, i.e. the boxes which belong to the diagram of λ but not to the diagram of μ. A semi-standard Young tableau of shape λ/μ is a filling of the skew Young diagram of λ/μ with positive integers, such that the entries increase weakly along rows and strictly down columns.

For example, if $\lambda = (6, 5, 3, 2)$ and $\mu = (4, 2, 1)$, then

$$\begin{array}{cccccc} &&&&\square&\square\\ &&\square&\square&\square&\\ &\square&\square&&&\\ \square&\square&&&& \end{array} \quad \text{and} \quad \begin{array}{cccccc} &&&&2&3\\ &&1&2&2&3\\ &2&4&&&\\ 2&&&&& \end{array}$$

are the Young diagram of λ/μ and a skew SSYT of shape λ/μ.

In dealing with these objects we will introduce a coordinate system where rows are indexed top to bottom by $1, 2, \ldots$, columns are indexed left to right and the entry in a tableau or plane partition T at row i and column j will be denoted by $T[i, j]$. If the multiset of entries of an SSYT consists of α_i entries equal to i we will say that the sequence $\alpha = (\alpha_1, \alpha_2, \ldots)$ is the **type** of this SSYT.

Standard Young tableaux of a certain shape, $\lambda \vdash n$, index a basis for the irreducible representation $\mathbb{S}^\lambda$ (so called Specht module) of the symmetric group on n letters, S_n. Similarly, the semi-standard Young tableaux of a given shape λ are directly related to the irreducible polynomial representations of $GL(V)$, these are V^λ and $V^\lambda = V^{\otimes n} \otimes_{\mathbb{C}[S_n]} \mathbb{S}^\lambda$. This correspondence is best understood through the characters of the V^λs, the Schur functions s_λ.

Definition 1.4. The Schur function $s_\lambda(x)$ is the trace of the image the diagonal matrix $\mathrm{diag}(x_1, \ldots, x_m)$ of $GL(\mathbb{C}^m)$ in $GL(V_\lambda)$. Equivalently, it is

$$s_\lambda(x_1, \ldots, x_m) = \sum_{T\text{: SSYT, } \mathrm{sh}(T)=\lambda} x^T,$$

where the sum goes over all SSYTs T of shape λ and entries in $[1, \ldots, m]$ and $x^T = \prod x_{T[i,j]}$ is the product of all entries in T where i is replaced with x_i. The Schur functions corresponding to a skew shape are defined similarly, namely

$$s_{\lambda/\mu}(x_1, \ldots, x_m) = \sum_{T\text{: skew SSYT,}\mathrm{sh}(T)=\lambda/\mu} x^T.$$

For example, we have that with $\lambda = (2, 2)$ the corresponding Schur function of 3 variables is

$$s_{\begin{array}{|c|c|}\hline \ & \ \\ \hline \ & \ \\ \hline\end{array}}(x_1, x_2, x_3) = \underset{\begin{array}{|c|c|}\hline 1&1\\ \hline 2&2\\ \hline\end{array}}{x_1^2 x_2^2} + \underset{\begin{array}{|c|c|}\hline 1&1\\ \hline 3&3\\ \hline\end{array}}{x_1^2 x_3^2} + \underset{\begin{array}{|c|c|}\hline 2&2\\ \hline 3&3\\ \hline\end{array}}{x_2^2 x_3^2} + \underset{\begin{array}{|c|c|}\hline 1&1\\ \hline 2&3\\ \hline\end{array}}{x_1^2 x_2 x_3} + \underset{\begin{array}{|c|c|}\hline 1&2\\ \hline 2&3\\ \hline\end{array}}{x_1 x_2^2 x_3} + \underset{\begin{array}{|c|c|}\hline 1&2\\ \hline 3&3\\ \hline\end{array}}{x_1 x_2 x_3^2}.$$

Similarly, we have

$$s_{(2,1)}(x_1, x_2, x_3) = x_1^2x_2 + x_1x_2^2 + x_2^2x_3 + x_2x_3^2 + x_1^2x_3 + x_1x_3^2 + 2x_1x_2x_3.$$

For the skew shape $(3,2)/(1)$ we have

$$s_{(3,2)/(1)}(x_1, x_2) = \underset{\begin{array}{|c|c|c|}\cline{2-3} \multicolumn{1}{c|}{} & 1 & 1 \\ \hline 1 & 2 & \multicolumn{1}{|c}{} \\ \cline{1-2}\end{array}}{x_1^3x_2} + \underset{\begin{array}{|c|c|c|}\cline{2-3} \multicolumn{1}{c|}{} & 1 & 2 \\ \hline 1 & 2 & \multicolumn{1}{|c}{} \\ \cline{1-2}\end{array}}{x_1^2x_2^2} + \underset{\begin{array}{|c|c|c|}\cline{2-3} \multicolumn{1}{c|}{} & 1 & 1 \\ \hline 1 & 2 & \multicolumn{1}{|c}{} \\ \cline{1-2}\end{array}}{x_1^3x_2} + \underset{\begin{array}{|c|c|c|}\cline{2-3} \multicolumn{1}{c|}{} & 1 & 1 \\ \hline 2 & 2 & \multicolumn{1}{|c}{} \\ \cline{1-2}\end{array}}{x_1^2x_2^2} + \underset{\begin{array}{|c|c|c|}\cline{2-3} \multicolumn{1}{c|}{} & 1 & 2 \\ \hline 2 & 2 & \multicolumn{1}{|c}{} \\ \cline{1-2}\end{array}}{x_1x_2^3}.$$

From the first definition as a character it follows directly that $s_\lambda(x)$ is a symmetric function, i.e. $s_\lambda(x_1, \ldots, x_m) = s_\lambda(x_{\sigma(1)}, \ldots, x_{\sigma(m)})$ for every permutation $\sigma \in S_m$ of the variables.

The fact that a lot of the information about representations is contained in their characters, makes the study of Schur functions on their own an important subject. This is also the main motivation of the theory of symmetric functions. We will now define the other relevant symmetric functions.

Definition 1.5. The n-th **homogenous** symmetric function is defined as

$$h_n(x) = \sum_{i_1 \le i_2 \le \cdots \le i_n} x_{i_1}x_{i_2}\cdots x_{i_n}.$$

For any partition λ we define $h_\lambda = h_{\lambda_1}h_{\lambda_2}\cdots$. Similarly, the n-th **elementary** symmetric function is defined as

$$e_n(x) = \sum_{i_1 < i_2 < \cdots < i_n} x_{i_1}x_{i_2}\cdots x_{i_n},$$

and $e_\lambda = e_{\lambda_1}e_{\lambda_2}\cdots$. The n-th **power sum** symmetric function is defined as

$$p_n(x) = \sum_i x_i^n$$

and similarly to the other two we have $p_\lambda = p_{\lambda_1}p_{\lambda_2}\cdots$. Last, but not least, we have the **monomial** symmetric functions $m_\lambda(x)$ defined as the sum of all *distinct* monomials of the kind $x_{i_1}^{\lambda_1}x_{i_2}^{\lambda_2}x_{i_3}^{\lambda_3}\cdots$ with $i_j \neq i_k$ for $j \neq k$.

Straight from their definitions we have that $e_n(x) = s_{1^n}(x)$ and $h_n(x) = s_{(n)}(x)$. We have, for example, that

$$h_3(x_1, x_2) = x_1^3 + x_1^2x_2 + x_1x_2^2 + x_2^3, \quad e_3(x_1, x_2) = 0, \quad e_3(x_1, x_2, x_3) = x_1x_2x_3,$$
$$p_{(2,1)}(x_1, x_2, x_3) = (x_1^2 + x_2^2 + x_3^2)(x_1 + x_2 + x_3),$$
$$m_{(2,1,1)}(x_1, x_2, x_3) = x_1^2x_2x_3 + x_1x_2^2x_3 + x_1x_2x_3^2.$$

The projective representations of the symmetric group and GL_n also give rise to certain tableaux called **shifted tableaux**. Let λ be a **strict** partition, i.e. $\lambda_i > \lambda_{i+1}$ for all i. The **shifted diagram** of λ is an array of boxes, such that each row contains λ_i boxes and row $i+1$ starts one box to the right of row i. A **standard shifted tableaux** of shape λ, $|\lambda| = n$, is an assignment of numbers to the shifted diagram of λ, such that the numbers increase along rows and down columns and each of the integers $1, 2, \ldots, n$ appears exactly once. For example,

$$\begin{array}{|c|c|c|c|c|c|c|} \hline & & & & & & \\ \hline \multicolumn{1}{c|}{} & & & & & \multicolumn{2}{|c}{} \\ \cline{2-6} \multicolumn{2}{c|}{} & & & & \multicolumn{2}{|c}{} \\ \cline{3-6} \multicolumn{3}{c|}{} & & & \multicolumn{2}{|c}{} \\ \cline{4-5} \end{array} \quad \text{and } T = \begin{array}{|c|c|c|c|c|c|c|} \hline 1 & 2 & 4 & 5 & 7 & 10 & 14 \\ \hline \multicolumn{1}{c|}{} & 3 & 6 & 9 & 12 & 16 & \multicolumn{1}{|c}{} \\ \cline{2-6} \multicolumn{2}{c|}{} & 8 & 11 & 17 & 19 & \multicolumn{1}{|c}{} \\ \cline{3-6} \multicolumn{3}{c|}{} & 13 & 18 & \multicolumn{2}{|c}{} \\ \cline{4-5} \end{array}$$

are,respectively, the shifted diagram of the strict partition $\lambda = (7,5,4,2)$ and a shifted standard tableaux of that shape.

2. Facts and theorems

We will now state some facts about tablaux, Schur functions and their relationships with other objects, mostly permutations. These facts will play a key role in the rest of this thesis.

2.1. The Robinson-Schensted-Knuth correspondence. The most important fact is the existence of a bijection, called RSK, between pairs of same shape SSYT and matrices with nonnegative integer entries. RSK provides a purely combinatorial interpretation of various Schur functions equalities arising in representation theory. It also translates certain properties between tableaux and permutations.

Let $A = [a_{ij}]_{i=1,j=1}^{n,m}$ be an $n \times m$ integer matrix with nonnegative entries. Form the generalized permutation w_A corresponding to this matrix as a two line array as follows. Proceeding from top to bottom on the rows of A and from right to left write $\begin{pmatrix} j \\ i \end{pmatrix}$ a_{ij} number of times. For example, if $A = \begin{bmatrix} 0 & 2 & 1 & 3 \\ 2 & 0 & 2 & 1 \\ 1 & 4 & 0 & 1 \end{bmatrix}$, then the first row gives $\begin{pmatrix} 2 & 2 & 3 & 4 & 4 & 4 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$. We then form w_A by writing these arrays one after the other for every row starting with the first one. For A in our example we obtain

$$w_A = \left(\begin{array}{ccccccccccccccccc} 2 & 2 & 3 & 4 & 4 & 4 & 1 & 1 & 3 & 3 & 4 & 1 & 2 & 2 & 2 & 2 & 4 \\ 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 3 & 3 & 3 & 3 & 3 & 3 \end{array} \right).$$

The correspondence RSK is actually an algorithm which inductively builds a pair of same shape tableaux from the two line array w_A. We will now describe this inductive step.

Let (P, Q) be two SSYTs of equal shapes. We add to them the pair of numbers a, b in the same column of w_A $\begin{pmatrix} a \\ b \end{pmatrix}$ as follows. We first add the number a into the SSYT P by a procedure called **insertion**. Find the smallest entry in the first row of P which is greater than a, call it a_1. If no such entry exists then just add a in its own box at the end of this row. Otherwise put a into the box of a_1 (we say a bumps a_1) and proceed by inserting a_1 into the next row of P. Let a_2 be the smallest entry in the second row that is greater than a_1. If no such exists then add a box with a_1 to the end of the second row, otherwise a_1 bumps a_2 and we insert a_2 into the next (third) row of P until no longer possible, i.e. we add a box with a_k to the end of the $k+1$st row (could be empty). At this point, since the shapes of P and Q were equal, we can add a box to Q at the same place where the box with a_k in P was added. Write b in the new box of Q, we call this **recording**.

We will illustrate this procedure by adding $\begin{pmatrix} 3 \\ 6 \end{pmatrix}$ to

$$(P, Q) = \left(\begin{array}{|c|c|c|c|c|c|c|} \hline 1 & 1 & 2 & 4 & 5 & 5 & 6 \\ \hline 2 & 3 & 3 & 5 & 6 \\ \cline{1-5} 4 & 4 & 6 \\ \cline{1-3} 5 \\ \cline{1-1} \end{array} , \begin{array}{|c|c|c|c|c|c|c|} \hline 1 & 3 & 3 & 4 & 4 & 5 & 5 \\ \hline 3 & 4 & 5 & 6 & 6 \\ \cline{1-5} 5 & 5 & 6 \\ \cline{1-3} 6 \\ \cline{1-1} \end{array} \right).$$

We have that $a = 3$ bumps $a_1 = 4$ from the first row, $a_1 = 4$ bumps the first 5 from the second row, so $a_2 = 5$. Then $a_2 = 5$ bumps $a_3 = 6$ from the third row and $a_3 = 6$ gets inserted at the end of the fourth row. At the same place we add $b = 6$ to Q. This insertion path is illustrated by the highlighted boxes in the newly obtained (P', Q'):

$$P = \begin{ytableau} 1 & 1 & 2 & *(gray) 4 & 5 & 5 & 6 \\ 2 & 3 & 3 & *(gray) 5 & 6 \\ 4 & 4 & *(gray) 6 \\ 5 \end{ytableau} \quad \leftarrow 3 \Longrightarrow \quad \begin{ytableau} 1 & 1 & 2 & *(gray) 3 & 5 & 5 & 6 \\ 2 & 3 & 3 & *(gray) 4 & 6 \\ 4 & 4 & *(gray) 5 \\ 5 & *(gray) 6 \end{ytableau} = P'$$

$$Q' = \begin{ytableau} 1 & 3 & 3 & 4 & 4 & 5 & 5 \\ 3 & 4 & 5 & 6 & 6 \\ 5 & 5 & 6 \\ 6 & *(gray) 6 \end{ytableau}$$

Definition 1.6. Let A be an integer matrix with nonnegative entries and let w_A be its corresponding double array. The Robinson-Schensted-Knuth (RSK) algorithm assigns to A the pair of tableaux $\mathrm{RSK}(A) = (P, Q)$ obtained by successively adding the columns of w_A via the insertion and recording procedure described above starting from the pair of empty tableaux $(\emptyset, \emptyset)$.

Again, this is best illustrated by an example. Let $A = \begin{pmatrix} 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix}$, then

$$w_A = \begin{pmatrix} 3 & 5 & 2 & 4 & 4 & 1 & 4 & 5 & 2 & 3 \\ 1 & 1 & 2 & 2 & 2 & 3 & 3 & 3 & 4 & 4 \end{pmatrix}.$$

We then add the pairs of numbers from the columns successively as follows:

$$(\emptyset, \emptyset) \leftarrow \begin{pmatrix} 3 \\ 1 \end{pmatrix} \qquad \left(\begin{ytableau} 3 \end{ytableau}, \begin{ytableau} 1 \end{ytableau} \right) \leftarrow \begin{pmatrix} 5 \\ 1 \end{pmatrix}$$

$$\left(\begin{ytableau} 3 & 5 \end{ytableau}, \begin{ytableau} 1 & 1 \end{ytableau} \right) \leftarrow \begin{pmatrix} 2 \\ 2 \end{pmatrix} \qquad \left(\begin{ytableau} 2 & 5 \\ 3 \end{ytableau}, \begin{ytableau} 1 & 1 \\ 2 \end{ytableau} \right) \leftarrow \begin{pmatrix} 4 \\ 2 \end{pmatrix}$$

$$\left(\begin{ytableau} 2 & 4 \\ 3 & 5 \end{ytableau}, \begin{ytableau} 1 & 1 \\ 2 & 2 \end{ytableau} \right) \leftarrow \begin{pmatrix} 4 \\ 2 \end{pmatrix} \qquad \left(\begin{ytableau} 2 & 4 & 4 \\ 3 & 5 \end{ytableau}, \begin{ytableau} 1 & 1 & 2 \\ 2 & 2 \end{ytableau} \right) \leftarrow \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$$\left(\begin{ytableau} 1 & 4 & 4 \\ 2 & 5 \\ 3 \end{ytableau}, \begin{ytableau} 1 & 1 & 2 \\ 2 & 2 \\ 3 \end{ytableau} \right) \leftarrow \begin{pmatrix} 4 \\ 3 \end{pmatrix} \qquad \left(\begin{ytableau} 1 & 4 & 4 & 4 \\ 2 & 5 \\ 3 \end{ytableau}, \begin{ytableau} 1 & 1 & 2 & 3 \\ 2 & 2 \\ 3 \end{ytableau} \right) \leftarrow \begin{pmatrix} 5 \\ 3 \end{pmatrix}$$

$$\left(\begin{ytableau} 1 & 4 & 4 & 4 & 5 \\ 2 & 5 \\ 3 \end{ytableau}, \begin{ytableau} 1 & 1 & 2 & 3 & 3 \\ 2 & 2 \\ 3 \end{ytableau} \right) \leftarrow \begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

$$\left(\begin{ytableau} 1 & 2 & 4 & 4 & 5 \\ 2 & 4 \\ 3 & 5 \end{ytableau}, \begin{ytableau} 1 & 1 & 2 & 3 & 3 \\ 2 & 2 \\ 3 & 4 \end{ytableau} \right) \leftarrow \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

$$\left(\begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 3 & 4 & 5 \\ \hline 2 & 4 & 4 \\ \cline{1-3} 3 & 5 \\ \cline{1-2} \end{array}, \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 2 & 3 & 3 \\ \hline 2 & 2 & 4 \\ \cline{1-3} 3 & 4 \\ \cline{1-2} \end{array} \right) = (P, Q).$$

Noticing that insertion paths move weakly to the left we have that the resulting P is an SSYT. It can be shown that the insertion path of the greater element lies strictly to the right of the insertion path of the previous smaller element. By definition, the top elements in the columns of w_A with equal second entries appear in increasing order. Hence, if the newly added boxes in Q contain equal numbers, they will not be on top of each other and so Qs columns are strictly increasing. This also tells us that the last box added to Q is the rightmost one with the largest entry. Knowing that the insertion procedure can be reversed starting from any pair of same-shape tableaux. Thus we have the following crucial theorem about RSK.

Theorem 1.7 (Robinson-Schensted-Knuth). The algorithm RSK is a bijection between matrices with nonnegative integer entries and pairs of same-shape SSYTs. More specifically, it is a bijection between matrices with row sums $\alpha = (\alpha_1, \alpha_2, \dots)$ and column sums $\beta = (\beta_1, \beta_2, \dots)$ and pairs of SSYT (P, Q), such that P is of type α and Q is of type β. Restricting to permutation matrices it follows that RSK is a bijection between permutations in S_n and pairs of same shape SYTs with n entries each.

RSK has many remarkable properties, some of which we will mention later on. To begin with, we have the following.

Theorem 1.8 (Symmetry of the RSK). Let A be a nonnegative integer matrix and $\mathrm{RSK}(A) = (P, Q)$ its image under RSK. Then if A^T is the transpose of A, we have that $\mathrm{RSK}(A^T) = (Q, P)$.

2.2. Permutations. The correspondence between permutations and pairs of SYT is particularly rich. While it is a special case of the already described RSK, we will give an example of it. Let $w = 634795182$ be a permutation in S_9, its corresponding permutation matrix and double array are, respectively

$$A = \begin{bmatrix} 0&0&0&0&0&1&0&0&0 \\ 0&0&1&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&0&1 \\ 0&0&0&0&1&0&0&0&0 \\ 1&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&1&0 \\ 0&1&0&0&0&0&0&0&0 \end{bmatrix} \text{ and } w_A = \begin{pmatrix} 6&3&4&7&9&5&1&8&2 \\ 1&2&3&4&5&6&7&8&9 \end{pmatrix};$$

the first row of w_A is equal to w and the second row is the identity permutation. We have that the image under RSK of w is

$$\mathrm{RSK}(w) = \mathrm{RSK}(A) = \left(\underbrace{\begin{array}{|c|c|c|c|} \hline 1 & 2 & 5 & 8 \\ \hline 3 & 4 & 9 \\ \cline{1-3} 6 & 7 \\ \cline{1-2} \end{array}}_{P}, \underbrace{\begin{array}{|c|c|c|c|} \hline 1 & 3 & 4 & 5 \\ \hline 2 & 6 & 8 \\ \cline{1-3} 7 & 9 \\ \cline{1-2} \end{array}}_{Q} \right).$$

A lot of important properties and statistics of a permutation translate to their images under RSK. One of them is the **descent set** of a permutation, defined as

$$D(w) = \{i \quad : \quad w_i > w_{i+1}\}.$$

For example,

$$D(634795182) = \{1, 5, 6, 8\}, \quad D(12 \ldots n) = \emptyset, \quad D(n(n-1)\ldots 1) = \{1, 2, \ldots, n-1\}.$$

The analogous set for an SYT T, $D(T)$, is defined as the set of i, such that $i+1$ appears in a lower row than i in T. For the tableaux above we have $D(P) = \{2, 5, 8\}$ and $D(Q) = \{1, 5, 6, 8\}$. Notice that when w_{i+1} is inserted via RSK into P right after a larger $w_i > w_{i+1}$, its insertion path will lie strictly to the left of the insertion path of w_i. The last box of this path will then be in a column to the left of the column of the last box of the insertion path of w_i, then the box with $i+1$ in Q will be to the left of the box with i in Q and so necessarily in a lower column. Similarly, if $w_i < w_{i+1}$, then i will be to the left of $i+1$ in Q and so we have the following theorem.

Theorem 1.9. Let $(P, Q) = \mathrm{RSK}(w)$. Then $D(w) = D(Q)$.

The **major index** of a permutation w is defined as the sum of position where descents happen, i.e.

$$\mathrm{maj}(w) = \sum_{i: i \in D(w)} i.$$

For example, $\mathrm{maj}(123\ldots n) = 0$ and $\mathrm{maj}(n(n-1)\ldots 1) = \frac{n(n-1)}{2}$. Analogously we define the major index of an SYT T by $\mathrm{maj}(T) = \sum_{i:i\in D(T)} i$ and by the last theorem we immediately have that $\mathrm{maj}(Q) = \mathrm{maj}(w)$.

An increasing subsequence of a permutation w is a sequence $w_{i_1} w_{i_2} \ldots w_{i_k}$ of entries of w, such that $i_j < i_{j+1}$ and $w_{i_j} < w_{i_{j+1}}$. For example, in $w = 634795182$ we have that 348 is an increasing subsequence. A **longest increasing subsequence** of a permutation w is an increasing subsequence of maximal possible length k, i.e. there is no other increasing subsequence of length larger than k. Denote the length of the longest increasing subsequence of w by $\mathrm{is}(w)$. In our example, $\mathrm{is}(w) = 4$ and a longest increasing subsequence is 3458. Notice that $\mathrm{sh}(P) = (4, 3, 2)$ and $\lambda_1 = \mathrm{is}(w)$, which actually holds in general. Similarly, we define the longest decreasing subsequence of a permutation and define $\mathrm{ds}(w)$ to be its length. For example, in w we have that 642 is a longest decreasing subsequence, so $\mathrm{ds}(w) = 3$ and $l(\lambda) = 3$ also.

Theorem 1.10 (Schensted). If $\mathrm{RSK}(w) = (P, Q)$ and $\mathrm{sh}(P) = \mathrm{sh}(Q) = \lambda$, then $\mathrm{is}(w) = \lambda_1$. Similarly $\mathrm{ds}(w) = l(\lambda)$. The analogous statements hold for double arrays w_A where the condition increasing is replaced with weakly increasing, decreasing remains strictly decreasing and the subsequences are subsequences of the top row of the array.

A more general statement gives the entire shape λ.

Theorem 1.11 (Greene). If $\mathrm{RSK}(w) = (P, Q)$ and $\lambda = \mathrm{sh}(P)$, then for every k the sum $\lambda_1 + \lambda_2 + \cdots + \lambda_k$ is equal to the maximal possible total length of k disjoint increasing subsequences of w.

For example, when $w = 634795182$ we have that $\lambda = (4, 3, 2)$. When $k = 1$ one increasing subsequence of maximal length is 3478 and $\lambda_1 = 4$. For $k = 2$ the maximal possible total length of 2 disjoint increasing subsequences is 7, achieved when they are

$\{3458, 679\}$. Finally, for $k = 3$ the whole permutation can be partitioned into 3 increasing subsequences $\{3458, 679, 12\}$. As another example, let $w' = 51267834$, then

$$\mathrm{RSK}(w') = \left(\begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 3 & 4 & 8 \\ \hline 5 & 6 & 7 \\ \cline{1-3} \end{array} , \begin{array}{|c|c|c|c|c|} \hline 1 & 3 & 4 & 5 & 6 \\ \hline 2 & 7 & 8 \\ \cline{1-3} \end{array} \right).$$

For $k = 1$, a longest increasing subsequence is 12678 and $\lambda_1 = 5$; for $k = 2$ there is a set of 2 disjoint increasing subsequences of maximal total length 8, namely $\{1234, 5678\}$.

2.3. Symmetric function identities. We will now review some facts about symmetric functions. Each class of symmetric functions (homogenous, elementary, power sum, monomial and Schur) forms a basis of the algebra of symmetric polynomials, Λ. While the origin of the symmetric function theory stands in representation theory they have also purely combinatorial proofs and interpretations, mostly due to the RSK correspondence.

Straight from the definitions of the elementary and homogenous symmetric functions we obtain the following formal identities.

Proposition 1.12. Let $x = (x_1, x_2, \ldots)$. Then

$$\sum_{n \geq 0} h_n(x) t^n = \prod_{i \geq 1} \frac{1}{1 - x_i t}, \tag{2}$$

$$\sum_{n \geq 0} e_n(x) t^n = \prod_{i \geq 1} (1 + x_i t). \tag{3}$$

For a nonnegative integer matrix $A = [a_{i,j}]$, assign x_i to column i and y_j to row j and let $(xy)^A = \prod_{ij} (x_i y_j)^{a_{i,j}}$. If $\mathrm{RSK}(A) = (P, Q)$, then $(xy)^A = x^P y^Q$ in the sense of $x^T = \prod x_{T[i,j]}$. Since RSK is a bijection between the set of all pairs of same-shaped SSYTs and nonnegative integer matrices, and since

$$\sum_A (xy)^A = \prod_{i,j} \sum (x_i y_j)^{a_{i,j}} = \prod_{i,j} \frac{1}{1 - x_i y_j},$$

the following main inequality follows immediately.

Theorem 1.13 (Cauchy's identity). Let $x = (x_1, x_2, \ldots)$ and $y = (y_1, y_2, \ldots)$, then the following formal identity holds

$$\sum_\lambda s_\lambda(x) s_\lambda(y) = \prod_{i,j} \frac{1}{1 - x_i y_j}.$$

By the symmetry of the RSK we have that if $A = A^T$ then $(P, Q) = \mathrm{RSK}(A) = \mathrm{RSK}(A^T) = (Q, P)$ and so $P = Q$. Thus RSK restricts to a bijection between symmetric matrices and single SSYTs. Analogously to Cauchy's identity we obtain an identity for sums of Schur functions.

Theorem 1.14. Let $x = (x_1, x_2, \ldots)$, then the following formal identity holds

$$\sum_\lambda s_\lambda(x) = \prod_i \frac{1}{1 - x_i} \prod_{i<j} \frac{1}{1 - x_i x_j}.$$

The classical definition for the Schur functions was given by Isaii Schur as a ratio of determinants.

THEOREM 1.15 (Determinantal formula for the Schur functions). Let $x = (x_1, \dots, x_n)$ and $a_\alpha = \det[x_i^{\alpha_j}]_{i,j=1}^n$. Then

$$s_\lambda(x) = \frac{a_{\lambda+\delta_{n-1}}(x)}{a_{\delta_{n-1}}(x)}.$$

It is worth noting that $a_{\delta_{n-1}}(x) = \det[x_i^{j-1}]_{i,j=1}^n$ is just the Vandermonde determinant and so $a_{\delta_{n-1}}(x) = \prod_{i<j}(x_i - x_j)$. As an example, let $\lambda = (2,1,1)$ and $n = 4$, then

$$s_{(2,1,1)}(x_1,x_2,x_3,x_4) = \frac{1}{\prod_{1\le i<j\le 4}(x_i - x_j)} \begin{vmatrix} x_1^5 & x_2^5 & x_3^5 & x_4^5 \\ x_1^3 & x_2^3 & x_3^3 & x_4^3 \\ x_1^2 & x_2^2 & x_3^2 & x_4^2 \\ 1 & 1 & 1 & 1 \end{vmatrix}.$$

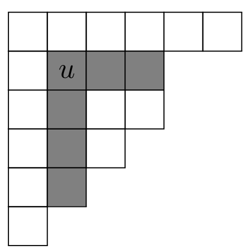
If we consider two sets of variables, x and y, we have the following identity involving skew Schur functions

$$s_\lambda(x,y) = \sum_\mu s_\mu(x) s_{\lambda/\mu}(y).$$

2.4. Tableaux enumeration. Another combinatorially interesting feature of Young tableaux is the fact that they possess nice enumerative properties. In particular, there is a simple closed-form product formula for the number of SYTs of a given shape.

DEFINITION 1.16. Let $D(\lambda)$ be the Young diagram of λ and let $u \in D$ be a box. The **hook** at u is defined as the set of boxes to the right of u in the same row, below u in the same column i and u itself. We denote by h_u the length of this hook, that is the number of boxes in the hook of u.

For example, if $\lambda = (5,4,4,3,2,1)$ and $u = [2,2]$, then the hook at u is



and $h_u = 6$.

THEOREM 1.17 (Hook-length formula of Frame, Robinson and Thrall.). The number of standard Young tableaux of shape λ, denoted by f^λ, is given by

$$f_\lambda = \frac{|\lambda|!}{\prod_{u\in D(\lambda)} h_u}.$$

For example, if $\lambda = (5,3,2)$ the hook lengths of all boxes in λ are given as

7	6	4	2	1
4	3	1		
2	1			

and the number of SYTs of shape λ is equal to

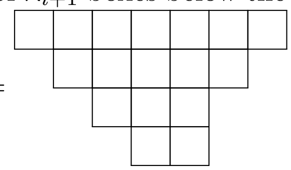
$$f_{(5,3,2)} = \frac{10!}{7\cdot 6\cdot 4\cdot 2\cdot 1\cdot 4\cdot 3\cdot 1\cdot 2\cdot 1} = 450.$$

The proof of this formula goes straightforwardly by induction. If $|\lambda| = n$, then for every tableau T, $\mathrm{sh}(T) = \lambda$, the largest entry n can be only in a corner box. Removing the box with n we obtain an SYT of size $n-1$. Thus

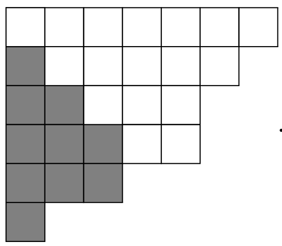
$$f_\lambda = \sum_{i:\lambda_i > \lambda_{i+1}} f_{(\lambda_1,\ldots,\lambda_i - 1,\lambda_{i+1},\ldots)}.$$

One checks that the same equation holds for the right-hand side of the hook-length formula.

A similar formula holds for standard tableaux of shifted shape. Let λ be a strict partition and u a box in its shifted diagram, $D_s(\lambda)$. Extend the diagram to a Young diagram $D'(\lambda)$ by drawing its reflection along the main diagonal, i.e. add a column of λ_{i+1} boxes below the first box in row i of $D_s(\lambda)$. For example, if $\lambda = (7,5,3,2)$, $D_s(\lambda) =$



and $D'(\lambda) =$



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Theorem 1.18. Let λ be a strict partition. The number of standard tableaux of shifted shape λ, denoted by g_λ, is given by

$$g_\lambda = \frac{|\lambda|!}{\prod_{u \in D_s(\lambda) \hookrightarrow D'(\lambda)} h_u},$$

where the product is over all boxes from the shifted diagram of λ and their hooks are considered in the extended Young diagram $D'(\lambda)$.

With $\lambda = (7,5,3,2)$ we have the relevant boxes with corresponding hooks as

12	10	9	7	6	3
	8	7	5	4	
		5	3	2	
			2	1	

.

The number of standard tableaux of this shape is then

$$g_{(7,5,3,2)} = \frac{17!}{12 \cdot 10 \cdot 9 \cdot 7 \cdot 6 \cdot 3 \cdot 8 \cdot 7 \cdot 5 \cdot 4 \cdot 5 \cdot 3 \cdot 2 \cdot 2 \cdot 1} = 38896.$$

CHAPTER 2

Bijective enumeration of permutations starting with a longest increasing subsequence

1. Introduction

In [**GG**], Adriano Garsia and Alain Goupil derived as a consequence of character polynomial calculations a simple formula for the enumeration of certain permutations. In his talk at the MIT Combinatorics Seminar [**Gar**], Garsia offered a $100 award for an 'elementary' proof of this formula. We give here such a proof of this formula and its q-analogue.

The object of interest will be the set of permutations in S_n whose first $n-k$ entries form a longest increasing subsequence as defined in 2.2.

Definition 2.1. Let $\Pi_{n,k} = \{w \in S_n | w_1 < w_2 < \cdots < w_{n-k}, \mathrm{is}(w) = n-k\}$.

The formula in question is the following theorem originally proven by A. Garsia and A.Goupil [**GG**].

Theorem 2.2. If $n \geq 2k$, the number of permutations in $\Pi_{n,k}$ is given by

$$\#\Pi_{n,k} = \sum_{r=0}^{k} (-1)^{k-r} \binom{k}{r} \frac{n!}{(n-r)!}.$$

This formula has a $q-$analogue, also due to Garsia and Goupil.

Theorem 2.3. For permutations in $\Pi_{n,k}$, if $n \geq 2k$, we have that

$$\sum_{w \in \Pi_{n,k}} q^{\mathrm{maj}(w^{-1})} = \sum_{r=0}^{k} (-1)^{k-r} \binom{k}{r} [n]_q \cdots [n-r+1]_q,$$

where $\mathrm{maj}(\sigma) = \sum_{i|\sigma_i > \sigma_{i+1}} i$ denotes the major index of a permutation, as in 2.2 and $[n]_q = \frac{1-q^n}{1-q}$.

In this chapter we will exhibit several bijections which will prove the above theorems. We will first define certain sets of permutations and pairs of tableaux which come in these bijections. We will then construct a relatively simple bijection showing a recurrence relation for the numbers $\#\Pi_{n,k}$. Using ideas from this bijection we will then construct a bijection proving Theorems 2.2 and 2.3 directly. We will also show a bijective proof which uses only permutations.

2. A few simpler sets and definitions.

We consider the RSK correspondence, as discussed in 2.1 and 2.2, between permutations and pairs of same-shape tableaux. For $w \in S_n$ we have $\mathrm{RSK}(w) = (P, Q)$, where P and Q are standard Young tableaux on $[n] = \{1, \ldots, n\}$ and of the same shape, with P the insertion tableau and Q the recording tableau of w.

Let $C_{n,s} = \{w \in S_n | w_1 < w_2 < \cdots < w_{n-s}\}$ be the set of all permutations on $[n]$ with their first $n-s$ entries forming an increasing sequence. A permutation in $C_{n,s}$ is bijectively determined by the choice of the first $n-s$ elements from $[n]$ in $\binom{n}{s}$ ways and the arrangement of the remaining s in $s!$ ways, so

$$\#C_{n,s} = \binom{n}{s} s! = \frac{n!}{(n-s)!}.$$

Let $C_{n,s}^{\mathrm{RSK}} = \mathrm{RSK}(C_{n,s})$. Its elements are precisely the pairs of same-shape SYTs (P,Q) such that the first row of Q starts with $1, 2, \ldots, n-s$: the first $n-s$ elements are increasing and so will be inserted in this order in the first row, thereby recording their positions $1, 2, \ldots, n-s$ in Q in the first row also.

Let also $\Pi_{n,s}^{\mathrm{RSK}} = \mathrm{RSK}(\Pi_{n,s})$. Its elements are pairs of SYTs (P,Q), such that, as with $C_{n,s}^{\mathrm{RSK}}$, the first row of Q starts with $1, 2, \ldots, n-s$. By Schensted's theorem, 1.10, the length of the first row in P and Q is the length of the longest increasing subsequence of w, denoted is(w) as in section 2.2. In the case of $\Pi_{n,s}$ we have is$(w) = n-s$, so the first row of Q is exactly $1, 2, \ldots, n-s$. That is, $\Pi_{n,s}^{\mathrm{RSK}}$ is the set of pairs of same-shape SYTs (P,Q), such that the first row of Q has length $n-s$ and elements $1, 2, \ldots, n-s$.

Finally, let $D_{n,k,s}$ be the set of pairs of same-shape tableaux (P,Q), where P is an SYT on $[n]$ and Q is a tableau filled with $[n]$, such that the first row of Q is $1, 2, \ldots, n-k, a_1, \ldots, a_s, b_1, \ldots$ where $a_1 > a_2 > \cdots > a_s$, $b_1 < b_2 < \cdots$ and the remaining elements of Q are increasing in rows and down columns. Thus Q without its first row is an SYT. Notice that when $s = 0$ we just have $D_{n,k,0} = C_{n,k}^{\mathrm{RSK}}$.

The three sets of pairs we defined are determined by their Q tableaux as shown below.

$$\begin{array}{|c|c|c|c|}\hline 1 & \cdots & n-s & \cdots \\ \hline & \cdots & & \\ \cline{1-3} \vdots & & \\ \cline{1-2}\end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & \cdots & n-s \\ \hline & \cdots & \\ \hline \vdots & & \\ \cline{1-2}\end{array} \qquad \begin{array}{|c|c|c|c|c|c|c|}\hline 1 & \cdots & n-k & a_1 & \cdots & a_s & \cdots \\ \hline & \cdots & & \\ \cline{1-3} \vdots & & \\ \cline{1-2}\end{array}$$

Q, for $(P,Q) \in C_{n,s}^{\mathrm{RSK}}$ $\qquad$ Q, for $(P,Q) \in \Pi_{n,s}^{\mathrm{RSK}}$ $\qquad$ Q, for $(P,Q) \in D_{n,k,s}$

3. A bijection.

We will first exhibit a simple bijection, which will give us a recurrence relation for the numbers $\#\Pi_{n,k}$ equivalent to Theorem 2.2.

We should remark that while the recurrence can be inverted to give the inclusion-exclusion form of Theorem 2.2, the bijection itself does not succumb to direct inversion. However, the ideas of this bijection will lead us to discover the necessary construction for Theorem 2.2.

Proposition 2.4. The number of permutations in $\Pi_{n,k}$ satisfies the following recurrence:

$$\sum_{s=0}^{k} \binom{k}{s} \#\Pi_{n,s} = \binom{n}{k} k!$$

Proof. Let $C_{n,k,s}^{\mathrm{RSK}}$ with $s \le k$ be the set of pairs of same-shape tableaux (P,Q), such that the length of their first rows is $n-k+s$ and the first row of Q starts with $1, 2, \ldots, n-k$;

clearly $C^{\mathrm{RSK}}_{n,k,s} \subset C^{\mathrm{RSK}}_{n,k}$. We have that

$$\bigcup_{s=0}^{k} C^{\mathrm{RSK}}_{n,k,s} = C^{\mathrm{RSK}}_{n,k}, \tag{4}$$

as $C^{\mathrm{RSK}}_{n,k}$ consists of the pairs (P,Q) with Q's first row starting with $1,\dots,n-k$ and if $n-k+s$ is this first row's length then $(P,Q) \in C^{\mathrm{RSK}}_{n,k,s}$.

There is a bijection $C^{\mathrm{RSK}}_{n,k,s} \leftrightarrow \Pi^{\mathrm{RSK}}_{n,k-s} \times \binom{[n-k+1,\dots,n]}{s}$ given as follows. If $(P,Q) \in C^{\mathrm{RSK}}_{n,k,s}$ and the first row of Q is $1,2,\dots,n-k,b_1,\dots,b_s$, let

$$f : [n-k+1,\dots,n] \setminus \{b_1,\dots,b_s\} \to [n-k+s+1,\dots,n]$$

be the order-preserving map. Let then Q' be the tableau obtained from Q by replacing every entry b not in the first row with $f(b)$ and the first row with $1,2,\dots,n-k,n-k+1,\dots,n-k+s$. Then Q' is an SYT, since f is order-preserving and so the rows and columns are still increasing, including the first row since its elements are smaller than any element below it. Then the bijection in question is $(P,Q) \leftrightarrow (P,Q',b_1,\dots,b_s)$. Conversely, if $b_1,\dots,b_s \in [n-k+1,\dots,n]$ (in increasing order) and $(P,Q') \in \Pi^{\mathrm{RSK}}_{n,k-s}$, then replace all entries b below the first row of Q' with $f^{-1}(b)$ and the the first row of Q' with $1,2,\dots,n-k,b_1,\dots,b_s$. We end up with a tableau Q, which is an SYT because: the entries below the first row preserve their order under f; and, since they are at most $k \le n-k$, they are all below the first $n-k$ entries of the first row of Q (which are $1,2,\dots,n-k$, and thus smaller).

So we have that $\#C^{\mathrm{RSK}}_{n,k,s} = \binom{k}{s}\#\Pi^{\mathrm{RSK}}_{n,k-s}$ and substituting this into (4) gives us the statement of the lemma. □

4. Proofs of the theorems.

We will prove Theorems 2.2 and 2.3 by exhibiting an inclusion-exclusion relation between the sets $\Pi^{\mathrm{RSK}}_{n,k}$ and $D_{n,k,s}$ for $s = 0,1,\dots,k$.

of Theorem 2.2. First of all, if $n \ge 2k$ we have a bijection $D_{n,k,s} \leftrightarrow C^{\mathrm{RSK}}_{n,k-s} \times \binom{[n-k+1,\dots,n]}{s}$, where the correspondence is $(P,Q) \leftrightarrow (P,Q') \times \{a_1,\dots,a_s\}$ given as follows.

Consider the order-preserving bijection

$$f : [n-k+1,\dots,n] \setminus \{a_1,\dots,a_s\} \to [n-k+s+1,\dots,n].$$

Then Q' is obtained from Q by replacing $a_1,\dots,a_s$ in the first row with $n-k+1,\dots,n-k+s$ and every other element b in Q, $b > n-k$ and $\neq a_i$, with $f(b)$. The first $n-k$ elements in the first row remain $1,2,\dots,n-k$. Since f is order-preserving, Q' without its first row remains an SYT (the inequalities within rows and columns are preserved). Since also $n-k \ge k$, we have that the second row of Q (and Q') has length less than or equal to k and hence $n-k$, so since the elements above the second row are among $1,2,\dots,n-k$ they are smaller than any element in the second row (which are all from $[n-k+1,\dots,n]$). Also, the remaining first row of Q' is increasing since it starts with $1,2,\dots,n-k,n-k+1,\dots,n-k+s$ and its remaining elements are in $[n-k+s+1,\dots,n]$ and are increasing because f is order-preserving. Hence Q' is an SYT with first row starting with $1,\dots,n-k+s$, so $(P,Q') \in C^{\mathrm{RSK}}_{n,k-s}$.

Conversely, if $(P,Q') \in C^{\mathrm{RSK}}_{n,k-s}$ and $\{a_1,\dots,a_s\} \in [n-k+1,\dots,n]$ with $a_1 > a_2 \cdots > a_s$, then we obtain Q from Q' by replacing $n-k+1,\dots,n = k+s$ with $a_1,\dots,a_s$ and the remaining elements $b > n-k$ with $f^{-1}(b)$, again preserving their order, and so $(P,Q) \in D_{n,k,s}$.

Hence, in particular,

$$\#D_{n,k,s} = \binom{k}{s}\#C^{\mathrm{RSK}}_{n,k-s} = \binom{k}{s}\#C_{n,k-s} = \binom{k}{s}\frac{n!}{(n-k+s)!}. \tag{5}$$

We have that $\Pi^{\mathrm{RSK}}_{n,k} \subset C^{\mathrm{RSK}}_{n,k}$ since $\Pi_{n,k} \subset C_{n,k}$. Then $C^{\mathrm{RSK}}_{n,k} \setminus \Pi^{\mathrm{RSK}}_{n,k}$ is the set of pairs of SYTs (P,Q) for which the first row of Q is $1,2,\dots,n-k,a_1,\dots$ for at least one a_1. So $E_{n,k,1} = C^{\mathrm{RSK}}_{n,k} \setminus \Pi^{\mathrm{RSK}}_{n,k}$ is then a subset of $D_{n,k,1}$. The remaining elements in $D_{n,k,1}$, that is $E_{n,k,2} = D_{n,k,1} \setminus E_{n,k,1}$, would be exactly the ones for which Q is not an SYT, which can happen only when the first row of Q is $1,2,\dots,n-k,a_1 > a_2,\dots$. These are now a subset of $D_{n,k,2}$ and by the same argument, we haven't included the pairs for which the first row of Q is $1,2,\dots,n-k,a_1 > a_2 > a_3,\dots$, which are now in $D_{n,k,3}$. Continuing in this way, if $E_{n,k,l+1} = D_{n,k,l} \setminus E_{n,k,l}$, we have that $E_{n,k,l}$ is the set of $(P,Q) \in D_{n,k,l}$, such that the first row of Q is $1,2,\dots,n-k,a_1 > \cdots > a_l < \cdots$. Then $E_{n,k,l+1}$ is the subset of $D_{n,k,l}$, for which the element after a_l is smaller than a_l and so $E_{n,k,l+1} \subset D_{n,k,l+1}$. Finally, $E_{n,k,k} = D_{n,k,k}$ and $E_{n,k,k+1} = \varnothing$. We then have

$$\Pi^{\mathrm{RSK}}_{n,k} = C^{\mathrm{RSK}}_{n,k} \setminus \left(D_{n,k,1} \setminus \left(D_{n,k,2} \setminus \cdots \setminus \left(D_{n,k,k-1} \setminus D_{n,k,k}\right)\right)\right), \tag{6}$$

or in terms of number of elements, applying (5), we get

$$\begin{aligned}\#\Pi_{n,k} = \#\Pi^{\mathrm{RSK}}_{n,k} &= \frac{n!}{(n-k)!} - \binom{k}{1}\frac{n!}{(n-k+1)!} + \binom{k}{2}\frac{n!}{(n-k+2)!} + \dots \\ &= \sum_{i=0}^{k}(-1)^i\binom{k}{i}\frac{n!}{(n-k+i)!},\end{aligned}$$

which is what we needed to prove. □

Theorem 2.3 will follow directly from (6) after we prove the following lemma.

Lemma 2.5. We have that

$$\sum_{w\in C_{n,s}} q^{\mathrm{maj}(w^{-1})} = [n]_q \dots [n-s+1]_q,$$

where $\mathrm{maj}(\sigma) = \sum_{i:\sigma_i>\sigma_{i+1}} i$ denotes the major index of σ.

Proof. We will first prove this using the elegant theory of $P-$partitions, as described in section 4.5 of [**Sta97**]. Let P be the poset on $[n]$ consisting of a chain $1,\dots,n-s$ and the single points $n-s+1,\dots,n$. Then $\sigma \in C_{n,s}$ if and only if $\sigma^{-1} \in \mathcal{L}(P)$, *i.e.* $\sigma : P \to [n]$ is a linear extension of P. We have that

$$\sum_{w\in C_{n,s}} q^{\mathrm{maj}(w^{-1})} = \sum_{w\in\mathcal{L}(P)} q^{\mathrm{maj}(w)};$$

denote this expression by $W_P(q)$. By theorem 4.5.8 from [**Sta97**] on P-partitions, we have that

$$W_P(q) = G_P(q)(1-q)\dots(1-q^n), \tag{7}$$

where $G_P(q) = \sum_{m\ge0} a(m)q^m$ with $a(m)$ denoting the number of P-partitions of m. That is, $a(m)$ is the number of order-reversing maps $\tau : P \to \mathbb{N}$, such that $\sum_{i\in P}\tau(i) = m$. In our particular case, these correspond to sequences $\tau(1),\tau(2),\dots$, whose sum is m and whose first $n-s$ elements are non-increasing. These correspond to partitions of at most $n-s$ parts and a sequence of s nonnegative integers, which add up to m. The partitions with at most $n-s$ parts are in bijection with the partitions with largest part $n-s$ (by transposing their Ferrers diagrams). The generating function for the latter is given by a well-known formula of Euler and is equal to

$$\frac{1}{(1-q)(1-q^2)\cdots(1-q^{n-s})}.$$

The generating function for the number of sequences of s nonnegative integers with a given sum is trivially $1/(1-q)^s$ and so we have that

$$G_P(q) = \frac{1}{(1-q)(1-q^2)\cdots(1-q^{n-s})}\frac{1}{(1-q)^s}.$$

After substitution in (7) we obtain the statement of the lemma.

We will now exhibit a longer, but more explicit proof of this result, without the use of the theory of $P-$partitions. We will use a bijection due to Foata, [**Foa68**], $F : S_n \to S_n$, which proves the equidistribution of the major index and the number of inversions of a permutation.

This bijection is defined inductively as follows. We define a sequence of permutations $\gamma_1, \dots, \gamma_n$, such that γ_k is a permutation of the first k letters in w, i.e. of $\{w_1, \dots, w_k\}$. First, $\gamma_1 = w_1$. To obtain γ_{k+1} from γ_k do the following. If $w_k < w_{k+1}$(respectively $w_k > w_{k+1}$), then put a $|$ after every letter a of γ_k, for which $a < w_{k+1}$(respectively $a > w_{k+1}$). These $|$ divide γ_k into compartments. Shift cyclically the letters in each compartment one unit to the right and add w_{k+1} at the end, this gives γ_{k+1}. For example, if $w = 415236$ we obtain $\gamma_1 = 4$, $\gamma_2 = 41$, $\gamma_3 = 4|1|5 = 415$, $\gamma_4 = 4|15|2 = 4512$, $\gamma_5 = 41|52|3 = 14253$, $\gamma_6 = 1|4|2|5|3|6 = 142536$. Set $F(w) = \gamma_n$.

The number of inversions of a permutation w is defined by $\text{inv}(w) = \#\{(i,j) : \quad i < j, \quad w_i > w_j\}$. It is a main corollary of Foata's bijection, [**Foa68**], that $\text{maj}(w) = \text{inv}(F(w))$.

Consider now the restriction of F to the subset of S_n $\{w|w^{-1} \in C_{n,s}$, which we denote by $C_{n,k}^{-1}$. Note that $C_{n,s}^{-1} = \{w|w^{-1}(1) < w^{-1}(2) < \cdots < w^{-1}(n-s)\}$, i.e. it is the set of all permutations, for which $1, 2, \dots, n-s$ appear in w in this order. Let $w \in C_{n,s}^{-1}$, we will show by induction on j that $1, 2, \dots, n-s$ appear (if at all) in increasing order in γ_j, where $\gamma_n = F(w)$. Let l be the largest number from $\{1, \dots, n-s\}$ to appear in γ_j. If $w_{j+1} > w_j$, then since $w^{-1}(l) \leq j$, we have that $l < w_{j+1}$. The delimiters $|$ will then be placed after each $1, \dots, l$ in γ_j. Thus $1, \dots, l$ will be in different compartments and the cyclic shifting will not change their relative order. Finally, w_{j+1} is either $l+1$, or greater than $n-s$, so adding it at the end of γ_j will produce a γ_{j+1} with the elements from $1, \dots, n-k$ in increasing order. If now, $w_{j+1} < w_j$, then we place the delimiters $|$ only after elements of γ_j greater than w_{j+1}, which is greater than l. Therefore, no element among $1, \dots, l$ will have a $|$ after it. The cyclic shifting to the right moves the last element of the compartment in front without changing the order of the other elements. Since none of $1, \dots, l$ is last in its compartment, their relative order is preserved. Again, adding w_{j+1} would not change the order either, so γ_{j+1} has $1, \dots, n-s$ in increasing order, thus completing the induction.

We have then that $F(C_{n,s}^{-1}) \subset C_{n,s}^{-1}$. Since F is injective and $C_{n,s}^{-1}$ is a finite set, we must have that $F(C_{n,s}^{-1}) = C_{n,s}^{-1}$ and so $F : C_{n,s}^{-1} \to C_{n,s}^{-1}$ is a bijection. Hence summing over $q^{\text{maj}(w)}$ turns into summing over $q^{\text{inv}(w)}$ for the same permutations and the left hand side of the equation in the lemma becomes

$$\sum_{w \in C_{n,s}} q^{\text{maj}(w^{-1})} = \sum_{w \in C_{n,s}^{-1}} q^{\text{maj}(w)} = \sum_{w \in C_{n,s}^{-1}} q^{\text{inv}(w)}. \tag{8}$$

We are now going to compute $\sum_{w\in C_{n,s}^{-1}} q^{\mathrm{inv}(w)}$. For a permutation $w \in S_n$, let σ_w be the subsequence of w consisting of the numbers $1, 2, \ldots, n-s$, let w' be the permutation in $C_{n,s}^{-1}$, s.t. $w'_i = w_i$ if $w_i > n-s$, and $1, 2, \ldots, n-s$ in w' occupy the same set of positions a in w, but are arranged in increasing order (i.e. $w' = \sigma_w^{-1}(w)$, as $\sigma_w \in S_{n-s} \hookrightarrow S_n$). Let $\pi : S_n \to C_{n,s}^{-1} \times S_{n-s}$ be given by $\pi(w) = (w', \sigma_w)$, for example if $s = 3$, then $\pi(435612) = (415623, 312)$. The map π is clearly a bijection, as $\pi^{-1}(w', \sigma) = \sigma(w')$. Now let (i,j) be an inversion of w, i.e. $i < j$ and $w_i > w_j$. If $w_i \leq n-s$, then as w_i and w_j appear in the same order in σ_w and $(\sigma_w^{-1}(w_i), \sigma_w^{-1}(w_j))$ will be an inversion in σ and vice versa - an inversion in σ leads to an inversion in w. Similarly, if $w_i > n-s$, then either $w_j > n-s$ too, in which case $w'_i = w_i, w'_j = w_j$, or $w_j \leq n-s$, in which case w'_j will be some other number no greater than $n-s$ after we rearrange the $1, 2, \ldots, n-s$ among themselves, so $w'_j < w'_i$ again. Either way, (i,j) will be an inversion in w' and vice versa, an inversion in w' comes from an inversion in w. We thus have that

$$\mathrm{inv}(w) = \mathrm{inv}(\sigma) + \mathrm{inv}(w').$$

We can now rewrite the sum of $q^{\mathrm{inv}(w)}$ over S_n as follows:

$$(9)\quad \sum_{w\in S_n} q^{\mathrm{inv}(w)} = \sum_{(w',\sigma), w'\in C_{n,s}^{-1}, \sigma\in S_{n-s}} q^{\mathrm{inv}(w')+\mathrm{inv}(\sigma)} = \left(\sum_{w'\in C_{n,s}^{-1}} q^{\mathrm{inv}(w')}\right)\left(\sum_{\sigma\in S_{n-s}} q^{\mathrm{inv}\,\sigma}\right).$$

We have by MacMahon's theorem that

$$\sum_{w\in S_m} q^{\mathrm{inv}(w)} = [m]_q \ldots [1]_q$$

(see e.g. [**Sta97**]). Substituting this in (9) for the sums over S_n and S_{n-s} we get that

$$[n]_q \ldots [1]_q = \left(\sum_{w'\in C_{n,s}^{-1}} q^{\mathrm{inv}(w')}\right) [n-s]_q \ldots [1]_q$$

and after cancellation

$$\left(\sum_{w'\in C_{n,s}^{-1}} q^{\mathrm{inv}(w')}\right) = [n]_q \ldots [n-s+1]_q,$$

which together with (8) becomes the statement of the lemma.

□

OF THEOREM 2.3. The descent set of a tableau T is the set of all i, such that $i+1$ is in a lower row than i in T, denote it by $D(T)$. By the properties of RSK, theorem 1.9, we have that the descent set of a permutation, $D(w) = \{i : w_i > w_{i+1}\}$ is the same as the descent set of its recording tableau. By the symmetry of RSK, theorem 1.8, $D(w^{-1})$ is the same as the descent set of the insertion tableau P. Write $\mathrm{maj}(T) = \sum_{i\in D(T)} i$. Hence we have that

$$(10)\qquad \sum_{w\in\Pi_{n,k}} q^{\mathrm{maj}(w^{-1})} = \sum_{w\in\Pi_{n,k}, \mathrm{RSK}(w)=(P,Q)} q^{\mathrm{maj}(P)} = \sum_{(P,Q)\in\Pi_{n,k}^{\mathrm{RSK}}} q^{\mathrm{maj}(P)}$$

From the proof of Theorem 2.2 we have the equality (6) on sets of pairs (P,Q),

$$\Pi_{n,k}^{\mathrm{RSK}} = C_{n,k}^{\mathrm{RSK}} \setminus (D_{n,k,1} \setminus (D_{n,k,2} \setminus \cdots \setminus (D_{n,k,k-1} \setminus D_{n,k,k}))),$$

or alternatively, $\Pi^{\mathrm{RSK}}_{n,k} = C^{\mathrm{RSK}}_{n,k} \setminus E_{n,k,1}$ and $E_{n,k,l} = D_{n,k,l} \setminus E_{n,k,l+1}$. Hence the statistic $q^{\mathrm{maj}(P)}$ on these sets will also respect the equalities between them; *i.e.* we have

$$\begin{aligned}
\sum_{(P,Q)\in\Pi^{\mathrm{RSK}}_{n,k}} q^{\mathrm{maj}(P)} &= \sum_{(P,Q)\in C^{\mathrm{RSK}}_{n,k}\setminus E_{n,k,1}} q^{\mathrm{maj}(P)} \\
&= \sum_{(P,Q)\in C^{\mathrm{RSK}}_{n,k}} q^{\mathrm{maj}(P)} - \sum_{(P,Q)\in E_{n,k,1}} q^{\mathrm{maj}(P)} \\
&= \sum_{(P,Q)\in C^{\mathrm{RSK}}_{n,k}} q^{\mathrm{maj}(P)} - \sum_{(P,Q)\in D_{n,k,1}} q^{\mathrm{maj}(P)} + \sum_{(P,Q)\in E_{n,k,2}} q^{\mathrm{maj}(P)} = \cdots \\
&= \sum_{(P,Q)\in C^{\mathrm{RSK}}_{n,k}} q^{\mathrm{maj}(P)} - \sum_{(P,Q)\in D_{n,k,1}} q^{\mathrm{maj}(P)} + \cdots + (-1)^k \sum_{(P,Q)\in D_{n,k,k}} q^{\mathrm{maj}(P)}. \qquad (11)
\end{aligned}$$

Again, by the RSK correspondence, $\mathrm{maj}(P) = \mathrm{maj}(w^{-1})$ and Lemma 2.5 we have that

$$\sum_{(P,Q)\in C^{\mathrm{RSK}}_{n,k}} q^{\mathrm{maj}(P)} = \sum_{w\in C_{n,k}} q^{\mathrm{maj}(w^{-1})} = [n]_q \cdots [n-k+1]_q. \qquad (12)$$

In order to evaluate $\sum_{(P,Q)\in D_{n,k,s}} q^{\mathrm{maj}(P)}$ we note that pairs $(P,Q) \in D_{n,k,s}$ are in correspondence with triples $(P, Q', \mathbf{a} = \{a_1, \dots, a_s\})$, where P remains the same and $(P,Q') \in C^{\mathrm{RSK}}_{n,k-s}$. Hence

$$\begin{aligned}
\sum_{(P,Q)\in D_{n,k,s}} q^{\mathrm{maj}(P)} &= \sum_{(P,Q',\mathbf{a})} q^{\mathrm{maj}(P)} \\
&= \sum_{\mathbf{a}\in\binom{[n-k+1,\dots,n]}{s}} \sum_{(P,Q')\in C^{\mathrm{RSK}}_{n,k-s}} q^{\mathrm{maj}(P)} \\
&= \binom{k}{s} [n]_q \cdots [n-k+s+1]_q. \qquad (13)
\end{aligned}$$

Substituting the equations for (12) and (13) into (11) and comparing with (10) we obtain the statement of the theorem. □

We can apply the same argument for the preservation of the insertion tableaux and their descent sets to the bijection $T_{n,k,s} \leftrightarrow \Pi^{\mathrm{RSK}}_{n,k-s} \times \binom{[n-k+1,\dots,n]}{s}$ in Proposition 2.4. We see that the insertion tableaux P in this bijection, $(P,Q) \leftrightarrow (P, Q', b_1, \dots, b_s)$ remains the same and so do the corresponding descent sets and major indices

$$\sum_{(P,Q)\in T_{n,k,s}} q^{\mathrm{maj}(P)} = \binom{k}{s} \sum_{(P,Q')\in \Pi^{\mathrm{RSK}}_{n,k-s}} q^{\mathrm{maj}(P)} = \binom{k}{s} \sum_{w\in\Pi_{n,k-s}} q^{\mathrm{maj}(w^{-1})}.$$

Hence we have the following corollary to the bijection in Proposition 2.4 and Lemma 2.5.

Proposition 2.6. We have that

$$\sum_{s=0}^{k} \binom{k}{s} \sum_{w\in\Pi_{n,k-s}} q^{\mathrm{maj}(w^{-1})} = [n]_q \cdots [n-k+1]_q. \qquad (14)$$

5. Permutations only.

Since the original question was posed only in terms of permutations, we will now give proofs of the main theorems without passing on to the pairs of tableaux. The constructions we will introduce is inspired from application of the inverse RSK to the pairs of tableaux considered in our proofs so far. However, since the pairs (P,Q) of tableaux in the sets $D_{n,k,s}$ are not pairs of Standard Young Tableaux we cannot apply directly the inverse RSK to the bijection in the proof of Theorem 2.2. This requires us to find new constructions and sets of permutations.

We will say that an increasing subsequence of length m of a permutation π satisfies the LLI-m (Least Lexicographic Indices) property if it is the first appearance of an increasing subsequence of length m (*i.e.* if a is the index of its last element, then $\bar{\pi} = \pi_1, \ldots, \pi_{a-1}$ has $\mathrm{is}(\bar{\pi}) < m$) and the indices of its elements are smallest lexicographically among all such increasing subsequences. For example, in $\pi = 2513467$, 234 is LLI-3. Let $n \geq 2s$ and let $C_{n,s,a}$ with $a \in [n-s+1, \ldots, n]$ be the set of permutations in $C_{n,s}$, for which there is an increasing subsequence of length $n-s+1$ and whose LLI-$(n-s+1)$ sequence has its last element at position a.

We define a map $\Phi : C_{n,s} \setminus \Pi^{\mathrm{RSK}}_{n,s} \to C_{n,s-1} \times [n-s+1, \ldots, n]$ for $n \geq 2s$ as follows. A permutation $\pi \in C_{n,s} \setminus \Pi^{\mathrm{RSK}}_{n,s}$ has a LLI-$(n-s+1)$ subsequence σ which would necessarily start with π_1 since $n \geq 2s$ and $\pi \in C_{n,s}$. Let $\sigma = \pi_1, \ldots, \pi_l, \pi_{i_{l+1}}, \ldots, \pi_{i_{n-s+1}}$ for some $l \geq 0$; if $a = i_{n-s+1}$ then $\pi \in C_{n,s,a}$. Let w be obtained from π by setting $w_{i_j} = \pi_{i_{j+1}}$ for $l+1 \leq j \leq n-s$, and then inserting $\pi_{i_{l+1}}$ right after π_l, all other elements preserve their (relative) positions. For example, if $\pi = 12684357 \in C_{8,4} \setminus \Pi^{\mathrm{RSK}}_{8,4}$, then 12457 is LLI-5, $a = 8$ and $w = 12468537$. Set $\Phi(\pi) = (w, a)$.

LEMMA 2.7. The map Φ is well-defined and injective. We have that

$$C_{n,s-1} \times a \setminus \Phi(C_{n,s,a}) = \bigcup_{n-s+2 \leq b \leq a} C_{n,s-1,b}.$$

PROOF. Let again $\pi \in C_{n,s,a}$ and $\Phi(\pi) = (w,a)$. It is clear by the LLI condition that we must have $\pi_{i_{l+1}} < \pi_{l+1}$ as otherwise

$$\pi_1, \ldots, \pi_{l+1}, \pi_{i_{l+1}}, \ldots, \pi_{i_{n-s}}$$

would be increasing of length $n-s+1$ and will have lexicographically smaller indices. Then the first $n-s+1$ elements of w will be increasing and $w \in C_{n,s-1}$.

To show injectivity and describe the coimage we will describe the inverse map $\Psi : \Phi(C_{n,s,a}) \to C_{n,s,a}$. Let $(w,a) \in \Phi(C_{n,s,a})$ with $(w,a) = \Phi(\pi)$ for some $\pi \in C_{n,s,a}$ and let $\bar{w} = w_1 \cdots w_a$.

Notice that $\bar{w}$ cannot have an increasing subsequence $\{y_i\}$ of length $n-s+2$. To show this, let $\{x_i\}$ be the subsequence of w which was the LLI-$(n-s+1)$ sequence of π. If there were a sequence $\{y_i\}$, this could have happened only involving the forward shifts of x_i and some of the x's and y's should coincide (in the beginning at least). By the pigeonhole principle there must be two pairs of indices $p_1 < q_1$ and $p_2 < q_2$ (q_1 and q_2 might be auxiliary, *i.e.* off the end of $\bar{w}$), such that in $\bar{w}$ we have $x_{p_1} = y_{p_2}$ and $x_{q_1} = y_{q_2}$ and between them there are strictly more elements of y and no more coincidences, *i.e.* $q_2 - p_2 > q_1 - p_1$. Then $x_{p_1 - 1} < x_{p_1} < y_{p_2+1}$ and in π (after shifting $\{x_i\}$ back) we will have the subsequence $x_1, \ldots, x_{p_1-1}, y_{p_2+1}, \ldots, y_{q_2-1}, x_{q_1} = y_{q_2}, \ldots, x_{n-s+1}$, which will be increasing and of length $p_1 - 1 + q_2 - p_2 + n - s + 1 - q_1 \geq n-s+1$. By the LLI property we must have that y_{p_2+1} appears after x_{p_1}, but then $x_1, \ldots, x_{p_1-1}, x_{p_1}, y_{p_2+1}, \ldots, y_{q_2-1}, x_{q_1} = y_{q_2}, \ldots, x_{n-s}$ will be increasing of length at least $n-s+1$ appearing before $\{x_i\}$ in π. This violates the other LLI condition of no $n-s+1$ increasing subsequences before x_{n-s+1}.

Now let $\sigma = w_1, \ldots, w_r, w_{i_{r+1}}, \ldots, w_{i_{n-s+1}}$ with $i_{r+1} > n-s+1$ be the $(n-s+1)$-increasing subsequence of $\bar{w}$ with largest lexicographic index sequence. Let w' be obtained from w by assigning $w'_{i_j} = w_{i_{j-1}}$ for $r+1 \leq j \leq n-s+1$, where $i_r = r$, $w'_{a_1} = w_{i_{n-s+1}}$ and then deleting the entry w_r at position r.

We claim that the LLI-$(n-s+1)$ sequence of w' is exactly σ. Suppose the contrary and let $\{y_i\}$ be the LLI-$(n-s+1)$ subsequence of w'. Since $n \geq 2s$ we have $y_1 = w_1 = \sigma_1, \ldots, y_{i_j} = w_{i_j} = \sigma_j$ for all $1 \leq j \leq l$ for some $l \geq r$. If there are no more coincidences between y and σ afterwards, then the sequence $w_1 = y_1, \ldots, w_{i_l} = y_l, y_{l+1}, \ldots, y_{n-s+1}$ is increasing of length $n-s+1$ in $\bar{w}$ (in the same order) and of lexicographically larger index than σ, since the index of y_{l+1} is after the index of y_l in w' equal to the index of σ_{l+1} in w.

Hence there must be at least one more coincidence, let $y_p = \sigma_q$ be the last such coincidence. Again, in $\bar{w}$ the sequence $\sigma_1, \ldots, \sigma_q, y_{p+1}, \ldots, y_{n-s+1}$ appears in this order and is increasing with the index of y_{p+1} larger than the one of σ_{q+1} in w, so its length must be at most $n-s$, *i.e.* $q + (n-s+1) - p \leq n-s$, so $q \leq p-1$. We see then that there are more y's between y_l and $y_p = \sigma_q$ than there are σ's there, so we can apply an argument similar to the one in the previous paragraph. Namely, there are indices p_1, q_1, p_2, q_2, such that $y_{p_1} = \sigma_{q_1}$, $y_{p_2} = \sigma_{q_2}$ with no other coincidences between them and $q_2 - q_1 < p_2 - p_1$. Then the sequence $\sigma_1, \ldots, \sigma_{q_1}, y_{p_1+1}, \ldots, y_{p_2-1}, \sigma_{q_2+1}, \ldots$ is increasing in this order in w, has length $q_1 + p_2 - 1 - p_1 + n - s + 1 - q_2 = (n-s+1) + (p_2 - p_1) - (q_2 - q_1) - 1 \geq n-s+1$ and the index of y_{p_1+1} in w is larger than the index of σ_{q_1+1} (which is the index of y_{p_1} in w'). We thus reach a contradiction, showing that we have found the inverse map of Φ is given by $\Psi(w,a) = w'$ and, in particular, that Φ is injective.

We have also shown that the image of Φ consists exactly of these permutations, which do not have an increasing subsequence of length $n-s+2$ within their first a elements. Therefore the coimage of Φ is the set of permutations in $C_{n,s-1}$ with $n-s+2$ increasing subsequence within its first a elements, so the ones in $C_{n,s-1,b}$ for $n-s+2 \leq b \leq a$. $\square$

We can now proceed to the proof of Theorem 2.2. We have that $C_{n,k} \setminus \Pi^{\mathrm{RSK}}_{n,k}$ is exactly the set of permutations in $C_{n,k}$ with some increasing subsequence of length $n-k+1$, hence

$$C_{n,k} \setminus \Pi^{\mathrm{RSK}}_{n,k} = \bigcup_{n-k+1 \leq a_1 \leq n} C_{n,k,a_1}.$$

On the other hand, applying the lemma we have that

$$\begin{aligned}
(15) \qquad \bigcup_{n-k+1 \leq a_1 \leq n} C_{n,k,a_1} &\simeq \bigcup_{n-k+1 \leq a_1 \leq n} \Phi(C_{n,k,a_1}) \\
&= \bigcup_{n-k+1 \leq a_1 \leq n} \left(C_{n,k-1} \times a_1 \setminus \bigcup_{n-k+2 \leq a_2 \leq a_1} C_{n,k-1,a_2} \right) \\
&= C_{n,k-1} \times \binom{[k]}{1} \setminus \left(\bigcup_{n-k+2 \leq a_2 \leq a_1 \leq n} C_{n,k-1,a_2} \times a_1 \right) \\
&\simeq C_{n,k-1} \times \binom{[k]}{1} \setminus \left(C_{n,k-1} \times \binom{[k]}{2} \setminus \left(\bigcup_{n-k+3 \leq a_3 \leq a_2 \leq a_1 \leq n} C_{n,k-2,a_3} \times (a_2, a_1) \right) \right)
\end{aligned}$$

$$= \cdots = C_{n,k-1} \times \binom{[k]}{1} \setminus \left(C_{n,k-2} \binom{[k]}{2} \setminus \cdots \setminus \left(C_{n,k-r} \times \binom{[k]}{r} \setminus \ldots \right) \ldots \right),$$

where $\simeq$ denotes the equivalence under Φ and $\binom{[k]}{r}$ represent the $r-$tuples $(a_r, \ldots, a_1)$ where $n - k + r \leq a_r \leq a_{r-1} \leq \cdots \leq a_1 \leq n$. Since $\#C_{n,k-r} \times \binom{[k]}{r} = \binom{n}{k-r}(k-r)!\binom{k}{r}$, Theorem 2.2 follows.

As for the $q-$analogue, Theorem 2.3, it follows immediately from the set equalities (15) and Lemma 2.5 once we realize that the map Φ does not change the major index of the inverse permutation, as shown in the following small lemma.

LEMMA 2.8. Let $D(w) = \{i + 1 \text{ before } i \text{ in } w\}$. Then $D(w) = D(\Phi(w))$, so the inverse major index $\text{maj}(w^{-1}) = \sum_{i \in D(w)} i$ is preserved by Φ.

PROOF. To see this, notice that i and $i + 1$ could hypothetically change their relative order after applying Φ only if exactly one of them is in the LLI-$(n - s + 1)$ sequence of w, denote this sequence by $\sigma = w_1, \ldots, w_{i_{n-s+1}}$.

Let $w_p = i$ and $w_q = i+1$. We need to check only the cases when $p = i_r$ and $i_{r-1} < q < i_r$ or $q = i_r$ and $i_{r-1} < p < i_r$, since otherwise i and $i + 1$ preserve their relative order after shifting σ one step forward by applying Φ. In either case, we see that the sequence $w_1, \ldots, w_{i_{r-1}}, w_p$ (or w_q), $w_{i_{r+1}}, \ldots, w_{i_{n-s+1}}$ is increasing of length $n - s + 1$ in w and has lexicographically smaller indices than σ, violating the LLI property. Thus these cases are not possible and the relative order of i and $i + 1$ is preserved, so $D(w) = D(\Phi(w))$. $\square$

We now have that the equalities and equivalences in (15) are equalities on the sets $D(w)$ and so preserve the $\text{maj}(w^{-1})$ statistic, leading directly to Theorem 2.3.

CHAPTER 3

Separable permutations and Greene's Theorem

1. Introduction

The Robinson-Schensted-Knuth (RSK) correspondence associates to any word w a pair of *Young tableaux*, each of equal partition shape $\lambda = (\lambda_1, \lambda_2, \ldots)$, as described in section 2.1. We say that w has *shape* $\mathrm{sh}(w) = \lambda$. As we have already seen in sections 2.2, many properties of w translate to natural properties of the tableaux and vice versa; the study of one object is made easier through the study of the other. We will now focus on the correspondences between increasing subsequences of w and its shape, $\mathrm{sh}(w)$. The most precise correspondence for any permutation is given by Greene's theorem, 1.11, which we will now quote again.

THEOREM 3.1 (Greene's Theorem). Let w be a word of shape λ. For any $d \geq 0$ the sum $\lambda_1 + \cdots + \lambda_d$ equals the maximum number of elements in a disjoint union of d weakly increasing subsequences of w.

However, it is *not* generally true for any w that one can find d disjoint increasing subsequences $u^1, u^2, \ldots, u^d$ of w with u^i of length λ_i for each i. In other words, the shape of a word does not tell you the lengths of the subsequences in a set of d disjoint increasing subsequences of maximum total length; it just tells you the maximum total length.

EXAMPLE 3.2. Consider the permutation $w = 236145$ of shape $(4, 2)$. The only increasing subsequence of length four is 2345. However, the remaining two entries appear in decreasing order. Greene's Theorem tells us that we should be able to find two disjoint increasing subsequences of total length 6. Indeed, 236 and 145 work.

The main result in this chapter is a sufficient condition for such a collection of subsequences $\{u^i\}$ to exist:

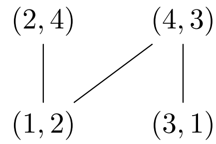
THEOREM 3.3. Let σ be a 3142, 2413-avoiding (i.e., *separable*) permutation of shape λ. For any $d \geq 1$, there exist d disjoint, increasing subsequences $u^1, \ldots, u^d$ such that the length of each u^i is given by λ_i. In other words, σ can be partitioned into $l(\lambda)$ disjoint increasing subsequences u^i, such that $|u^i| = \lambda_i$ for all $i = 1, \ldots, l(\lambda)$.

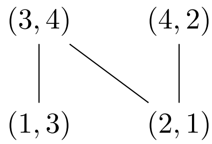
This chapter is based on our joint work with A.Crites and G.Warrington from [**CPW**].

2. Background and setup

Given permutations $\tau \in S_n$ and $\pi \in S_m$, $m \leq n$, we say that τ *contains the pattern* π if there exist indices $1 \leq i_1 < i_2 < \cdots < i_m \leq n$ such that, for all $1 \leq j, k \leq m$, $\tau(i_j) \leq \tau(i_k)$ if and only if $\pi_j \leq \pi_k$. If τ does not contain the pattern π, then we say τ *avoids* π. Central to this chapter will be permutations that simultaneously avoid the patterns 3142 and 2413. Being 3142, 2413-avoiding is one characterization of the class of *separable* permutations (see [**BBL98**]). Throughout this paper σ will denote a separable permutation with $\mathrm{sh}(\sigma) = \lambda = (\lambda_1, \lambda_2, \ldots)$.

Given a permutation $\pi \in S_n$, let $P(\pi)$ denote its *inversion poset*. $P(\pi)$ has elements $(i, \pi(i))$ for $1 \leq i \leq n$ under the partial order $\prec$, in which $(a, b) \prec (c, d)$ if and only if $a < c$ and $b < d$. Note that increasing subsequences in π correspond to chains in $P(\pi)$. A longest increasing subsequence of π corresponds to a maximal chain in $P(\pi)$.

EXAMPLE 3.4. The inversion poset of 2413 is 

and the inversion poset of 3142 is 

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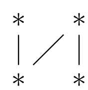
Example 3.4 above immediately gives the following fact.

LEMMA 3.5. A permutation π is separable if and only if its inversion poset $P(\pi)$ has no (induced) subposet isomorphic to 

.

Besides permutations we can, more generally, consider words on the set $[n]$, that is sequences of numbers from $[n] = \{1, \ldots, n\}$. As described in section 2.1, RSK is a correspondence between words (double arrays with bottom row $1, 2, 3 \ldots$) and pairs of same-shape SSYTs.

EXAMPLE 3.6. The permutation $\pi = 7135264$ contains the pattern 4231 (as the subsequence 7564) but avoids 3412. The word $w = 2214312$ is a supersequence of 132 but not of 321. In fact, w is a supersequence of the set $B = \{132, 312, 213\}$.

3. Proof of Theorem 3.3

Central to our proof of Theorem 3.3 will be the ability to exchange collections of disjoint increasing subsequences with other collections for which the number of intersections has, in a certain sense, been reduced. Lemma 3.7, which is the only place separability explicitly appears in our proof, allows us to do this.

LEMMA 3.7. Let u, w, and w' be increasing subsequences of a separable permutation σ, such that w and w' are disjoint. Then there exist two disjoint increasing subsequences α and β, such that $\alpha \bigcup \beta = w \bigcup w'$ and $\alpha \bigcap u = \emptyset$.

PROOF. We can assume that u is a subsequence of $w \bigcup w'$, since the elements of $u \setminus (w \bigcup w')$ won't play any role.

We give first a short proof by contradiction. Consider the inversion poset $P(\sigma)$ of the separable permutation σ. Increasing subsequences are in correspondence with chains and we will regard them as such. Assume there is no chain $\beta \subset (w \cup w')$, such that $u \subset \beta$ and $(w \cup w') \setminus \beta$ is also a chain. Let $\gamma \subset (w \cup w')$ be some maximal chain, such that $u \subset \gamma$. Then there exist two incomparable points $x, y \in (w \cup w') \setminus \gamma$, i.e. $x \ngtrless y$ and thus belonging to the two different chains, e.g. $x \in w$, $y \in w'$. By maximality, $x \cup \gamma$, $y \cup \gamma$ are not chains. Hence there exist $a, b \in \gamma$ for which $x \ngtrless a$, $y \ngtrless b$, so we must have $a \in w'$ and $b \in w$. Assume

$a \succ b$, then we must have $x \succ b$ and $y \prec a$. We have $\begin{array}{ccc} x & & a \\ | & / & | \\ b & & y \end{array}$ with $x \not\gtrless a, x \not\gtrless y, y \not\gtrless b$. This is a subposet of $P(\sigma)$ isomorphic to $\begin{array}{ccc} * & & * \\ | & / & | \\ * & & * \end{array}$, contradicting Lemma 3.5.

$\square$

Proposition 3.8. Let $d \geq 0$ and $s^1, \ldots, s^d$ be disjoint (possibly empty) increasing subsequences of the separable permutation σ. Then there exists an increasing subsequence u^{d+1}, disjoint from each s^i, such that $|u^{d+1}| \geq \lambda_{d+1}$.

Proof. Let $W = \{w^{0,0}, w^{0,1}, \ldots, w^{0,d}\}$ be a collection of $d+1$ disjoint increasing subsequences of σ of maximum total length. By Green'e theorem this length would be $\lambda_1 + \cdots + \lambda_{d+1}$. Apply Lemma 3.7 with $u = s^d$, $w = w^{0,d}$, $w' = w^{0,d-1}$ to obtain two disjoint increasing subsequences α and β, such that $\alpha \bigcup \beta = w^{0,d} \bigcup w^{0,d-1}$ and $\alpha \bigcap s^d = \emptyset$. Set $w^{1,d-1} = \alpha$ and $\beta^{1,0} = \beta$ and apply the Lemma to s^d, $\beta^{1,0}$ and $w^{0,d-2}$. We obtain two disjoint increasing subsequences $\beta^{1,1}$ and $w^{1,d-2}$, such that $w^{1,d-2} \bigcap s^d = \emptyset$. Continue applying the Lemma with s^d, $\beta^{1,k}$ and $w^{0,d-2-k}$ to obtain $\beta^{1,k+1}$ and $w^{1,d-k-1} \bigcap s^d = \emptyset$ for $k = 2, \ldots, d-2$. We end up with $\{w^{1,0}, \ldots, w^{1,d-1}\}$ disjoint increasing subsequences not intersecting s^d, such that $\beta^{1,d-1} \bigcup w^{1,0} \bigcup \cdots \bigcup w^{1,d-1} = w^{0,0} \bigcup w^{0,1} \bigcup \cdots \bigcup w^{0,d}$.

Next repeat the procedure above with $\{w^{1,0}, \ldots, w^{1,d-1}\}$ and s^{d-1}, obtaining another set of d disjoint increasing susbequences with the same elements $\{\beta^{2,d-2}, w^{2,0}, \ldots, w^{2,d-2}\}$, such that $w^{2,i} \bigcap s^{d-1} = \emptyset$ for all i. Repeat this procedure inductively for $\{w^{k,0}, \ldots, w^{k,d-k}\}$ and s^{d-k} to get $\{\beta^{k+1,d-k-1}, w^{k+1,0}, \ldots, w^{k+1,d-k-1}\}$ for $k = 2, \ldots, d-1$. By construction we have that $w^{k,i}$ $(i = 0, \ldots, d-k)$ is disjoint from $s^d, \ldots, s^{d-k+1}$ for every k. Hence $w^{d,0}$ is disjoint from $s^1, \ldots, s^d$ and from our construction we have $\beta^{1,d-1} \bigcup \cdots \bigcup \beta^{d,0} \bigcup w^{d,0} = w^{0,0} \bigcup \cdots \bigcup w^{0,d}$ as sets. By Greene's theorem the collection of d disjoint increasing susbequences $\{\beta^{i,d-i}\}_{i=1}^{d}$ has total length no larger than $\lambda_1 + \cdots + \lambda_d$ and so $|w^{d,0}| = \sum |w^{0,i}| - \sum |\beta^{i,d-i}| \geq \sum_{i=1}^{d+1} \lambda_i - \sum_{i=1}^{d} \lambda_i = \lambda_{d+1}$. The desired sequence is thus $u^{d+1} = w^{d,0}$. $\square$

Theorem 3.3 now follows easily.

Proof of Theorem 3.3. We can construct such a sequence via $l(\lambda)$ applications of Proposition 3.8. In particular, given the $u^1, \ldots, u^i$ for some $0 \leq i < l(\lambda)$, produce u^{i+1} by applying the proposition with $d = i$ and $s^j = u^j$ for $1 \leq j \leq d$. $\square$

4. A consequence to Theorem 3.3

The discovery of Theorem 3.3 happened in our attempts to prove a conjecture of G.Warrington, which is now the following corollary.

Corollary 3.9. If a word w contains a separable permutation σ as a pattern, then $\operatorname{sh}(w) \supseteq \operatorname{sh}(\sigma)$.

Proof. Let $\operatorname{sh}(w) = \mu = (\mu_1, \mu_2, \ldots)$. Let σ' be any subsequence of w in the same relative order as the elements of σ; i.e., w contains σ at the positions of σ'. By Greene's Theorem applied to w, for any $k \geq 1$ there exist k disjoint increasing subsequences $w^1, \ldots, w^k$ with $|w^1| + \cdots + |w^k| = \mu_1 + \cdots + \mu_k$. The intersection $\sigma' \cap w^i$ induces a subsequence of σ we denote by s^i. These s^i are then k disjoint increasing subsequences of σ. By Proposition 3.8, there is an increasing subsequence u of σ, disjoint from the s^is, with length at least λ_{k+1}. The mapping $\sigma \mapsto \sigma'$ induces a corresponding map of u to a subsequence u' of w. It follows then that u' is disjoint from each w^i as well. Then $w^1, \dots, w^k, u'$ are $k+1$ disjoint increasing subsequences in w. By Greene's Theorem, 1.11,

$$|w^1| + \cdots + |w^k| + |u'| \le \mu_1 + \cdots + \mu_k + \mu_{k+1}.$$

Hence $|u'| \le \mu_{k+1}$. We also know by construction that $\lambda_{k+1} \le |u| = |u'|$. Combining these equalities and running over all k yields $\lambda \subseteq \mu$ as desired. $\square$

Example 3.10. We illustrate the necessity of avoiding 2413: Let $\sigma = 2413$ and $w = 24213$. Then $(P(\sigma), Q(\sigma)) = \left(\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline\end{array}, \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline\end{array}\right)$, so $\mathrm{sh}(\sigma) = (2,2)$. However, $(P(w), Q(w)) = \left(\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 2 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array}, \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array}\right)$, so $\mathrm{sh}(w) = (3,1,1) \not\supseteq (2,2)$.

CHAPTER 4

Tableaux and plane partitions of truncated shapes

1. Introduction

A central topic in the study of tableaux and plane partition are their enumerative properties. The number of standard tableaux is given by the famous hook-length formula of Frame, Robinson and Thrall, as discussed in Section 2.4. Not all tableaux have such nice enumerative properties, for example their main generalizations as skew tableaux are not counted by any product type formulas.

In this chapter we find product formulas for special cases of a new type of tableaux and plane partitions, namely ones whose diagrams are not straight or shifted Young diagrams of integer partitions. The diagrams in question are obtained by removing boxes from the north-east corners of a straight or shifted Young diagram and we say that the shape has been truncated by the shape of the boxes removed. We discover formulas for the number of tableaux of specific truncated shapes: a rectangle truncated by a staircase shape and a shifted staircase truncated by one box. The proofs rely on several steps of interpretations. Truncated shapes are interpreted as (tuples of) SSYTs, which translates the problem into specializations of sums of restricted Schur functions. The number of standard tableaux is found as a polytope volume and then a certain limit of these specializations whose computations involve complex integration, the Robinson-Schensted-Knuth correspondence, etc.

The consideration of these objects started after R. Adin and Y. Roichman asked for a formula for the number of linear extensions of the poset of triangle-free triangulations, which are equivalent to standard tableaux of shifted staircase shape with upper right corner box removed, [**AR**]. We find and prove the formula in question, namely

THEOREM 4.1. The number of shifted standard tableaux of shape $\delta_n \setminus \delta_1$ is equal to

$$g_n \frac{C_n C_{n-2}}{2C_{2n-3}},$$

where $g_n = \frac{\binom{n+1}{2}!}{\prod_{0\leq i<j\leq n}(i+j)}$ is the number of shifted staircase tableaux of shape $(n, n-1, \ldots, 1)$ and $C_m = \frac{1}{m+1}\binom{2m}{m}$ is the $m-$th Catalan number.

We also prove formulas for the cases of rectangles truncated by a staircase.

THEOREM 4.2. The number of standard tableaux of truncated straight shape $\underbrace{(n, n, \ldots, n)}_{m} \setminus \delta_k$ (assume $n \leq m$), is

$$(mn - \binom{k+1}{2})! \times \frac{f_{(n-k-1)^m}}{(m(n-k-1))!} \times \frac{g_{(m,m-1,\ldots,m-k)}}{((k+1)m - \binom{k+1}{2}))!} \frac{E_1(k+1, m, n-k-1)}{E_1(k+1, m, 0)},$$

where

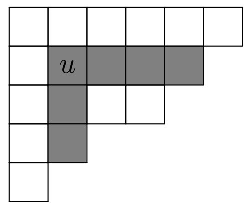
$$E_1(r,p,s) = \begin{cases} \displaystyle\prod_{r<l<2p-r+2} \frac{1}{(l+2s)^{r/2}} \prod_{2\le l\le r} \frac{1}{((l+2s)(2p-l+2+2s))^{\lfloor l/2\rfloor}}, & r \text{ even},\\ \displaystyle\frac{((r-1)/2+s)!}{(p-(r-1)/2+s)!} E_1(r-1,p,s), & r \text{ odd}. \end{cases}$$

In [**AKR**] Adin, King and Roichman have independently found a formula for the case of shifted staircase truncated by one box by methods different from the methods developed here. This chapter is based on our work in [**Pan10b**].

2. Definitions

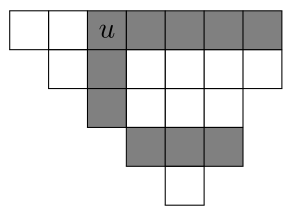
As discussed in Section 2.4, the number of straight standard Young tableaux (SYT) of shape λ is given by a hook-length formula, 1.17:

$$f_\lambda = \frac{|\lambda|!}{\prod_{u\in\lambda} h_u}, \qquad \text{hook } h_u:$$



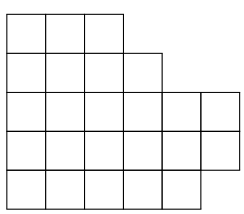
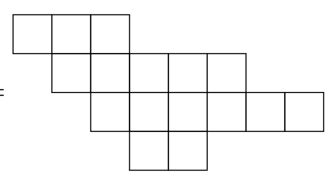
and so is the number of standard Young tableaux of shifted shape λ, 1.18:

$$g_\lambda = \frac{|\lambda|!}{\prod_u h_u} \qquad \text{hook } h_u:$$



We are now going to define our main objects of study.

Let $\lambda = (\lambda_1, \lambda_2, \ldots)$ and $\mu = (\mu_1, \mu_2, \ldots)$ be integer partitions, such that $\lambda_i \ge \mu_i$. A **straight diagram of truncated shape** $\lambda \setminus \mu$ is a left justified array of boxes, such that row i has $\lambda_i - \mu_i$ boxes. If λ has no equal parts we can define a **shifted diagram of truncated shape** $\lambda \setminus \mu$ as an array of boxes, where row i starts one box to the right of the previous row $i-1$ and has $\lambda_i - \mu_i$ boxes. For example

$D_1 =$  , $D_2 =$ 

D_1 is of straight truncated shape $(6,6,6,6,5) \setminus (3,2)$ and D_2 is of shifted truncated shape $(8,7,6,2) \setminus (5,2)$.

We define standard and semistandard Young tableaux and plane partitions of truncated shape the usual way except this time they are fillings of truncated diagrams.

Definition 4.3. A **standard truncated Young tableau** of shape $\lambda \setminus \mu$ is a filling of the corresponding truncated diagram with the integers from 1 to $|\lambda| - |\mu|$, such that the entries across rows and down columns are increasing and each number appears exactly once. A **plane partition of truncated shape** $\lambda \setminus \mu$ is a filling of the corresponding truncated diagram with integers such that they weakly increase along rows and down columns.

For example

$$T_1 = \begin{ytableau} 1 & 2 & 4 \\ \none & 3 & 5 & 7 & 8 \\ \none & \none & 6 & 9 \end{ytableau}, T_2 = \begin{ytableau} 1 & 1 & 3 \\ \none & 2 & 4 & 4 & 5 \\ \none & \none & 5 & 6 \end{ytableau}, T_3 = \begin{ytableau} 1 & 1 & 3 \\ \none & 1 & 4 & 4 & 6 \\ \none & \none & 4 & 5 \end{ytableau}$$

are respectively a standard Young tableaux (SYT), a semi-standard Young tableaux (SSYT) and a plane partition (PP) of shifted truncated shape $(5,4,2) \setminus (2)$. We also define **reverse** PP, SSYT, SYT by reversing all inequalities in the respective definitions (replacing weakly/strictly increasing with weakly/strictly decreasing). For this paper it will be more convenient to think in terms of the reverse versions of these objects.

In this chapter we will consider truncation by staircase shape $\delta_k = (k, k-1, k-2, \ldots, 1)$ of shifted staircases and straight rectangles. We will denote by $T[i,j]$ the entry in the box with coordinate (i,j) in the diagram of T, where i denotes the row number counting from the top and j denotes the $j-$th box in this row counting from the beginning of the row.

For any shape D, let

$$F_D(q) = \sum_{T:\mathrm{sh}(T)=D} q^{\sum T[i,j]}$$

be the generating function for the sum of the entries of all plane partitions of shape D.

3. A bijection with skew SSYT

We will consider a map between truncated plane partitions and skew Semi-Standard Young Tableaux which will enable us to enumerate them using Schur functions.

As a basic setup for this map we first consider truncated shifted plane partitions of staircase shape $\delta_n \setminus \delta_k$. Let T be such a plane partition. Let $\lambda^j = (T[1,j], T[2,j], \ldots, T[n-j,j])$ - the sequence of numbers in the jth diagonal of T. For example if $T = \begin{ytableau} 8 & 6 & 5 \\ \none & 6 & 4 & 3 \\ \none & \none & 4 & 2 \\ \none & \none & \none & 1 \end{ytableau}$, then $\lambda^1 = (8,6,4,1)$, $\lambda^2 = (6,4,2)$ and $\lambda^3 = (5,3)$.

Let P be a reverse skew tableau of shape λ^1/λ^{n-k}, such that the entries filling the subshape λ^j/λ^{j+1} are equal to j, i.e. it corresponds to the sequence $\lambda^{n-k} \subset \lambda^{n-k-1} \subset \cdots \subset \lambda^1$. The fact that this is all well defined follows from the inequalities that the $T[i,j]$'s satisfy by virtue of T being a plane partition. Namely, $\lambda^{j+1} \subset \lambda^j$, because $\lambda^j_i = T[i,j] \geq T[i,j+1] = \lambda^{j+1}_i$. Clearly the rows of P are weakly decreasing. The columns are strictly decreasing, because for each j the entries j in the ith row of P are in positions $T[i,j+1]+1$ to $T[i,j] < T[i-1,j+1]+1$, so they appear strictly to the left of the js in the row above $(i-1)$.

Define $\phi(T) = P$, ϕ is the map in question. Given a reverse skew tableaux P of shape $\lambda \setminus \mu$ and entries smaller than n we can obtain the inverse shifted truncated plane partition $T = \phi^{-1}(P)$ as $T[i,j] = \max(s | P[i,s] \geq j)$; if no such entry of P exists let $s = 0$.

For example we have

$$T = \begin{ytableau} 8 & 7 & 6 & 5 \\ \none & 7 & 5 & 4 & 3 \\ \none & \none & 5 & 3 & 2 \\ \none & \none & \none & 3 & 1 \\ \none & \none & \none & \none & 1 \end{ytableau} \quad \xrightarrow{\phi} \quad P = \begin{ytableau} \none & \none & \none & \none & \none & 3 & 2 & 1 \\ \none & \none & \none & 3 & 2 & 1 & 1 \\ 3 & 3 & 2 & 1 & 1 \\ 2 & 1 & 1 \\ 1 \end{ytableau} \tag{16}$$

Notice that

$$\sum T[i,j] = \sum P[i,j] + |\lambda^{n-k}|(n-k), \tag{17}$$

where $\mathrm{sh}(P) = \lambda^1/\lambda^{n-k}$.

The map ϕ can be extended to any truncated shape; then the image will be tuples of SSYTs with certain restrictions. For the purposes of this paper we will extend it to truncated plane partitions of shape $(n^m) \setminus \delta_k$ as follows.

Let T be a plane partition of shape $n^m \setminus \delta_k$ and assume that $n \le m$ (otherwise we can reflect about the main diagonal). Let

$$\lambda = (T[1,1], T[2,2], \ldots, T[n,n]) \ , \ \mu = (T[1,n-k], T[2,n-k+1], \ldots, T[k+1,n])$$

and let T_1 be the portion of T above and including the main diagonal, hence of shifted truncated shape $\delta_n \setminus \delta_k$, and T_2 the transpose of the lower portion including the main diagonal, a shifted PP of shape $(m, m-1, \ldots, m-n+1)$.

Extend ϕ to T as $\phi(T) = (\phi(T_1), \phi(T_2))$. Here $\phi(T_2)$ is a SSYT of at most n rows (shape λ) and filled with $[1, \ldots, m]$ the same way as in the truncated case.

As an example with $n = 5, m = 6, k = 2$ we have

$$T = \begin{array}{|c|c|c|c|c|} \hline 7 & 6 & 4 & \multicolumn{2}{c}{} \\ \cline{1-4} 6 & 6 & 4 & 4 & \multicolumn{1}{c}{} \\ \hline 4 & 4 & 3 & 3 & 2 \\ \hline 4 & 3 & 2 & 2 & 2 \\ \hline 3 & 2 & 2 & 1 & 1 \\ \hline 2 & 1 & 1 & 1 & 1 \\ \hline \end{array} \quad T_1 = \begin{array}{ccccc} \cline{1-3} \multicolumn{1}{|c|}{7} & \multicolumn{1}{c|}{6} & \multicolumn{1}{c|}{4} & & \\ \cline{1-4} & \multicolumn{1}{|c|}{6} & \multicolumn{1}{c|}{4} & \multicolumn{1}{c|}{4} & \\ \cline{2-5} & & \multicolumn{1}{|c|}{3} & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{2} \\ \cline{3-5} & & & \multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{2} \\ \cline{4-5} & & & & \multicolumn{1}{|c|}{1} \\ \cline{5-5} \end{array} \quad T_2 = \begin{array}{cccccc} \hline \multicolumn{1}{|c|}{7} & \multicolumn{1}{c|}{6} & \multicolumn{1}{c|}{4} & \multicolumn{1}{c|}{4} & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{2} \\ \hline & \multicolumn{1}{|c|}{6} & \multicolumn{1}{c|}{4} & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{1} \\ \cline{2-6} & & \multicolumn{1}{|c|}{3} & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{1} \\ \cline{3-6} & & & \multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{1} & \multicolumn{1}{c|}{1} \\ \cline{4-6} & & & & \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{1} \\ \cline{5-6} \end{array},$$

$$\phi(T) = \left(\begin{array}{ccccccc} \cline{5-7} & & & & \multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{1} \\ \cline{5-7} & & & & \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{1} & \\ \cline{3-3} \cline{5-6} & & \multicolumn{1}{|c|}{2} & & & & \\ \cline{1-3} \multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{2} & & & & & \\ \cline{1-2} \multicolumn{1}{|c|}{1} & & & & & & \\ \cline{1-1} \end{array} \ , \ \begin{array}{|c|c|c|c|c|c|c|} \hline 6 & 6 & 5 & 4 & 2 & 2 & 1 \\ \hline 5 & 4 & 3 & 2 & 1 & 1 & \multicolumn{1}{c}{} \\ \cline{1-6} 4 & 3 & 1 & \multicolumn{4}{c}{} \\ \cline{1-3} 3 & 1 & \multicolumn{5}{c}{} \\ \cline{1-2} 2 & \multicolumn{6}{c}{} \\ \cline{1-1} \end{array} \right)$$

We thus have the following

Proposition 4.4. The map ϕ is a bijection between shifted truncated plane partitions T of shape $\delta_n \setminus \delta_k$ filled with nonnegative integers and (reverse) skew semi-standard Young tableaux with entries in $[1, \ldots, n-k-1]$ of shape λ/μ with $l(\lambda) \le n$ and $l(\mu) \le k+1$. Moreover, $\sum_{i,j} T[i,j] = \sum_{i,j} P[i,j] + |\mu|(n-k)$. Similarly ϕ is also a bijection between truncated plane partitions T of shape $n^m \setminus \delta_k$ and pairs of SSYTs (P, Q), such that $\mathrm{sh}(P) = \lambda/\mu$, $\mathrm{sh}(Q) = \lambda$ with $l(\lambda) \le n$, $l(\mu) \le k+1$ and P is filled with $[1, \ldots, n-k-1]$, Q with $[1, \ldots, m]$. Moreover, $\sum T[i,j] = \sum P[i,j] + \sum Q[i,j] - |\lambda| + |\mu|(n-k)$.

4. Schur function identities

We will now consider the relevant symmetric function interpretation arising from the map ϕ. Substitute the entries $1, \ldots$ in the skew SSYTs in the image with respective variables $x_1, \ldots$ and $z_1, \ldots$. The idea is to evaluate the resulting expressions at $x = (q, q^2, \ldots)$ and $z = (1, q, q^2, \ldots)$ to obtain generating functions for the sum of entries in the truncated plane partitions, which will later allow us to derive enumerative results.

For the case of shifted truncated shape $\delta_n \setminus \delta_k$ we have the corresponding sum

$$S_{n,k}(x;t) = \sum_{\lambda,\mu | l(\lambda)\le n, l(\mu)\le k+1} s_{\lambda/\mu}(x_1,\ldots,x_{n-k-1})t^{|\mu|}, \tag{18}$$

and for the straight truncated shape $n^m \setminus \delta_k$

$$D_{n,k}(x;z;t) = \sum_{\lambda,\mu | l(\lambda)\le n, l(\mu)\le k+1} s_{\lambda}(z)s_{\lambda/\mu}(x)t^{|\mu|}. \tag{19}$$

We need to find formulas when $x_i = 0$ for $i > n-k-1$ and $z_i = 0$ for $i > m$. Keeping the restriction $l(\mu) \le k+1$ we have $s_{\lambda/\mu}(x) = 0$ if $l(\lambda) > n$ and this allows us to drop the length restriction on λ in both sums.

From now on the different sums will be treated separately. Consider another set of variables $y = (y_1, \ldots, y_{k+1})$ which together with $(x_1, \ldots, x_{n-k-1})$ form a set of n variables. Using Cauchy's identity 1.14 we have

$$\sum_{\lambda | l(\lambda)\le n} s_\lambda(x_1,\ldots,x_{n-k-1},ty_1,\ldots,ty_{k+1})$$
$$= \prod \frac{1}{1-x_i} \prod_{i<j\le n-k-1} \frac{1}{1-x_ix_j} \prod_{i<j\le k+1} \frac{1}{1-t^2y_iy_j} \prod_{i,j} \frac{1}{1-x_ity_j} \prod \frac{1}{1-ty_i},$$

where the length restriction drops because $s_\lambda(u_1,\ldots,u_n) = 0$ when $l(\lambda) > n$. On the other hand we have

$$\sum_{\lambda | l(\lambda)\le n} s_\lambda(x_1,\ldots,x_{n-k-1},ty_1,\ldots,ty_{k+1})$$
$$= \sum_{\lambda,\mu | l(\mu)\le k+1} s_{\lambda/\mu}(x_1,\ldots,x_{n-k-1})s_\mu(y_1,\ldots,y_{k+1})t^{|\mu|}$$

since again $s_\mu(yt) = 0$ if $l(\mu) > k+1$. We thus get

$$\prod \frac{1}{1-x_i} \prod_{i<j\le n-k-1} \frac{1}{1-x_ix_j} \prod_{i<j\le k+1} \frac{1}{1-t^2y_iy_j} \prod_{i,j} \frac{1}{1-x_ity_j} \prod \frac{1}{1-ty_i} \tag{20}$$
$$= \sum_{\lambda,\mu | l(\mu)\le k+1} s_{\lambda/\mu}(x_1,\ldots,x_{n-k-1})s_\mu(y_1,\ldots,y_{k+1})t^{|\mu|}$$

We now need to extract the coefficients of $s_\mu(y)$ from both sides of (20) to obtain a formula for $S_{n,k}(x;t)$. To do so we will use the determinantal formula for the Schur functions 1.15, namely that

$$s_\nu(u_1,\ldots,u_p) = \frac{a_{\nu+\delta_p}(u)}{a_{\delta_p}(u)} = \frac{\det[u_i^{\nu_j+p-j}]_{i,j=1}^p}{\det[u_i^{p-j}]_{i,j=1}^p}.$$

We also have that $a_{\delta_p}(z_1,\ldots,z_p) = \prod_{i<j}(z_i - z_j)$. Substituting $a_{\delta_{k+1}+\mu}(y)/a_{\delta_{k+1}}(y)$ for $s_\mu(y)$ in the right-hand side of (20) and multiplying both sides by $a_{\delta_{k+1}}(y)$ we obtain

$$\sum_{\lambda,\mu} s_{\lambda/\mu}(x_1,\ldots,x_{n-k-1})a_{\mu+\delta_{k+1}}(y_1,\ldots,y_{k+1})t^{|\mu|} = \tag{21}$$
$$\prod \frac{1}{1-x_i} \prod_{i<j\le n-k-1} \frac{1}{1-x_ix_j} \prod_{i<j\le k+1} \frac{y_i-y_j}{1-t^2y_iy_j} \prod_{i,j} \frac{1}{1-x_ity_j} \prod \frac{1}{1-ty_i}$$

Observe that $a_\alpha(u_1,\ldots,u_p) = \sum_{w\in S_p}\operatorname{sgn}(w)u_{w_1}^{\alpha_1}\cdots u_{w_p}^{\alpha_p}$ with $\alpha_i > \alpha_{i+1}$ has exactly $p!$ different monomials in y, each with a different order of the degrees of u_i (determined by w). Moreover, if $\alpha \neq \beta$ are partitions of distinct parts, then $a_\alpha(u)$ and $a_\beta(u)$ have no monomial in common. Let

$$A(u) = \sum_{\alpha|\alpha_i>\alpha_{i+1}}\sum_{w\in S_p}\operatorname{sgn}(w)u_{w_1}^{\alpha_1}\cdots u_{w_p}^{\alpha_p}.$$

For every β of strictly decreasing parts, every monomial in $a_\beta(u)$ appears exactly once and with the same sign in $A(u)$, so $a_\beta(u)A(u^{-1})$ has coefficient at u^0 equal to $p!$. Therefore we have

$$[y^0]\left(\sum_{\lambda,\mu} s_{\lambda/\mu}(x_1,\ldots,x_{n-k-1})a_{\mu+\delta_{k+1}}(y_1,\ldots,y_{k+1})t^{|\mu|}A(y^{-1})\right) \tag{22}$$

$$= (k+1)!\sum_{\lambda,\mu|l(\mu)\le k+1} s_{\lambda/\mu}(x_1,\ldots,x_{n-k-1})t^{|\mu|} = (k+1)!S_{n,k}(x;t). \tag{23}$$

Using the fact that $a_\alpha(u) = s_\nu(u)a_\delta(u)$ for $\nu = \alpha-\delta_p$ by the determinantal formula 1.15 and Cauchy's formula for the sum of the Schur functions 1.14 we have

$$A(u) = \sum_\nu s_\nu(u)a_{\delta_p}(u) = \prod\frac{1}{1-u_i}\prod_{i<j}\frac{(u_i-u_j)}{1-u_iu_j}.$$

In order for $A(y^{-1})$ and $\sum s_\mu(y)t^{|\mu|}$ to converge we need $1<|y_i|<|t^{-1}|$ for every i. For $S_{n,k}(x;t)$ to also converge, let $|t|<1$ and $|x_j|<1$ for all j.

We also have that for any doubly infinite series $f(y)$, $[y^0]f(y) = \frac{1}{2\pi i}\int_C f(y)y^{-1}dy$, on a circle C in the $\mathbb{C}$ plane centered at 0 and within the region of convergence of f. Hence we have the formula for $S_{n,k}(x;t)$ through a complex integral.

Proposition 4.5. We have that

$$S_{n,k}(x;t) = (-1)^{\binom{k+1}{2}}\prod\frac{1}{1-x_i}\prod_{i<j\le n-k-1}\frac{1}{1-x_ix_j}$$

$$\frac{1}{(2\pi i)^{k+1}}\int_T\prod_{i<j\le k+1}\frac{(y_i-y_j)^2}{1-t^2y_iy_j}\prod_{i,j}\frac{1}{1-x_ity_j}\prod\frac{1}{1-ty_i}\prod\frac{1}{y_i-1}\prod_{i<j}\frac{1}{y_iy_j-1}dy_1\cdots dy_{k+1},$$

where $T = C_1\times C_2\times\cdots C_p$ and $C_i = \{z\in\mathbb{C}: |z| = 1+\epsilon_i\}$ for $\epsilon_i < |t^{-1}|-1$.

For the **straight shape case** and $D_{n,k}$ we consider the sum, whose formula is given by Cauchy's identity 1.13,

$$\sum_\lambda s_\lambda(z)s_\lambda(x,ty) = \prod_{i,j}\frac{1}{1-x_iz_j}\prod_{i,j}\frac{1}{1-y_itz_j}. \tag{24}$$

Let $x_i = 0$ if $i > n-k-1$, $z = (z_1,\ldots,z_m)$ and $y = (y_1,\ldots,y_{k+1})$. Then $s_\lambda(x,ty) = 0$ if $l(\lambda) > n$, so this sum ranges over λ with $l(\lambda)\le n$. Also, we have

$$s_\lambda(x,yt) = \sum_\mu s_{\lambda/\mu}(x)s_\mu(yt),$$

and since $s_\mu(yt) = 0$ if $l(\mu) > k+1$, the sum ranges only over μ, s.t. $l(\mu) \leq k+1$. Thus we have

$$\prod_{i,j} \frac{1}{1-x_iz_j} \prod_{i,j} \frac{1}{1-y_itz_j} = \sum_{\lambda,\mu | l(\lambda)\leq n, l(\mu)\leq k+1} s_\lambda(z) s_{\lambda/\mu}(x) t^{|\mu|} s_\mu(y)$$

If we expand the left-hand side of the above equation as a linear combination of $s_\mu(y)$, summing all the coefficients on both sides will give us the desired formula for $D_{n,m,k}(x, z; t)$.

We have that

$$\prod_{i,j} \frac{1}{1-y_iz_jt} = \sum_\nu s_\nu(zt)s_\nu(y),$$

and the other factor contains only x and z, a constant over the ring of symmetric polynomials in y. Comparing coefficients we get

Proposition 4.6. We have that

$$D_{n,m,k}(x, z; t) = \prod_{i,j} \frac{1}{1-x_iz_j} \left(\sum_{\nu | l(\nu)\leq k+1} s_\nu(zt) \right).$$

For the purpose of enumeration of SYTs we will use this formula as it is. Even though there are formulas, e.g. of Gessel and King, for the sum of Schur functions of restricted length, they would not give the enumerative answer any more easily.

5. A polytope volume as a limit

Plane partitions of specific shape (truncated or not) of size N can be viewed as integer points in a cone in $\mathbb{R}^N$. Let D be the diagram of a plane partition T, with the coordinates of $\mathbb{R}^{|D|}$ indexed by the boxes present in T. Then

$$\begin{aligned} C_D = &\{(\cdots, x_{i,j}, \cdots) \in \mathbb{R}^N_{\geq 0} : \\ &[i,j] \in D,\ x_{i,j} \leq x_{i,j+1} \text{ if } [i,j+1] \in D, x_{i,j} \leq x_{i+1,j} \text{ if } [i+1,j] \in D\} \end{aligned}$$

is the corresponding cone.

Let $P(C)$ be the section of a cone C in $\mathbb{R}^N_{\geq 0}$ with the hyperplane $H = \{x | \sum_{[i,j]\in D} x_{i,j} = 1\}$. Consider the standard tableaux of shape T; these correspond to all linear orderings of the points in C_D and thus also $P_D = P(C_D)$. Considering T as a bijection $D \to [1, \ldots, N]$, P_D is thus subdivided into chambers $\{x : 0 \leq x_{T^{-1}(1)} \leq x_{T^{-1}(2)} \leq \cdots \leq x_{T^{-1}(N)}\} \cap H$ of equal volume, namely $\frac{1}{N!} \operatorname{Vol}(\Delta_N)$, where $\Delta_N = H \cap \mathbb{R}^N$ is the N-simplex. Hence the volume of P_D is

$$\operatorname{Vol}_{N-1}(P_D) = \frac{\#T : SYT, \operatorname{sh}(T) = D}{N!} \operatorname{Vol}(\Delta_N). \tag{25}$$

The following lemma helps determine the volume and thus the number of SYTs of shape D.

Lemma 4.7. Let P be a $(d-1)$-dimensional rational polytope in $\mathbb{R}^d_{\geq 0}$, such that its points satisfy $a_1 + \cdots + a_d = 1$ for $(a_1, \ldots, a_d) \in P$, and let

$$F_P(q) = \sum_n \sum_{(a_1,\ldots,a_d)\in nP\cap\mathbb{Z}^d} q^{a_1+a_2+\cdots+a_d}.$$

Then the $(d-1)$-dimensional volume of P is

$$\operatorname{Vol}_{d-1}(P) = (\lim_{q\to 1} (1-q)^d F_P(q)) \operatorname{Vol}(\Delta_d),$$

where Δ_d is the $(d-1)$-dimensional simplex.

Proof. Let $f_P(n) = \#\{(a_1, \ldots, a_d) \in nP\}$. Then

$$F_P(q) = \sum_n f_P(n) q^n.$$

Moreover, $\mathrm{Vol}_{d-1}(P) = \lim_{n\to\infty} \frac{f_P(n)}{n^{d-1}} \mathrm{Vol}(\Delta_d)$ since subdividing an embedding of P in $\mathbb{R}^{d-1}$ into $(d-1)$-hypercubes with side $\frac{1}{n}$ we get $f_P(n)$ cubes of total volume $f_P(n)/n^{d-1}$ which scaled by $\mathrm{Vol}(\Delta_d)$ approximate P as $n \to \infty$.

Moreover, since $\displaystyle\lim_{n\to\infty} \frac{n^{d-1}}{(n+1)\cdots(n+d-1)} = 1$, we also have $\displaystyle\frac{\mathrm{Vol}_{d-1}(P)}{\mathrm{Vol}(\Delta_d)} = \lim_{n\to\infty} \frac{f_P(n)}{(n+1)\cdots(n+d-1)}$. Let $\displaystyle G(q) = \sum_n \frac{f_P(n)}{(n+1)\cdots(n+d-1)} q^{n+d-1}$. Then

$$\tilde{G}(q) = (1-q)G(q) = \underbrace{\left(\frac{f_P(n)}{(n+1)\cdots(n+d-1)} - \frac{f_P(n-1)}{(n)\cdots(n+d-2)}\right)}_{b_n} q^{n+d-1}$$

and $\displaystyle\tilde{G}(1) = \sum b_n = \lim_{n\to\infty} b_1 + \cdots + b_n = \lim_{n\to\infty} \frac{f_P(n)}{(n+1)\cdots(n+d-1)} = \frac{\mathrm{Vol}_d(P)}{\mathrm{Vol}(\Delta_d)}$. On the other hand

$$\frac{\mathrm{Vol}_d(P)}{\mathrm{Vol}(\Delta_d)} = \tilde{G}(1) = \lim_{q\to 1}(1-q)G(q) = \lim_{q\to 1} \frac{G'(q)}{(\frac{1}{1-q})'}$$

$$= \cdots = \lim_{q\to 1} \frac{G^{(d-1)}(q)}{(\frac{1}{1-q})^{(d-1)}} = \lim_{q\to 1} F_p(q)(1-q)^d$$

by L'Hopital's rule. □

Notice that if $P = P(C_D)$ for some shape D, then

$$F_P(q) = \sum_n \sum_{a\in nP\cap\mathbb{Z}^N} q^n = \sum_{a\in C_D\cap\mathbb{Z}^N} q^{|a|} = \sum_{T:PP,\mathrm{sh}(T)=D} q^{\sum T[i,j]} = F_D(q).$$

Using (25) and this Lemma we get the key fact to enumerating SYTs of truncated shapes using evaluations of symmetric functions.

Proposition 4.8. The number of standard tableaux of (truncated) shape D is equal to

$$N! \lim_{q\to 1}(1-q)^N F_D(q).$$

6. Shifted truncated SYTs

We are now going to use Propositions 4.5 and 4.8 to find the number of standard shifted tableaux of truncated shape $\delta_n \setminus \delta_1$. Numerical results show that a product formula for the general case of truncation by δ_k does not exist.

First we will evaluate the integral in Proposition 4.5 by iteration of the residue theorem. For simplicity, let $u_0 = t$ and $u_i = tx_i$, so the integral becomes

$$\frac{1}{(2\pi i)^2} \int_T \frac{(y_1-y_2)^2}{1-t^2y_1y_2} \frac{1}{y_2-1} \frac{1}{y_1-1} \frac{1}{y_1y_2-1} \prod_{i\geq 0, j=1,2} \frac{1}{1-u_iy_j} dy_1 dy_2$$

Integrating by y_1 we have poles at $1, y_2^{-1}t^{-2}$, y_2^{-1} and u_i^{-1}. Only 1 and y_2^{-1} are inside C_1 and the respective residues are

$$Res_{y_1=1} = -\prod_{i\geq 0}\frac{1}{1-u_i}\frac{1}{2\pi i}\int_{C_2}\frac{1}{(1-t^2y_2)}\prod_i\frac{1}{1-u_iy_2}dy_2 = 0,$$

since now the poles for y_2 are all outside C_2.

For the other residue we have

$$\begin{aligned}Res_{y_1=y_2^{-1}} &= \frac{1}{2\pi i}\int_{C_2}\frac{(y_2^{-1}-y_2)^2}{1-t^2}\frac{1}{y_2-1}\frac{1}{y_2^{-1}-1}\frac{1}{y_2}\prod_{i\geq 0}\frac{1}{1-u_iy_2}\prod_{i\geq 0}\frac{1}{1-u_iy_2^{-1}}dy_2\\ &= \frac{1}{1-t^2}\frac{1}{2\pi i}\int_{C_2}(1+y_2)^2y_2^{n-3}\prod_{i\geq 0}\frac{1}{y_2-u_i}\prod_{i\geq 0}\frac{1}{1-u_iy_2}dy_2\end{aligned}$$

If $n\geq 3$ the poles inside C_2 are exactly $y_2=u_i$ for all i and so we get a final answer

$$-\sum_{i\geq 0}\frac{1}{1-t^2}(1+u_i)^2u_i^{n-3}\prod_{j\neq i}\frac{1}{u_i-u_j}\prod_{j\geq 0}\frac{1}{1-u_iu_j}$$

and

$$\begin{aligned}S_{n,1}(x;t) = \frac{(-1)^{\binom{k+1}{2}+1}}{(k+1)!}\prod_i\frac{1}{1-x_i}\prod_{1\leq i<j\leq n-k-1}\frac{1}{1-x_ix_j}\times\\ \times\sum_{i\geq 0}\frac{1}{1-t^2}(1+u_i)^2u_i^{n-3}\prod_{j\neq i}\frac{1}{u_i-u_j}\prod_{j\geq 0}\frac{1}{1-u_iu_j},\end{aligned}$$

where $u_i = tx_i$ with $x_0 = 1$.

We can simplify the sum above as follows. Notice that for any p variables $v=(v_1,\ldots,v_p)$ we have

$$\begin{aligned}\sum_{i=1}^{p}\frac{v_i^r}{\prod_{i<j}(v_i-v_j)} &= \frac{\sum_i(-1)^{i-1}v_i\prod_{s<l;l,s\neq i}(v_s-v_l)}{\prod_{s<l}(v_s-v_l)}\\ &= \frac{a_{(r-p+1)+\delta_p}(v)}{a_{\delta_p}(v)} = s_{(r-p+1)}(v) = h_{r-p+1}(v),\end{aligned}$$

where $h_s(v)=\sum_{i_1\leq i_2\leq\cdots\leq i_s}v_{i_1}v_{i_2}\cdots v_{i_s}$ is the s-th homogenous symmetric function defined in Section 2.3.

Then we have

$$\begin{aligned}\sum_{i=0}^{n-k-1}\frac{u_i^s}{\prod(u_i-u_j)}\prod\frac{1}{1-u_iu_j} &= \sum_{i=0}^{n-k-1}\frac{u_i^s}{\prod(u_i-u_j)}\sum_p u_i^ph_p(u)\\ &= \sum_{p\geq 0}\sum_{i=0}^{n-k-1}\frac{u_i^{s+p}}{\prod(u_i-u_j)}h_p(u)\\ &= \sum_{p\geq 0}h_{s-n+k+1+p}(u)h_p(u) = c_{s-n+k+1}(u), \qquad (26)\end{aligned}$$

where $c_i=\sum_{n\geq 0}h_nh_{n+i}$. We then have the new formulas

$$S_{n,1}(x;t) = \prod\frac{1}{1-x_i}\prod_{1\leq i<j\leq n-k-1}\frac{1}{1-x_ix_j}\frac{1}{1-t^2}(c_1(u)+c_0(u)), \qquad (27)$$

where $(1+u_i)^2u_i^{n-3} = u_i^{n-3} + 2u_i^{n-2} + u_i^{n-1}$ contributed to $c_{-1}(u) + 2c_0(u) + c_1(u)$ and by its definition $c_{-1} = c_1$. We are now ready to prove the following.

THEOREM 4.9. The number of shifted standard tableaux of shape $\delta_n \setminus \delta_1$ is equal to

$$g_n \frac{C_n C_{n-2}}{2C_{2n-3}},$$

where $g_n = \frac{\binom{n+1}{2}!}{\prod_{0\le i<j\le n}(i+j)}$ is the number of shifted staircase tableaux of shape $(n, n-1, \dots, 1)$ and $C_m = \frac{1}{m+1}\binom{2m}{m}$ is the $m-$th Catalan number.

PROOF. We will use Proposition 4.8 and the formula (27).
By the properties of ϕ we have for the shape $D = \delta_n \setminus \delta_k$,

$$F_D(q) = \sum_{T|\,\mathrm{sh}(T)=D} q^{\sum T[i,j]} = \sum_{P=\phi(T)} q^{(n-k)|\mu|+\sum P[i,j]} = S_{n,k}(q, q^2, \dots, q^{n-k-1}; q^{n-k}).$$

Now let $k = 1$. For the formula in Proposition 4.8 we have $N = \binom{n+1}{2} - 1$ and plugging $x = (q, \dots, q^{n-2})$, $t = q^{n-1}$ in (27) we get

$$\lim_{q\to 1}(1-q)^N F_D(q) = \prod \frac{1}{1-q^i} \prod_{1\le i<j\le n-2} \frac{1}{1-q^{i+j}} \frac{1}{1-q^{2(n-1)}}(c_1(u) + c_0(u)),$$

where $u = (q^{n-1}, q^n, \dots, q^{2n-3})$.

We need to determine $\lim_{q\to 1}(1-q)^{2n-3}c_s(u)$. Let $c_s(x;y) = \sum_l h_l(x)h_{l+s}(y)$ where $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_m)$. We have

$$\begin{aligned} c_s(x;y) &= \sum_l \sum_{i_1\le\cdots\le i_l; j_1\le\cdots\le j_{l+s}} x_{i_1}\cdots x_{i_l} y_{j_1}\cdots y_{j_{s+l}} \\ &= \sum_p h_s(y_1, \dots, y_p) \sum_{P:(1,p)\to(n,m)} (-1)^{m+n-p-\#P} \sum_l h_l((xy)_P), \end{aligned}$$

where the sum runs over all fully ordered collections of lattice points P in between $(1,p)$ to (n,m) and $(xy)_P = (x_{i_1}y_{j_1}, \dots)$ for all points $(i_1, j_1), \dots \in P$ and the (-1)s indicate the underlying inclusion-exclusion process. We also have that

$$\sum_l h_l((xy)_P) = \frac{1}{\prod_{(i,j)\in P}(1-x_iy_j)}.$$

The degree of $1-q$ dividing the denominators after substituting $(x_i, y_j) = (u_i, u_j) = (q^{n-2+i}, q^{n-2+j})$ for the evaluation of $c_s(u)$ is equal to the number of points in P. $\#P$ is maximal when the lattice path is from $(1,1)$ to $(n-1, n-1)$ and is saturated, so $\max(\#P) = 2(n-1) - 1 = 2n-3$. The other summands will contribute 0 when multiplied by the larger power of $(1-q)$ and the limit is taken. For each maximal path we have $\{i+j|(i,j)\in P\} = \{2, \dots, 2n-2\}$ and the number of these paths is $\binom{2n-4}{n-2}$, so

$$(1-q)^{2n-3}c_s(q^{n-1}, \dots, q^{2n-3}) = \binom{2n-4}{n-2}\prod_{i=2}^{2n-2}\frac{1-q}{1-q^{2(n-2)+i}} + (1-q)...,$$

where the remaining terms are divisible by $1-q$, hence contribute 0 when the limit is taken.

Now we can proceed to compute $\lim_{q\to 1}(1-q)^N S_{n,1}(q^1, q^2, \ldots, ; q^{n-1})$. Putting all these together we have that

$$\lim_{q\to 1}(1-q)^{\binom{n+1}{2}-1} S_{n,1}(q^1, \ldots, q^{n-2}; q^{n-1}) =$$

$$\lim_{q\to 1} \prod_{1\le i\le n-2} \frac{1-q}{1-q^i} \prod_{1\le i<j\le n-2} \frac{1-q}{1-q^{i+j}} \frac{1-q}{1-q^{2(n-1)}} \left((1-q)^{2n-3}(c_1(u)+c_0(u))\right) =$$

$$\prod_{0\le i<j\le n-2} \frac{1}{i+j} \frac{1}{2(n-1)} 2\binom{2n-4}{n-2} \prod_{i=2}^{2n-2} \frac{1}{2n-4+i} = \frac{g_{\delta_{n-2}}}{\binom{n-1}{2}!} \frac{1}{(n-1)} \binom{2n-4}{n-2} \frac{(2n-3)!}{(4n-6)!},$$

where $g_{n-2} = \frac{\binom{n-1}{2}!}{\prod_{0\le i<j\le n-2}(i+j)}$ is the number of shifted staircase tableaux of shape $(n-2, \ldots, 1)$. After algebraic manipulations we arrive at the desired formula. □

7. Straight truncated SYTs

We will compute the number of standard tableaux of straight truncated shape $D = n^m \setminus \delta_k$ using propositions 4.4, 4.8 and 4.6.

By Proposition 4.4 we have

$$\begin{aligned} F_D(q) &= \sum_{T:\mathrm{sh}(T)=D} q^{\sum T[i,j]} \\ &= \sum_{\lambda,\mu | l(\mu)\le k+1, l(\lambda)\le n} \sum_{P,\mathrm{sh}(P)=\lambda/\mu, Q,\mathrm{sh}(Q)=\lambda} q^{\sum P[i,j]+\sum Q[i,j]-|\lambda|+(n-k)|\mu|} \\ &= \sum_{\lambda,\mu|l(\lambda)\le n, l(\mu)\le k+1} s_\lambda(1, q, q^2, \ldots, q^{m-1}) s_{\lambda/\mu}(q, q^2, \ldots, q^{n-k-1}) q^{(n-k)|\mu|}, \end{aligned}$$

which is $D_{n,m,k}(x, z; t)$ for $x = (q, q^2, \ldots, q^{n-k-1})$, $z = (1, q, \ldots, q^{m-1})$ and $t = q^{n-k}$ and from its simplified formula from Proposition 4.6 and Proposition 4.8 the number of standard tableaux of shape $n^m \setminus \delta_k$ is

(28)

$$\lim_{q\to 1}(1-q)^{nm-\binom{k+1}{2}} F_D(q) =$$

$$\lim_{q\to 1} \left(\prod_{i=1,j=0}^{n-k-1,m-1} \frac{1-q}{1-q^{i+j}} (1-q)^{m(k+1)-\binom{k+1}{2}} \Big(\sum_{\nu|l(\nu)\le k+1} s_\nu(q^{n-k}, q^{n-k+1}, \ldots, q^{m-1+n-k}) \Big) \right).$$

We are thus going to compute the last factor.

Lemma 4.10. Let $p \ge r$ and $N = rp - \binom{r}{2}$. Then for any s we have

$$\lim_{q\to 1}(1-q)^N \sum_{\lambda|l(\lambda)\le r} s_\lambda(q^{1+s}, \ldots, q^{p+s}) = \frac{g_{(p,p-1,\ldots,p-r+1)}}{N!} \frac{E_1(r,p,s)}{E_1(r,p,0)},$$

where

$$E_1(r,p,s) = \prod_{r<l<2p-r+2} \frac{1}{(l+2s)^{r/2}} \prod_{2\le l\le r} \frac{1}{((l+2s)(2p-l+2+2s))^{\lfloor l/2\rfloor}}$$

for r even and $E_1(r,p,s) = \frac{((r-1)/2+s)!}{(p-(r-1)/2+s)!} E_1(r-1,p,s)$ when r is odd and g_λ is the number of shifted SYTs of shape λ, given by formula 1.18.

PROOF. Consider the Robinson-Schensted-Knuth (RSK) correspondence between SSYTs with no more than r rows filled with $x_1, \ldots, x_p$ and symmetric $p \times p$ integer matrices A, as defined in Section 2.1. The limit on the number of rows translates through Schensted's Theorem 1.10 to the fact that there are no $m+1$ nonzero entries in A with coordinates $(i_1, j_1), \ldots, (i_{r+1}, j_{r+1})$, s.t. $i_1 < \cdots < i_{r+1}$ and $j_1 > \cdots > j_{r+1}$ (i.e. a decreasing subsequence of length $r+1$ in the generalized permutation w_A corresponding to A). Let $\mathcal{A}$ be the set of such matrices. Let $\mathcal{B} \subset \mathcal{A}$ be the set of $0-1$ matrices satisfying this condition, we will refer to them as allowed configurations. Notice that $A \in \mathcal{A}$ if and only if $B \in \mathcal{B}$, where $B[i,j] = \begin{cases} 1, & \text{if } A[i,j] \neq 0 \\ 0, & \text{if } A[i,j] = 0 \end{cases}$. We thus have that

$$\begin{aligned}
\sum_{\lambda | l(\lambda) \leq r} s_\lambda(x_1, \ldots, x_p) &= \sum_{A \in \mathcal{A}} \prod_i x_i^{A[i,i]} \prod_{i>j} (x_i x_j)^{A[i,j]} \\
&= \sum_{B \in \mathcal{B}} \prod_{i:B[i,i]=1} \Big(\sum_{a_{i,i}=1}^{\infty} x_i^{a_{i,i}} \Big) \prod_{i>j:B[i,j]=1} \Big(\sum_{a_{i,j}=1}^{\infty} (x_i x_j)^{a_{i,j}} \Big) \\
&= \sum_{B \in \mathcal{B}} \prod_{i:B[i,i]=1} \frac{x_i}{1-x_i} \prod_{i>j:B[i,j]=1} \frac{x_i x_j}{1 - x_i x_j}.
\end{aligned}$$

Notice that B cannot have more than N nonzero entries on or above the main diagonal. No diagonal $i+j=l$ (i.e. the antidiagonals) can have more than r nonzero entries on it because of the longest decreasing subsequence condition. Also if $l \leq r$ or $l > 2p-r+1$, the total number of points on such diagonal are $l-1$ and $2p-l+1$ respectively. Since B is also symmetric the antidiagonals $i+j=l$ will have $r-1$ entries if $l \equiv r-1 \pmod 2$ and r is odd. Counting the nonzero entries on each antidiagonal on or above the main diagonal gives always exactly N in each case for the parity of r and p.

If B has less than N nonzero entries, then

$$\lim_{q\to 1} (1-q)^N \prod_{i:B[i,i]=1} \frac{q^{i+1}}{1-q^{i+s}} \prod_{i>j:B[i,j]=1} \frac{q^{i+j+2s}}{1-q^{i+j+2s}} =$$

$$\lim_{q\to 1} (1-q)^{N-|B|>0} \prod_{i:B[i,i]=1} \frac{q^{i+1}(1-q)}{1-q^{i+s}} \prod_{i>j:B[i,j]=1} \frac{q^{i+j+2s}(1-q)}{1-q^{i+j+2s}} = 0,$$

so such B's won't contribute to the final answer.

Consider now only B's with maximal possible number of nonzero entries (i.e. N), which forces them to have exactly r (or $r-1$) nonzero entries on every diagonal $i+j=l$ for $r < l \leq 2p-r+1$ and all entries in $i+j \leq r$ and $i+j > 2p-r+1$.

If r is even, then there are no entries on the main diagonal when $r < l < 2p-r+2$ and so there are $r/2$ terms on each diagonal $i+j=l$. Thus every such B contributes the same factor when evaluated at $x = (q^{1+s}, \ldots)$:

$$E_q(r,p,s) := \prod_{r<l<2p-r+2} \frac{q^{(l+2s)r/2}}{(1-q^{l+2s})^{r/2}} \prod_{2 \leq l \leq r} \frac{q^{(l+4s+2p-l+2)\lfloor l/2 \rfloor}}{((1-q^{l+2s})(1-q^{2p-l+2+2s}))^{\lfloor l/2 \rfloor}}.$$

If r is odd, then the entries on the main diagonal will all be present with the rest being as in the even case with $r-1$, so the contribution is

$$E_q(r,p,s) := \prod_{\frac{r+1}{2} \leq i \leq p - \frac{r+1}{2}+1} \frac{q^{i+s}}{1-q^{i+s}} E_q(r-1,p,s).$$

Let now M be the number of such maximal B's in $\mathcal{A}_0$. The final answer after taking the limit is $ME_1(r,p,s)$, where we have $E_1(r,p,s) = \lim_{q\to 1}(1-q)^N E_q(r,p,s)$ as defined in the statement of the lemma.

In order to find M observe that the case of $s=0$ gives

$$\lim_{q\to 1}(1-q)^N \sum_{\lambda | l(\lambda)\le r} s_\lambda(q^1,\dots,q^p) = ME_1(r,p,0), \tag{29}$$

on one hand. On the other hand via the bijection ϕ we have

$$\sum_{\lambda | l(\lambda)\le m} s_\lambda(q^1,\dots,q^n) = \sum_T q^{\sum T[i,j]},$$

where the sum on the right side goes over all shifted plane partitions T of shape $(p, p-1,\dots,p-r+1)$. Multiplying by $(1-q)^N$ and taking the limit on the right hand side gives us, by the inverse of Proposition 4.8, $\frac{1}{N!}$ times the number of standard shifted tableaux of that shape. This number is well known and is $g_{(p,p-1,\dots,p-r+1)} = \dfrac{N!}{\prod_u h_u}$, where the product runs over the hook-lengths of all boxes on or above the main diagonal of (p^r, r^{p-r}). Putting all this together gives

$$ME_1(r,p,0) = \frac{g_{(p,p-1,\dots,p-r+1)}}{N!}.$$

Solving for M we obtain the final answer:

$$\frac{g_{(p,p-1,\dots,p-r+1)}}{N!}\frac{E_1(r,p,s)}{E_1(r,p,0)}.$$

□

We can now put all of this together and state

THEOREM 4.11. The number of standard tableaux of truncated straight shape $(n^m)\setminus\delta_k$ (assume $n\le m$), is

$$(mn - \binom{k+1}{2})! \times \frac{f_{(n-k-1)^m}}{(m(n-k-1))!} \times \frac{g_{(m,m-1,\dots,m-k)}}{((k+1)m-\binom{k+1}{2})!}\frac{E_1(k+1,m,n-k-1)}{E_1(k+1,m,0)},$$

where

$$E_1(r,p,s) = \begin{cases} \displaystyle\prod_{r<l<2p-r+2}\frac{1}{(l+2s)^{r/2}}\prod_{2\le l\le r}\frac{1}{((l+2s)(2p-l+2+2s))^{\lfloor l/2\rfloor}}, & r \text{ even},\\ \dfrac{((r-1)/2+s)!}{(p-(r-1)/2+s)!}E_1(r-1,p,s), & r \text{ odd}.\end{cases}$$

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