Dyck tilings, linear extensions, descents, and inversions

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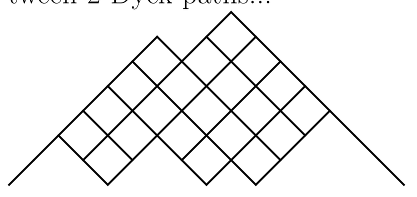
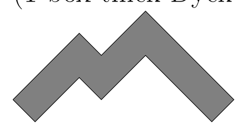
Cover-inclusive Dyck tilings, definition

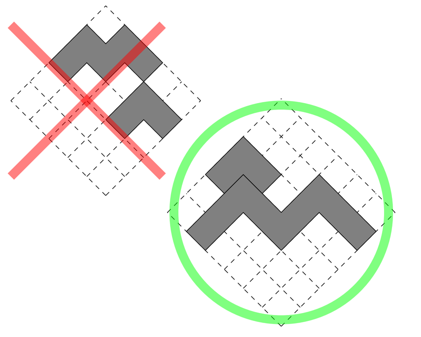
A **tiling** of a skew [rotated] shape between 2 Dyck paths...

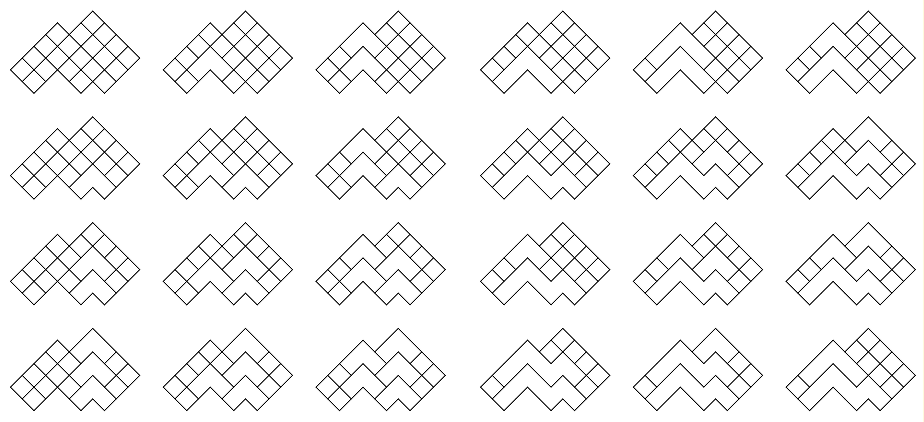


with **Dyck tiles**.... (1-box thick Dyck path)



... such that: If one Dyck tile covers another (i.e. has a box whose center lies straight above the center of a box of the second tile), then the horizontal extent of this tile is a subset of the horizontal extent of second tile.





Example – all cover-inclusive Dyck tilings of a given shape.

For a Dyck tiling T of shape $\mathrm{sh}(T) = \lambda/\mu$, we define $\mathrm{dis}(T) = \mathrm{dis}(\lambda,\mu)$ – the **discrepancy** between λ and μ, i.e. number of places where λ has a Down step while μ has an Up step (half the Hamming distance between the Dyck words of λ and μ). Let **tiles**(T) be the number of tiles in T, and **area**(T) be the number of unit squares of the skew shape λ/μ. We define

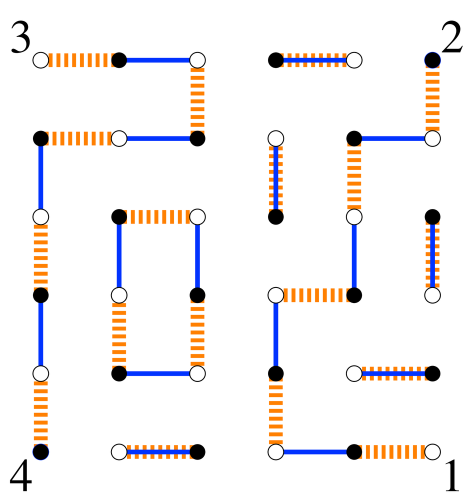
$$\mathbf{art}(T) := (\underline{\mathbf{a}}\mathrm{rea}(T) + \underline{\mathbf{t}}\mathrm{iles}(T))/2.$$

Background: the Double-Dimer model

$G = (V, E)$ - bipartite graph, $V = \{B, W\}$, $E \subset \{(i, j) | i \in B, j \in W\}$

Dimer cover = perfect matching.

$\mathcal{N} = \{1, 2, \ldots, 2n\} \subset V$ (nodes on boundary, n black, n white)



$G^{BW} = G \setminus ((\{1, 3, .., 2n-1\} \cap B) \cup (\{2, 4, ..., 2n\} \cap W))$
$G^{WB} = G \setminus ((\{1, 3, .., 2n-1\} \cap W) \cup (\{2, 4, .., 2n\} \cap B))$

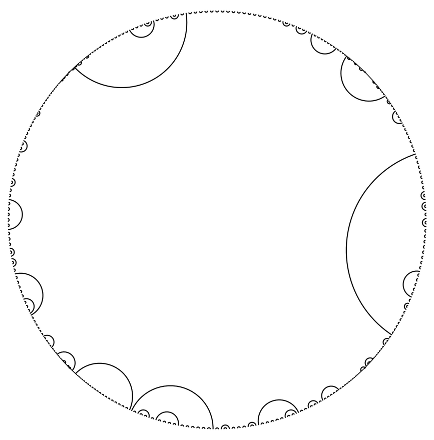
Double-dimer$(G, \mathcal{N})$ = **superposition of Dimer**(G^{BW}) + **Dimer**(G^{WB})

[In the Double-dimer model the odd nodes in $\mathcal{N}$ will always be connected (paired) with the even nodes - **pairing** σ. E.g. $\sigma = \{\{1, 2\}, \{3, 4\}\}$.]

Theorem 4 (Kenyon-Wilson). **The probability of a certain pairing σ of the $2n$ nodes $\mathcal{N}$ is a rational function of $X_{i,j} = \frac{Z_{i,j}}{Z}$, where $Z_{i,j}$ is the weighted sum of dimer covers of $G \setminus \{i, j\}$, and Z it the weighted sum of dimer covers of G.**

E.g. $\Pr(\{\{1, 2\}, \{3, 4\}\}) = \frac{X_{1,2}X_{3,4}}{X_{1,2}X_{3,4}+X_{1,4}X_{2,3}}$

Marginal pairing probabilities Probability that node i is paired with node j when $\#V$ is large?



Theorem 5 (Kenyon-Wilson). **Let S be a set of equal number of white and black nodes, then**

$$\frac{D_S}{D_\emptyset} = \sum_{\pi:\text{ pairing of nodes}} M_{S,\pi} \Pr(\pi),$$

where $D_S = \det[(1_{i,j\in S} + 1_{i,j\notin S}) \times (-1)^{(|i-j|-1)/2} X_{i,j}]^{j=2,4,\ldots,2n}_{i=1,3,\ldots,2n-1}$ **and**

$$M_{S,\pi} = \begin{cases} 0 & \pi \text{ connects some nodes } S \leftrightarrow S^c \\ 1 & \text{otherwise} \end{cases}$$

$$\Pr(\pi) = \sum M^{-1}_{S,\pi} \frac{D_S}{D_\emptyset}$$

Inverting M — counting Dyck tilings

$S \leftrightarrow$ balanced parentheses $\leftrightarrow$ noncrossing pairings $\leftrightarrow$ Dyck paths!

$S = \{1, 2, 3, 6\} \leftrightarrow P = ()(()) \leftrightarrow \pi = (1, 2)(3, 6)(4, 5) \leftrightarrow$ [1 2 3 4 5 6]

$$M_{S,\pi} = M_{P_1,P_2} = \begin{cases} 1 & P_1 \overset{()}{\leftarrow} P_2 \\ 0 & \text{otherwise} \end{cases}$$

$P_1 \overset{()}{\leftarrow} P_2$ if P_1 can be obtained from P_2 by reversing several pairs of matched (), e.g. $()() \overset{()}{\leftarrow} (())$, but $()(()) \overset{()}{\not\leftarrow} ((()))$.

		()()()	()(())	(())()	(()())	((()))
{1, 2, 3, 4, 5, 6}	()()()	1	1	1	1	1
{1, 2, 3, 6}	()(())	0	1	0	1	0
{1, 4, 5, 6}	(())()	0	0	1	1	0
{1, 6}	(()())	0	0	0	1	1
{1, 3, 4, 6}	((()))	0	0	0	0	1

Theorem 6 (Kenyon-Wilson). **The inverse matrix of M is given by** $M^{-1}_{\lambda/\mu} = (-1)^{|\lambda/\mu|} \times$ **cover-inclusive Dyck tilings of shape λ/μ**

231-avoiding permutations

Theorem 9. The maps $\mathrm{DTS}(\mathrm{zigzag}_n, \cdot)$ and $\mathrm{DTR}(\mathrm{zigzag}_n, \cdot)$ restrict to bijections between 231-avoiding permutations in S_n and Dyck tilings whose lower path is zigzag_n and which contain only one-box tiles (so, in particular these correspond to Dyck paths, being determined just by the upper path μ).

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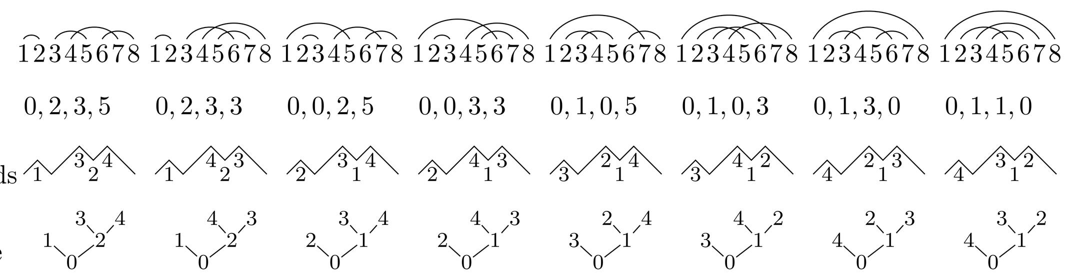
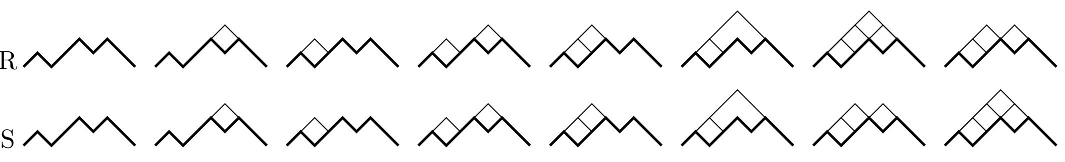
The many faces and aspects of Dyck tilings

min-word w	1346 2578	1354 2678	2146 3578	2154 3678	2316 4578	2314 5678	2351 4678	2431 5678
$p_1, \ldots, p_n$	0, 2, 3, 5	0, 2, 3, 3	0, 0, 2, 5	0, 0, 3, 3	0, 1, 0, 5	0, 1, 0, 3	0, 1, 3, 0	0, 1, 1, 0
preorder word	1234	1243	2134	2143	3124	3142	4123	4132
inverse-preorder σ	1234	1243	2134	2143	2314	2413	2341	2431
left-endpoints ℓ	1346	1364	3146	3164	3416	3614	3461	3641
right-endpoints r	2857	2875	8257	8275	8527	8725	8572	8752
$\mathrm{des}(\sigma) = \mathrm{dis}(\mathrm{DTR}) = \mathrm{des}(w)$	0	1	1	2	1	1	1	2
$\mathrm{inv}(\sigma) = \mathrm{art}(\mathrm{DTS})$	0	1	1	2	2	3	3	4
$\mathrm{inv}(w) = \mathrm{tiles}(\mathrm{DTS})$ = nestings(matching)	0	1	1	2	2	2	3	4





Main Results

With the definitions from Sections *Cover-inclusive Dyck tilings, definition* and *Chord posets, statistics*:

Theorem 1 (Conjecture 1 in (Kenyon-Wilson 2011)). **Given a Dyck path λ of order n, we have**

$$\sum_{\text{Dyck tilings } T \in \mathcal{D}(\lambda,*)} q^{\mathrm{art}(T)} = \frac{[n]_q!}{\prod_{\text{chords } c \text{ of } \lambda} [|c|]_q}, \quad (1)$$

where the sum is over all cover-inclusive Dyck tilings T with fixed lower path λ.

Theorem 2. Given a Dyck path λ of order n, we have

$$\sum_{\text{Dyck tilings } T \in \mathcal{D}(\lambda,*)} z^{\mathrm{dis}(T)} = \sum_{\sigma \in \mathcal{L}(P_\lambda)} z^{\mathrm{des}(\sigma)}, \quad (2)$$

where $\mathcal{L}$ is the Jordan-Hölder set (set of linear extensions) of the chord poset P_λ of λ.

Proof. The two bijections DTS and DTR, defined below, and the q−hook-length formula: $\sigma \in \mathcal{L}(P_\lambda) \longleftrightarrow$ Dyck tilings with lower shape λ: $\mathrm{DTR}(\lambda, \sigma)$ and $\mathrm{DTS}(\lambda, \sigma)$, s.t.

$$\mathrm{art}(\mathrm{DTS}(\lambda, \sigma)) = \mathrm{inv}(\sigma) \quad \text{and} \quad \mathrm{dis}(\mathrm{DTR}(\lambda, \sigma)) = \mathrm{des}(\sigma). \quad \square$$

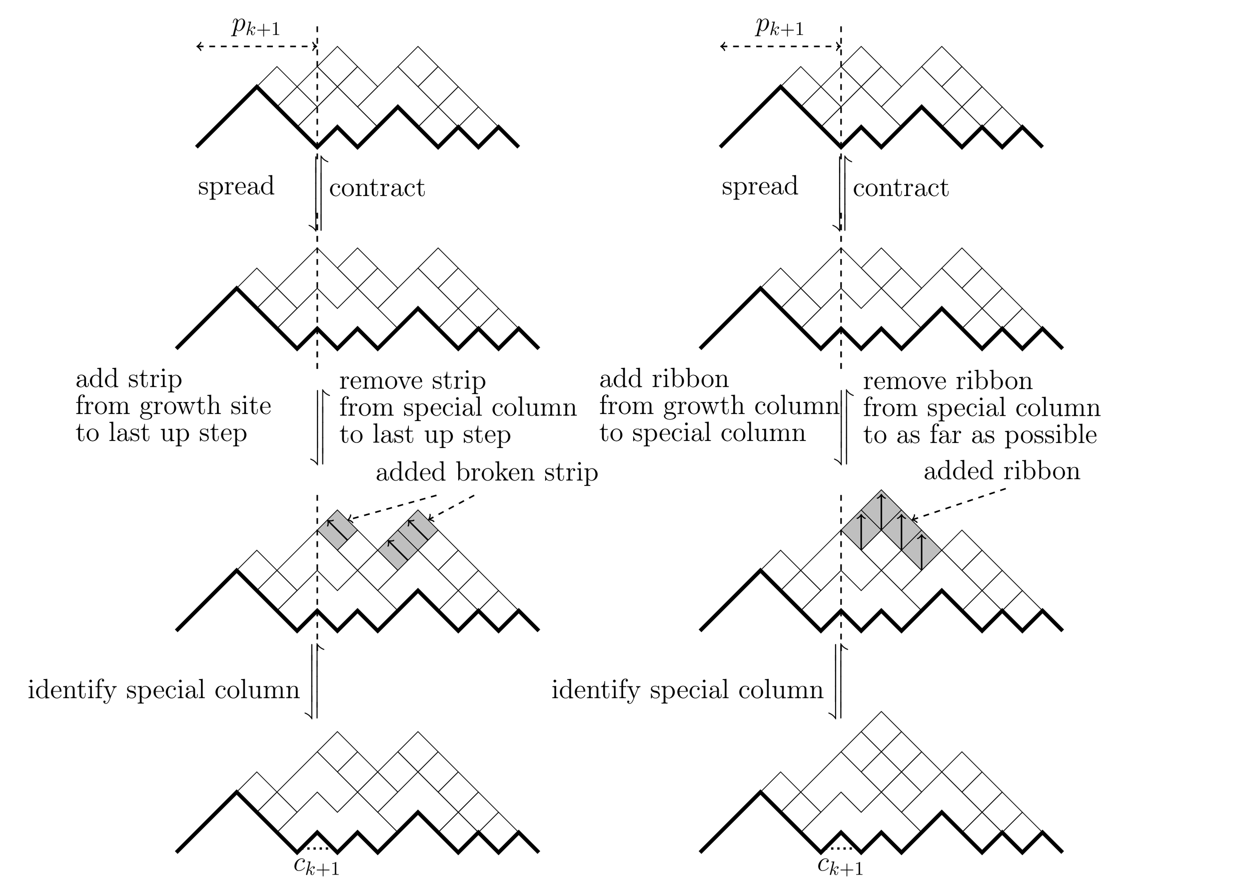
Theorem 3. The maps DTS and DTS are bijections between integer sequences $p_1, \ldots, p_n$, such that $0 \le p_i \le 2(i-1)$ and cover-inclusive Dyck tilings of order n.

Bijections: Linear extenstions ↔ Dyck tilings

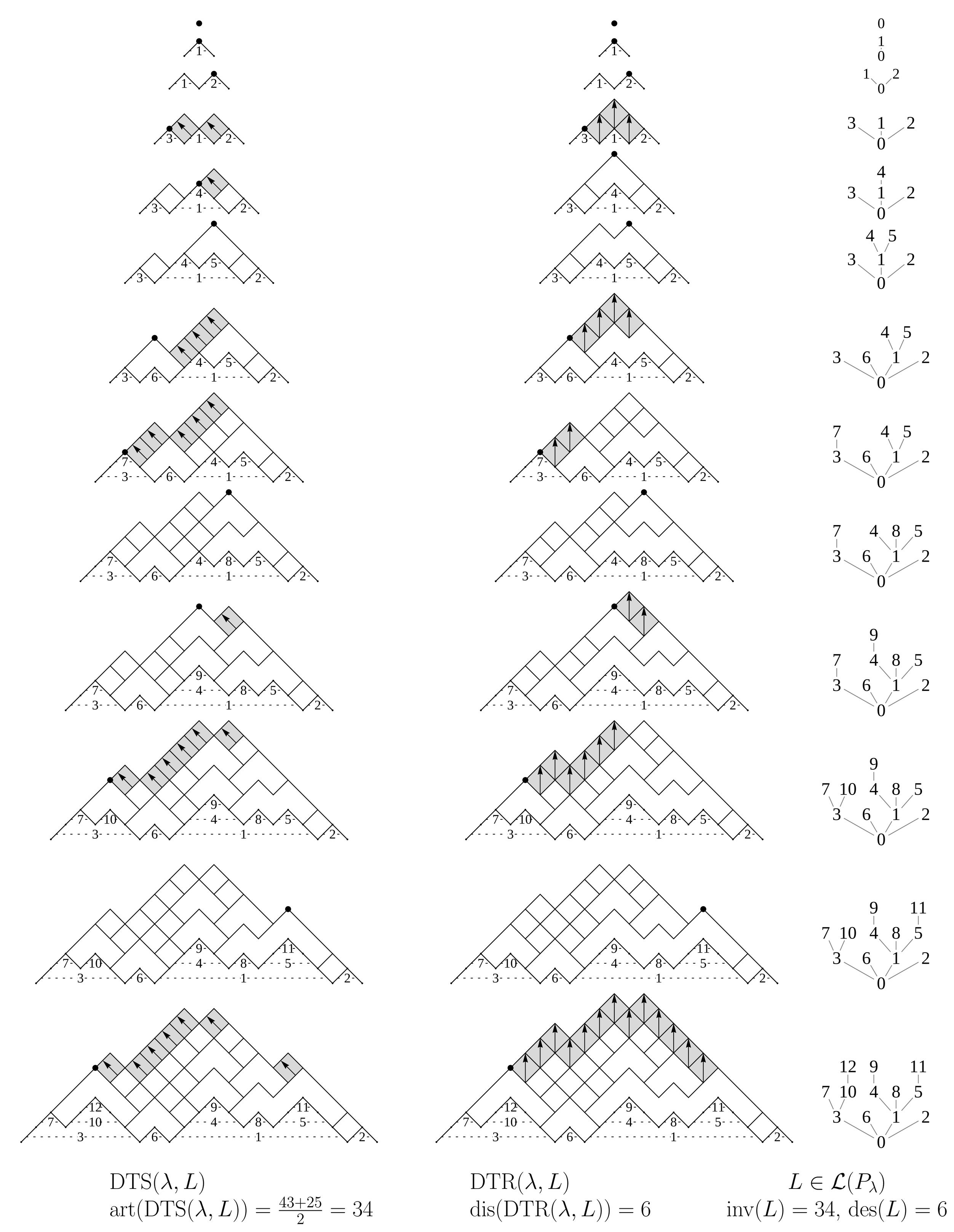
Two bijections: Linear extensions L of $P_\lambda \leftrightarrow (p_1, \ldots, p_n) \leftrightarrow$ Dyck tilings $T \in \mathcal{D}(\lambda, *)$.

Algorithm: Let P_k be the subposet of P_λ of the nodes labelled $0, 1, \ldots, k$ in L, corresponding to Dyck path λ_k, and let T_k be the corresponding tiling. $P_{k+1} = P_k + c_{k+1}$, where c_{k+1} is the node/chord labelled $k+1$. Then λ_{k+1} is obtained from λ_k by expansion at the position of c_{k+1} relative to λ_k, at horizontal distance p_{k+1} from the beginning of λ_k. Then T_{k+1} is obtained from T_k by:

Step $T_k \to T_{k+1}$: **growing** at p_{k+1}, the insertion point of chord labeled $k+1$: spreading columns and adding strips (DTS) or ribbons (DTR):



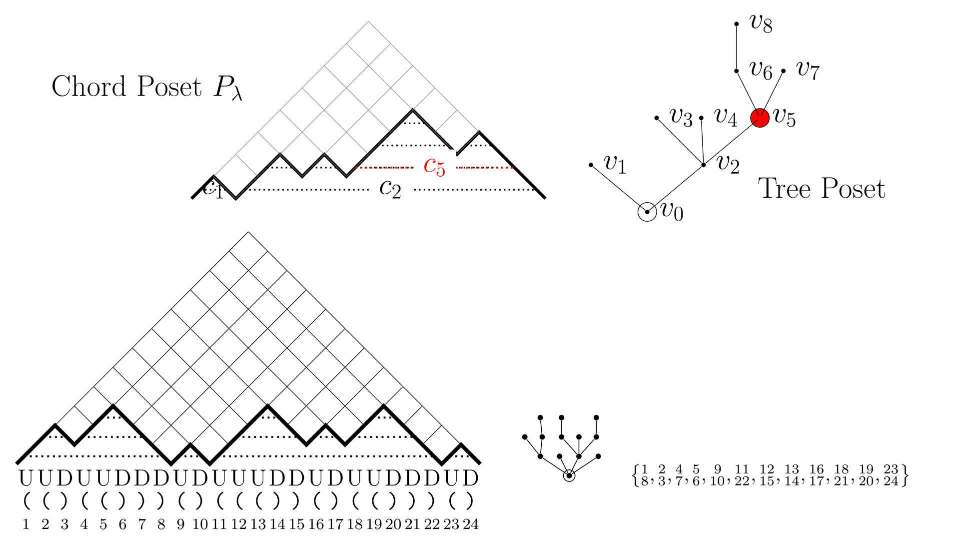
Bijections: DTS: $\mathrm{art}(\mathrm{DTS}(\lambda, L)) = \mathrm{inv}(L)$ DTR: $\mathrm{dis}(\mathrm{DTR}(\lambda, \sigma)) = \mathrm{des}(L)$

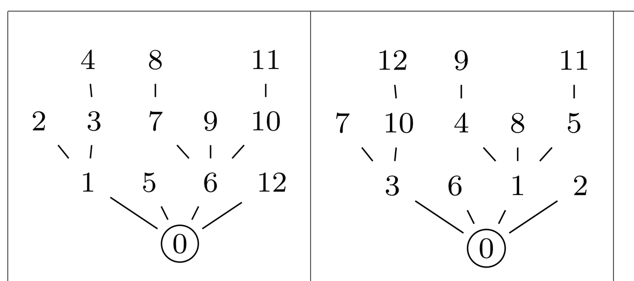


$\mathrm{DTS}(\lambda, L)$, $\mathrm{art}(\mathrm{DTS}(\lambda, L)) = \frac{43+25}{2} = 34$ — $\mathrm{DTR}(\lambda, L)$, $\mathrm{dis}(\mathrm{DTR}(\lambda, L)) = 6$ — $L \in \mathcal{L}(P_\lambda)$, $\mathrm{inv}(L) = 34$, $\mathrm{des}(L) = 6$

Chord Posets, statistics

The **chords** of a Dyck path λ are the segments between matching Up and Down steps (matching parentheses in the corresponding balanced parentheses expression). **Partial order on chords:** $c_i \prec c_j$ if c_j is vertically above c_i (the ( ) of c_j nest inside the ( ) of c_i). **Chord length** $|c|$ = number of Up steps on or above c.





Reference **natural labeling** $\mathbf{L_0}$ of $\mathbf{P_\lambda}$ (left-to-right depth-first search) — Sample natural labeling L of tree — Left and right endpoints of chords $\frac{\ell_1,\ldots,\ell_n}{r_1,\ldots,r_n}$ of λ listed in the order given by L.

$3, 7, 10, 12, 6, 1, 4, 9, 8, 5, 11, 2$
Preorder word $L \circ L_0^{-1}$ (list of labels of L encountered in left-to-right depth first search)

$6, 12, 1, 7, 10, 5, 2, 9, 8, 3, 11, 4$
Element of $\mathcal{L}(P_\lambda)$: $L_0 \circ L^{-1}$, **inverse preorder word**, standardization of $\ell_1, \ldots, \ell_n$.

Linear extensions of naturally labeled poset P_λ:

$$\mathcal{L}(P) = \{L_0 \circ L^{-1} : L \text{ is a natural labeling of } P\}.$$

Inversions statistic of a permutation σ on $[n]$ (element of $\mathcal{L}(P)$): $\mathrm{inv}(\sigma) = \#\{(i, j) : 1 \le i < j \le n, \sigma(i) > \sigma(j)\}$, **descent** statistic: $\mathrm{des}(\sigma) = \#\{i < n : \sigma_i > \sigma_{i+1}\}$.

Labeled tree/chord poset (P_λ, L)
$\leftrightarrow$ **sequence of insertion locations of chords** $(p_1, \ldots, p_n)$, s.t. $0 \le p_i < 2i-1$: $p_i = \#\{j < i : \ell_j < \ell_i\} + \#\{j < i : r_j < \ell_i\}$
$\leftrightarrow$ **Perfect matchings on** $1, \ldots, 2n$:

$$\mathrm{match}(p_1, .., p_n) = I_{p_n} \circ \mathrm{match}(p_1, .., p_{n-1}) \cup \{(p_n+1, 2n)\},\ I_m(a) = \begin{cases} a, & m \ge a \\ a+1, & \text{o.w.} \end{cases}$$

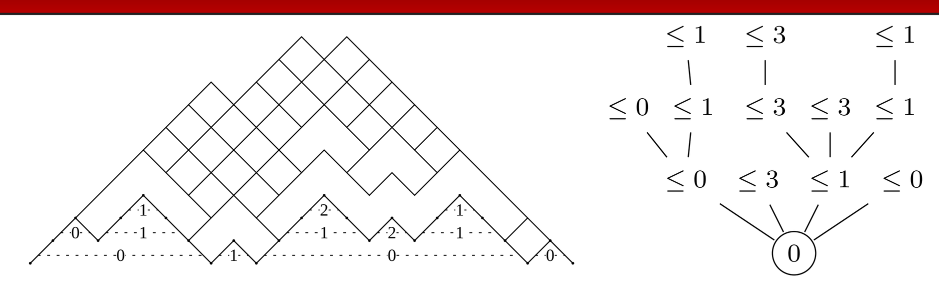
Hook-length formula

Knuth: the number of linear extensions of a tree poset P is $\frac{n!}{\prod_{v\in P} h_v}$, where h_v is the number of descendants of v plus 1. q-analog [Björner and Wachs] :

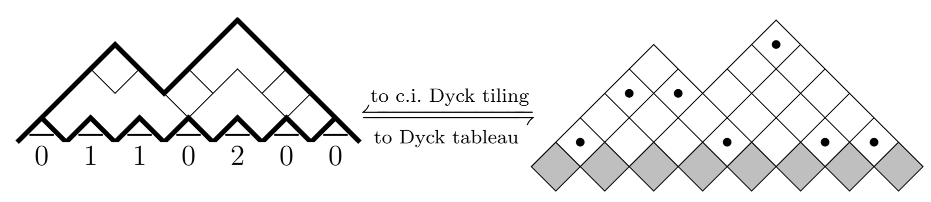
$$\sum_{\sigma\in\mathcal{L}(P_\lambda)} q^{\mathrm{inv}(\sigma)} = \frac{[n]_q!}{\prod_{\text{vertices } v \in P_\lambda} [|\text{subtree rooted at } v|]_q},$$

where $[n]_q = 1 + q + \cdots + q^{n-1}$ and $[n]_q! = [1]_q \cdots [n]_q$.

Dyck tableaux and Dyck tilings

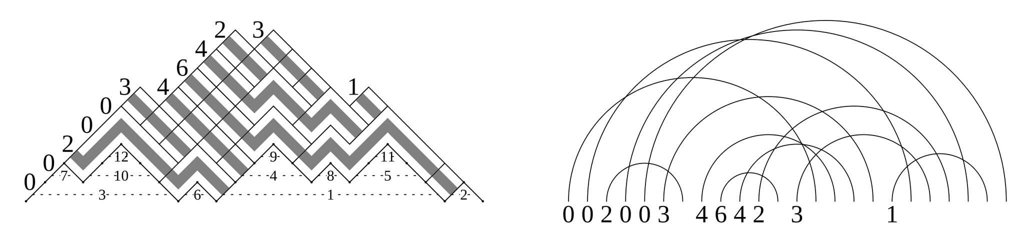


Proposition 7. There is a bijection between the cover-inclusive Dyck tilings T of a skew shape λ/μ and the weakly increasing assignments of nonnegative integers to the poset of chords P_λ, such that the number g_c assigned to chord c satisfies $0 \le g_c \le h_c$, where h_c is the maximum number of Dyck tiles that can cover the chord c and fit within the shape λ/μ. This bijection satisfies $\sum_c g_c = (\mathrm{area}(T) - \mathrm{tiles}(T))/2$.



Proposition 8. There is a bijection between the cover-inclusive Dyck tilings whose lower path is the zig-zag path $\mathrm{zigzag}_n = (\mathrm{UD})^n$ of n up-down steps and Dyck tableaux [introduced by Aval, Boussicault, and Dasse-Hartaut in ARXIV:1109.0370v2] of order n.

Hermite histories



Transforming a cover-inclusive Dyck tiling into an *histoire d'Hermite* [Kim, Konvalinka]. Left: each number on the up step counts the number of tiles of the tiling which are encountered by the exploration process of gray paths and each tile is encountered exactly once. Right: the beginning of each arc labeled with the number of arcs containing it (nesting number).

Theorem 10. The *histoire d'Hermite* arising from exploring $\mathrm{DTS}(p_1, \ldots, p_n)$ from the left is the same as the *histoire d'Hermite* of $\mathrm{match}(p_1, \ldots, p_n)$ recording the nesting numbers on the up steps.

The MAD statistic

Clarke, Steingrímsson, and Zeng define **mad** of a permutation σ as:

$$\mathrm{desdif}(\sigma) = \sum_{i\in \mathrm{DES}(\sigma)} (\sigma_i - \sigma_{i+1}),$$

$$\mathrm{res}(\sigma) = \sum_{i\in \mathrm{DES}(\sigma)} \#\{k < i : \sigma_i > \sigma_k > \sigma_{i+1}\},$$

$$\mathrm{mad}(\sigma) = \mathrm{desdif}(\sigma) + \mathrm{res}(\sigma).$$

$$\mathrm{mad}(\sigma) = \sum_{i\in \mathrm{DES}(\sigma)} [1+\#\{k > i+1 : \sigma_i > \sigma_k > \sigma_{i+1}\} + 2\times\#\{k < i : \sigma_i > \sigma_k > \sigma_{i+1}\}].$$

Theorem 11. For each permutation σ of order n,

$$\mathrm{art}(\mathrm{DTR}(\mathrm{zigzag}_n, \sigma)) = \mathrm{mad}(\sigma).$$

Remark: Combining Theorems 11 and 1, we obtain an involution

$$\mathrm{DTS}(\mathrm{zigzag}_n, \cdot)^{-1} \circ \mathrm{DTR}(\mathrm{zigzag}_n, \cdot)$$

on permutations of order n which goes by way of Dyck tilings and which maps mad to inv. In particular, this shows that mad is Mahonian.

Corollary: generalization of mad for linear extensions of tree posets P_λ, $\mathrm{mad}(L) := \mathrm{art}(\mathrm{DTR}(\lambda, L))$.