Separable equations and blowing up

Suppose that g and h are nonnegative functions and

$$\frac{dy}{dt} = \frac{g(t)}{h(y)}\,, \tag{1}$$

as on page 637 of Stewart. We can solve it by integrating both sides, but we may have to leave one side or both as an integral: $H(y) = G(t) + C$ where

$$\begin{aligned} G(t) &= \int_0^t g(s)\,ds\,; \\ H(y) &= \int_0^y h(s)\,ds\,. \end{aligned}$$

Whether or not we can explicitly evaluate these integrals, it is important to identify whether either or both of these has a finite limit as the upper limit of the integral goes to infinity. For each solution in the family, the value of $H(y(t)) - G(t)$ will be a constant C; for example if $y(t_0) = y_0$, then $C = H(y_0) - G(t_0)$. To see why finiteness of the integrals is important, let us see what happens as we increase y and t. In case you are not aware: if g is a continuous positive function, then $\int_a^\infty g(s)\,ds$ will either be finite for all choices of the lower limit a or will be infinite for all choices.

Case 1: both G and H increase without bound. In other words, both $\int_a^\infty g(s)\,ds$ and $\int_a^\infty h(s)\,ds$ are infinite. Then the common value of $H(y)$ and $G(t)+C$ can increase without bound, so no matter what the initial conditions, in all solutions, $y(t)$ will increase to infinity as t increases to infinity.

Case 2: G increases without bound but $\int_0^\infty h(s)\,ds = B < \infty$. In this case, as $G(t)$ increases toward $B - C$, we see that y must get arbitrarily large in order that $H(y)$ remain equal to $G(t) + C$, and for $t \geq B - C$ it is no longer possible to find a y big enough to have $H(y) = G(t) + C$. This means that y "blows up", going to $+\infty$ at time $B - C$. In the graph, we see a vertical asymptote.

Case 3: H increases without bound but $\int_0^\infty g(s)\,ds = B < \infty$. In this case, as $t \to \infty$, $G(t) \to B$, so $H(y(t)) \to B + C$, meaning that y approaches the value $H^{-1}(B + C)$. In otherwords, there is a horizontal asymptote.

Case 4: $\int_0^\infty h(s)\,ds = A < \infty$ and $\int_0^\infty g(s)\,ds = B < \infty$. In this case, the common value of $H(y)$ and $G(t) + C$ cannot increase beyond either A or $B + C$. The solutions with $C > A - B$ then H blows up first. This is like case 2: the solution blows up in finite time and there is a vertical asymptote. The solutions with $C < A - B$ see G blow up at a finite H value, meaning that there is a horizontal asymptote as in Case 3. Finally, in the solution with $C = A - B$, as $G(t)$ increases to B and $H(y)$ increases to A, both t and y will increase without bound, as in Case 1.

Examples:

1. Example 1 from Stewart. Here $h(y) = y^2, g(t) = t^2$, and so $H(y) = y^3/3, G(t) = t^3/3$. Both of these integrate to infinity so the solutions, which we know to have the explicit equation $y = \sqrt[3]{x^3 + K}$ all increase with no asymptote.

2. $y' = y^2$. Then $dy/y^2 = dt$ so $-1/y = t - C$ and $y = 1/(C - t)$. Here I used $-C$ rather than C in order to get a nicer final form. Note that dy/y^2 is integrable but dt is not. This is Case 2. There is a vertical asymptote at $t = C$. If you check what C is for any positive initial conditions, you see that C is always greater than t_0. For example, if $y(t_0) = y_0$ with $y_0, t_0 > 0$, then $y_0 = 1/(C - t_0)$ so $C = t_0 + 1/y_0$ which is greater than t_0.

3. $y' = y/t^2$. Then $dy/y = dt/t^2$ so $\log y = C - 1/t$ and $y = e^{C-1/t}$. As you can see, we are in Case 3. There is a horizontal asymptote corresponding to the fact that as $t \to \infty$, $y(t)$ approaches e^C.

In general, if you cannot integrate one or both exactly, you can use a comparison test to see which are finite.

Example:

4 $y' = y/\sqrt{1 + t^3}$. Then $dy/y = dt/\sqrt{1 + t^3}$. We can't integrate $(1 + t^3)^{-1/2}$ but this is less than $1/\sqrt{t^3}$ and we know $t^{-3/2}$ is integrable, so we are in Case 3.