Zeros of polynomials and their importance in combinatorics and probability

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Zero sets are important throughout mathematics!

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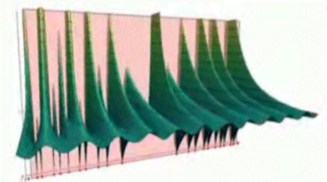

A 3-D representation of Riemann's zeta function. All the points where the function has zero value seem to lie on the same "critical" line (the black downward spikes), an observation known as the Riemann Hypothesis.

SSI*

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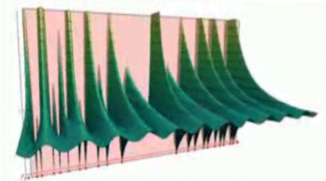

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Musical digression...

Zeta song lyrics

Last verse:

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Oh, where are the zeros of zeta of s?
We must know exactly, we cannot just guess,
In orer to strengthen the prime number theorem,
The integral's contour must not get too near 'em.
```

Lyrics (SSI) credited to Tom Apostol

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Curricular Digression: Gröbner bases are the new Gaussian elimination. In the future, every math major will learn the algorithmic soultion of systems of polynomial equations, just as they now learn algorithmic soultion of systems of linear equations.

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ODE's: Solutions to

$$p_n(x)\frac{d^n}{dx^n}f + \cdots + p_0(x)f = 0$$

fail to be regular (have a phase change) where p_n vanishes.

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PDE's: The evolution of an equation governed by the linear operator $\mathbf{P}(\partial/\partial\mathbf{x})\mathbf{F} = 0$ is determined by the geometry of the zero set of the polynomial $\mathbf{P}$, cf. Gårding's theory of hyperbolic polynomials and operators.

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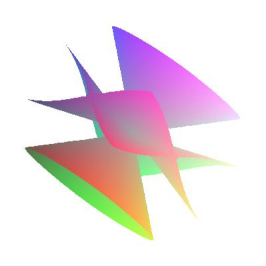
Certain components of the complement of the real zero set of a hyperbolic polynomial are convex, leading to many useful properties.



Combinatorics

Algebraic Combinatorics (Gian-Carlo Rota, 1985):

> *"The one contribution of mine that I hope will be remembered has consisted in pointing out that all sorts of problems of combinatorics can be viewed as problems of the location of the zeros of certain polynomials..."*

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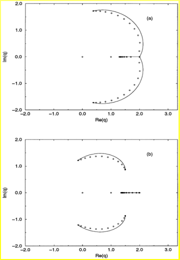

Real roots

Long Digression: univariate polynomials with real roots.

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Suppose $\mathbf{f}(\mathbf{x}) = \mathbf{a}_0 + \mathbf{a}_1\mathbf{x} + \cdots + \mathbf{a}_\mathbf{n}\mathbf{x}^\mathbf{n}$ is a polynomial with nonnegative coefficients and real roots. Then the sequence $\{\mathbf{a}_\mathbf{k} : 0 \leq \mathbf{k} \leq \mathbf{n}\}$ is unimodal, in fact it is log-concave, and in fact

$$\left\{ \frac{\mathbf{a}_\mathbf{k}}{\binom{\mathbf{n}}{\mathbf{k}}} \right\}$$

is log-concave. This is a theorem of I. Newton (1707).

Closure properties

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It is not only closed under products (obvious) but under coefficient-wise products (Hadamard products – not obvious!).

$$\begin{aligned} f(x) &= a_0 + a_1 x + \cdots + a_n x^n \\ g(x) &= b_0 + b_1 x + \cdots + b_n x^n \\ (f * g)(x) &:= (a_0 b_0) + (a_1 b_1) x + \cdots + (a_n b_n) x^n \end{aligned}$$

This is a consequence of the Pólya-Schur Theorem (1914).

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If $\mathbf{n}$ complex numbers are chosen independently from a law μ and $\mathbf{f}$ is the polynomial with these roots, then the empirical distribution of the roots of $\mathbf{f}'$ converges to μ as $\mathbf{n} \to \infty$ [PR12].

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Combinatorial Enumeration: Topic #1 is the dependence of the Taylor series coefficients of $1/\mathbf{Q}$ on the geometry of the zero set of the polynomial $\mathbf{Q}$.

Probability: Topic #2 concerns properties of a discrete probability measure which follow from the location of the zeros of the probability generating function.

RATIONAL SERIES: coefficients and poles

Generating functions

A generating function for an array $\{\mathbf{a_r} : \mathbf{r} \in (\mathbb{Z}^+)^\mathbf{d}\}$ of numbers is a power series in $\mathbf{d}$ variables

$$\mathbf{F}(\mathbf{x}_1, \ldots, \mathbf{x_d}) = \sum_{\mathbf{r} \in \mathbb{Z}^\mathbf{d}} \mathbf{a_r x^r} .$$

For any array $\{\mathbf{a_r}\}$, this exists as a formal power series, but when $\{\mathbf{a_r}\}$ grow at most exponentially in $|\mathbf{r}|$, then the series $\mathbf{F}$ has a positive radius of convergence and $\mathbf{F}$ is an analytic object as well.

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Example: $\mathbf{d} = 1$, $\mathbf{F}$ is a polynomial; real nonpositive zeros imply unimodality/log-concavity.

Rational generating functions

Rational generating functions $\mathbf{F} = \mathbf{P}/\mathbf{Q}$ correspond to arrays $\{\mathbf{a_r}\}$ that obey linear recurrences. For example, the number of lattice paths from (0,0) to $(\mathbf{i}, \mathbf{j})$ that move only North and East satisfies $\mathbf{a_{i,j}} - \mathbf{a_{i-1,j}} - \mathbf{a_{i,j-1}} = 0$ as long as $(\mathbf{i}, \mathbf{j}) \neq (0, 0)$.

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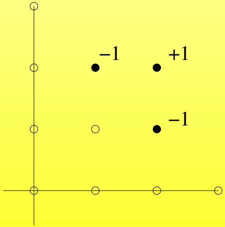

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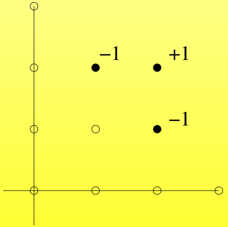

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This leads to $(1 - \mathbf{x} - \mathbf{y})\mathbf{F}(\mathbf{x}, \mathbf{y}) = 1$, corresponding to the fact that the $(\mathbf{i}, \mathbf{j})$-coefficient of $(1 - \mathbf{x} - \mathbf{y})\mathbf{F}$ is zero when $(\mathbf{i}, \mathbf{j}) \neq (0, 0)$ and one when $\mathbf{i} = \mathbf{j} = 0$.

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Phenomena: Quantum Walks

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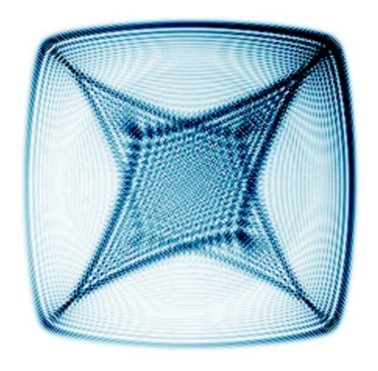
Intensity plot of quantum walk at time 200

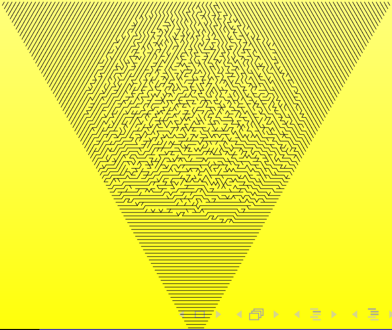


Phenomena: Cube Groves

$$F(x,y,z) = \frac{1}{(1-z)(3-x-y-z-xy-xz-yz+3xyz)}$$

is the generating function for the probabilities $\{a_{rst}\}$ of horizontal line elements at the barycentric coordinate (r,s,t) in the order $r+s+t$ **cube grove**.

Randomly sampled order-100 cube grove



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The first is to find the exponential rate of growth or decay, namely the function $g(r)$ such that

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The second step, arriving at a true asymptotic formula by nailing down the $o(|r|)$ term to within $o(1)$, I will only hint at. It requires, essentially, the computation of an inverse Fourier transform.

Domains of convergence

The power series $P/Q = \sum_r a_r z^r$ converges for some $z \in \mathbb{C}^d$ if and only if it converges for any z' whose coordinates differ from z by unit complex multiples. The **logarithmic** domain of convergence is the set of all $x \in \mathbb{R}^d$ such that $\sum_r a_r z^r$ converges when $\Re\{\log z\} = x$.

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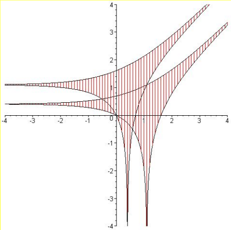
Example: if $Q = (3 - 2x - y)(3 - x - 2y)$ then the power series converges for $(\log x, \log y)$ in the white region in the lower left of the figure.

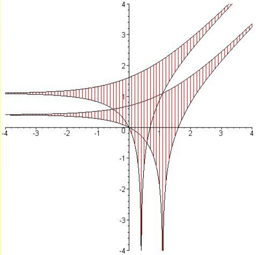


Amoebas

Digression on amoebas:

The figure in red is the set of all points $\log z$ with $Q(z) = 0$. Of course these points cannot be in the domain of convergence (the whit must be disjoint from the red) and in fact the boundary the domain of convergence must lie in the red set.

Amoeba of $(3-2x-y)(3-x-2y)$



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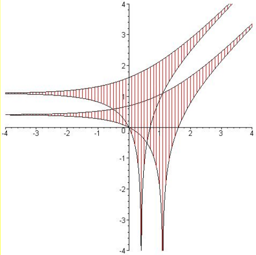

For extra credit: what are the other white regions?

Legendre inequality

Pick any $x \in B$. Absolute convergence of $\sum_r a_r z^r$ means that the terms must tend in magnitude to zero, which implies that

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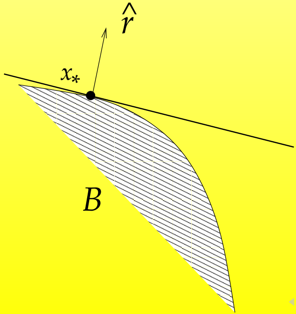
The limsup exponential rate in direction r is at most $-\hat{r} \cdot x$.

Legendre transform

Let x vary over B and optimize $\limsup_{r \to \infty} \frac{\log |a_r|}{|r|} \leq -\hat{r} \cdot x$:

$$\limsup_{r \to \infty} \frac{\log |a_r|}{|r|} \leq \phi(\hat{r}) := -\hat{r} \cdot x_*$$

where x_* is the support point of B normal to $\hat{r}$. The **Legendre transform** ϕ is thus an upper bound for the exponential growth.



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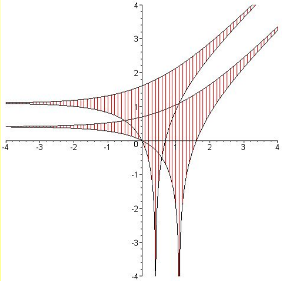
In general, the determination of $g(r)$ requires a topological computation on the complex hypersurface $\{Q = 0\}$.

Feasible regions

Let's see what happens when the boundary of the logartihmic domain of convergence passes through the origin, as is common in combinatorial applications.



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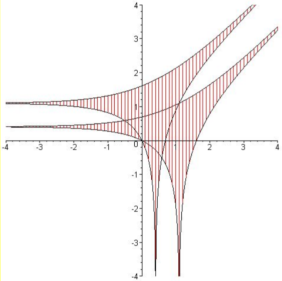
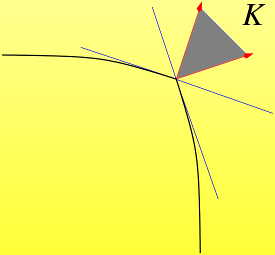
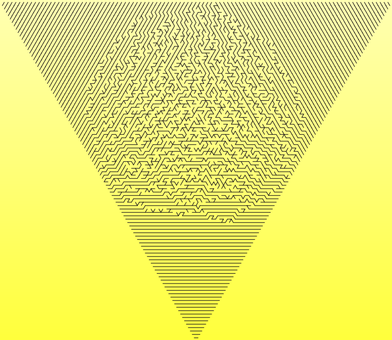




This results in a Legendre transform that is zero on the dual cone K at x_* and strictly negative elsewhere. There is exponential decay of a_r when r is not in the **feasible cone**, K.

Examples of feasible cones

Feasible cone
is circular



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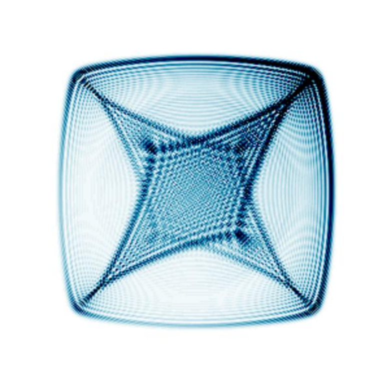
Feasible cone is
a rounded square



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This is an inverse Fourier problem; I will give you a reference at the end where you can read further. [Hint: let $x \to x_*$ within B.]

ZEROS OF PROBABILITY GENERATING FUNCTIONS

Real root property

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This was recently accomplished and sheds light on some very natural classes of probability generating functions.

Review of Newton's inequalities

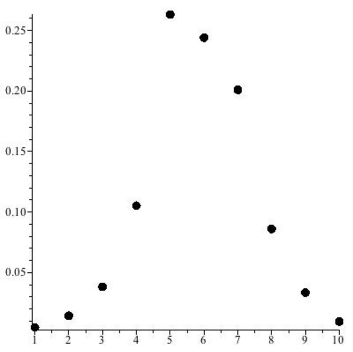
To summarize the univariate case, a sequence of positive numbers $b_0, \ldots, b_d$ is **log concave** if $b_{j-1} b_{j+1} \leq b_j^2$ for $1 \leq j \leq d-1$.

Theorem 1 ([New07])

Let $f(x) := \sum_{n=0}^{d} a_n x^n$ be a real polynomial all of whose roots are real and nonpositive. Then the sequence $a_0, \ldots, a_d$ is **ultra-log concave***, meaning that the sequence $\{a_n / \binom{d}{n}\}$ is log concave.*

Unimodality

A consequence of log concavity, hence of ultra-log concavity is unimodality, meaning the sequence increases to a greatest value (or possibly two consecutive equal values) then decreases.



Real-rooted polynomials

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Real-rooted polynomials

What makes the real root property particularly useful is that the class of real-rooted polynomials has some nice closure properties, in particular, under term by term multipication by a so-called Pólya-Schur multiplier sequence.

In the remainder of this talk I will explain the recent (2006–present) success of various researchers (Borcea, Brändén, Wagner, Gurvits, Liggett) to extend to the multivariate setting.

HALF PLANE PROPERTY

Half-plane property

Let $F(x_1, \ldots, x_n)$ be a polynomial in n variables. Say that F is **stable** (alternatively, F has the **upper half-plane** property) if it has no zeros in the n-fold product of upper half-planes:

$$\Im\{x_j\} > 0 \text{ for all } 1 \le j \le n \;\Rightarrow\; F(x_1, \ldots, x_n) \neq 0\,.$$

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When $n = 1$ and all coefficients are real, zeros come in conjugate pairs, so no zeros in the upper half-plane is equivalent to all real zeros. Mysteriously, among the many possible such equivalent formulations, this one appears to have extraordinary closure properties.

Easy properties

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(e) *Inversion: if the degree of x_1 in f is m and f is stable then $x_1^m f(-1/x_1, x_2, \ldots, x_d)$ is stable;*

Differentiation

Theorem 3 (differentiation)

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PROOF: Fix any values of $\{x_i : i \neq j\}$ in the upper half plane. As a function of x_j, f has no zeros in the upper half plane. By the Gauss-Lucas theorem, the zeros of f' are in the convex hull, therefore not in the upper half plane. □

Symmetrization

Theorem 4 (partial symmetrization)

Let f be stable and let $\tau_{ij} f$ denote f with the roles of x_i and x_j swapped. If f is stable, then so is $(1-\theta) f + \theta \tau_{ij} f$ for any $\theta \in [0,1]$ and any i and j.

The proof of this is the culmination of a series of lemmas in Borcea-Brändén-Liggett (2009).

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The function $(1-\theta)f + \theta\tau_{ij}f$ is the PGF for the measure resulting from first flipping a θ-coin to see whether to swap i and j, then drawing from the (possibly swapped) distribution.

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For collections of **binary** random variables, stability of the PGF implies a strong analogue of Newton's inequalities.

The last part of the talk explains what binary variables are, what is special about their PGF's, and what inequalities follow from their stability.

BINARY VALUED RANDOM VARIABLES

Multi-affine functions

Suppose $X_1, \ldots, X_n$ are random variables taking the values zero or one. The joint distribution of such a collection is a probability distribution on the Boolean lattice $\mathcal{B}_n := \{0,1\}^n$. The **probability generating function**

$$\sum_{\omega \in \mathcal{B}_n} \mu(\omega) x^\omega$$

is a **multi-affine** polynoimal (every variable appears with degree at most one in every monomial).

Nice things happen in the multi-affine case.

One reason the multi-affine theory is nice

Theorem 5 (Borcea-Branden)

A multiaffine polynoimal f in n variables is stable if and only if for all real x, and for all $i, j \leq n$,

$$\frac{\partial f}{\partial x_i}(x)\frac{\partial f}{\partial x_j}(x) \geq f(x)\frac{\partial^2 f}{\partial x_i \partial x_j}(x)\,.$$

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Corollary 6

If the PGF for $X_1, \ldots, X_n$ is stable, then each pair $\{X_i, X_j\}$ is negatively correlated.

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*If the PGF for $X_1, \ldots, X_n$ is stable, then each pair $\{X_i, X_j\}$ is negatively correlated. In fact the family $X_1, \ldots, X_n$ is **negatively associated**, a much stronger negative dependence property than negative correlation.*

Uniform spanning trees

Let T be a uniformly chosen spanning tree of a finite graph.

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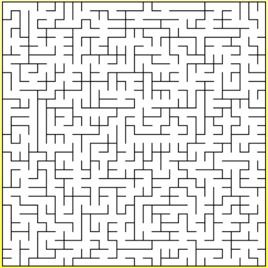
Digression on spanning trees and algorithms: Suppose G is a finite square grid. A famous algorithm attributed to Broder chooses a spanning tree for G in a reasonably short time by executing a random walk on G and deleting redundant edges. This algorithm was used in the early computer game RATMAZ, in which the computer constructs a uniform random maze which you (the rat) must navigate. This was played on teletypes at the Lawrence Hall of Science in the early 1970's.

Uniform spanning trees



A randomly generated maze (SSI)

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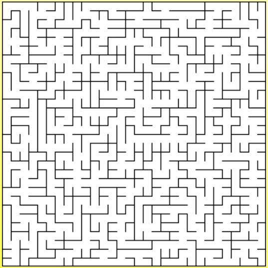

A randomly generated maze (SSI)

The variables X_e and X_f, recording the presence of edges e and f in T respectively, are always negatively correlated.

Conditioning on the sum

Flip n coins independently, with different biases, and condition on the sum being k. The resulting random variables $X_1, \ldots, X_n$ have a stable PGF.

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Flip n coins independently, with different biases, and condition on the sum being k. The resulting random variables $X_1, \ldots, X_n$ have a stable PGF.

In particular, the sum over any subset has a univariate stable distribution, thus satisfies Newton's inequalities.

Swap processes

On a finite graph, mark some of the vertices as occupied and leave the rest vacant. Each edge has an independent timer, programmed to go off randomly at a certain rate, and when it does, the endpoints swap. Thus, if one endpoint was occupied and the other was vacant, then the occupied point becomes vacant and the vacant point becomes occupied.

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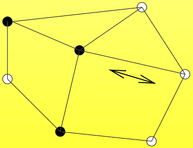

When the timer goes off on the marked edge, the token occupying the site at the left of the edge moves across to the right endpoint of the edge.

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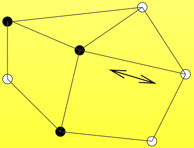

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Theorem [BBL09]: After any time t, the PGF of the random configuration is stable.

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