Generating Function Computations in Probability and Combinatorics

Robin Pemantle

ICERM tutorial, 13-15 November, 2012

Three lectures

- Overview of generating functions and the base case (smooth point computations)

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II Rate functions, convex duals and algebraic computation

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II Rate functions, convex duals and algebraic computation

III Analytic method for sharp asymptotics: saddle point integrals and inverse Fourier transforms

Overview of generating functions and the base case

Lecture I outline

(i) Purpose of these lectures

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(ii) Scope of GF method

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(iv) Examples and phenomena

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(vi) Application: Gaussian behavior and large deviations

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Analytic Combinatorics in Several Variables
Robin Pemantle and Mark C. Wilson

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The lectures are meant to be user-friendly and to focus on how one might actually carry out the computations. This involves some computational algebra and some complex integration, all of which will be explained with examples as it arises.

Arrays of numbers

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More generally, we might write $\{\mathbf{p}(\mathbf{r}) : \mathbf{r} \in \mathbb{Z}^{\mathbf{d}}\}$, where **d** always denotes the number of parameters (**dimension**) and the indices may be negative as well as positive (but always discrete); when $\mathbf{d} \leq 3$ we use letter alphabetically from **r** instead of subscripts.

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The method is most useful when the quantities $\mathbf{p}(\mathbf{r})$ obey some kind of recursion. Some examples are as follows.

Example: binomial coefficients

Binomial coefficients: use the symmetric form $\mathbf{C}(\mathbf{r},\mathbf{s}) := \binom{\mathbf{r}+\mathbf{s}}{\mathbf{r},\mathbf{s}}$.

These satisfy

$$\mathbf{C}(\mathbf{r},\mathbf{s}) = \mathbf{C}(\mathbf{r},\mathbf{s}-1) + \mathbf{C}(\mathbf{r}-1,\mathbf{s})$$

for $\mathbf{r},\mathbf{s} \geq 0, (\mathbf{r},\mathbf{s}) \neq (0,0)$, where coefficients with negative indices are taken to be zero by convention and the recursion fails at $(0,0)$.

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A probabilist might also consider normalized binomial coefficients $\mathbf{p}(\mathbf{r},\mathbf{s}) = 2^{-\mathbf{r}-\mathbf{s}}\mathbf{C}(\mathbf{r},\mathbf{s})$ satisfying

$$\mathbf{p}(\mathbf{r},\mathbf{s}) = \frac{\mathbf{p}(\mathbf{r},\mathbf{s}-1) + \mathbf{p}(\mathbf{r}-1,\mathbf{s})}{2}.$$

Example: random walk

Let μ be a measure on $\mathbb{Z}^{\mathbf{d}}$ and and let $\mathbf{p}(\mathbf{r},\mathbf{n}) := \mathbb{P}_{\mathbf{n}}(0,\mathbf{r})$ denote the probability of an $\mathbf{n}$-step transition from $\mathbf{0}$ to $\mathbf{r}$. Then

$$\mathbf{p}(\mathbf{r},\mathbf{n}) = \sum_{\mathbf{s}} \mathbf{p}(\mathbf{s},\mathbf{n})\mu(\mathbf{s}-\mathbf{r})\,.$$

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- lattice paths
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- queuing probabilities
- orientation probabilities in random tilings

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The point of these examples is that the method is both narrow and broad: narrow because it works only (mostly) for exactly solvable models; broad because of the many models and phenomena that are included under this.

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The existence of new tools such as computational algebra and topological methods of the 1970's and 80's paves the way for a renaissance of this genre.

Generating Functions

Multivariate generating function

The generating function for $\{p(r)\}$ is the formal series in d variables:

$$F(z) := F(z_1, \ldots, z_d) := \sum_r p(r) z^r .$$

Here, $z^r := z_1^{r_1} \cdots z_d^{r_d}$ is monomial power notation. If $r \in (\mathbb{Z}^+)^d$ then this is a **formal power series**; if coordinates of r may be negative, then it is a **formal Laurent series**.

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As long as $p(r)$ does not grow more than exponentially in r, the formal series F is also a **convergent series** on some domain in $\mathbb{C}^d$. If $p(r) \in [0,1]$ for all r, then F converges on at least the unit polydisk. If $p(r) \to 0$ faster than exponentially in $|r|$ then F is **entire**.

Obtaining generating functions

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The analytic properties are then used to *estimate* $p(r)$.

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First though, if we are to have any hope of using this to compute, we need to take a few minutes to carry out the step of obtaining the generating function.

I will so this by example. For details and theory you can consult [PW13, Chapter 2] or one of the many fine combinatorics texts dealing with this, my favorites being [Wil94] and [Sta97, Sta99].

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This is described in [PW13, Section 2.2].

Here follows a worked example.

Linear recursions

Example: lattice path counting. Let $a(r)$ denote the number of lattice paths from the origin to r whose steps are in the finite set $\mathbb{E} \subseteq (\mathbb{Z}^d)^+$. Let $P(z) := \sum_{x \in E} z^x$. The relation

$$a_r = \sum_{x \in E} a_{r-x}$$

with the single boundary conditions $a_0 = 1$ leads to

$$\left(1 - \sum_{m \in E} z^m \right) F(z) = \sum_r \delta_{0,r} z^r = 1 \, .$$

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Thus

$$F(z) = \frac{1}{1 - P(z)}\,.$$

Delannoy numbers

A sub-example of lattice path counting is the Delannoy numbers, which count N-E-NE paths.

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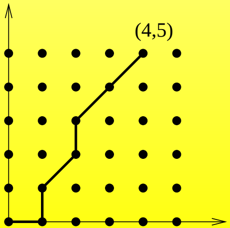
Example: The Delannoy numbers count N-E-NE paths.

$$F_{\mathrm{Del}}(z) = \frac{1}{1 - x - y - xy} .$$



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The generating function counting NE-rook paths is therefore

$$F(x,y) = \frac{1}{1-P(x,y)} = \frac{(1-x)(1-y)}{1-2x-2y+3xy}\,.$$

Kernel method

When the recursion is forward looking, the relation $a_r = \sum_{x \in E} a_{r-x}$ fails along a whose coordinate plane. This leads to

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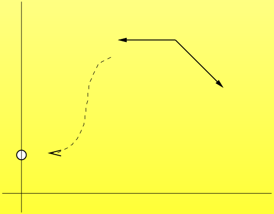
When the look-ahead in the recursion is well behaved, the generating function is still algebraic; this is the **kernel method**; see, e.g. [BMJ05]. I will give only a brief example; see [PW13, Section 2.3] for details.

Example: W-SE random walk

Example [LL99]. A random walker begins at $(r, s) \in (\mathbb{Z}^+)^2$ and moves by fair coin-flip either west $(-1, 0)$ or southeast $(1, -1)$. What is the probability of first hitting the axes at $(0, 1)$?



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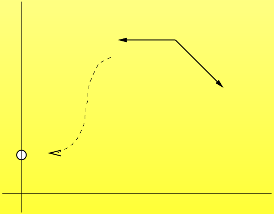

The recursion yields $(2 - x - y/x)F = R$ but R is not rational. The Laurent polynomial $(2 - x - y/x)$ is called the **kernel**.

Result of the kernel method

Setting the **kernel** $2 - x - y/x$ to zero yields $x = 1 \pm \sqrt{1-y}$. The kernel method yields the algebraic function

$$F(x,y) = \frac{2}{1 - \sqrt{1-y} - x}\,.$$

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Note: F has a branch singularity on the (complex) line $y = 1$ but also a pole at $x = \sqrt{1-y}$; some asymptotic directions are controlled by the branch and some by the pole (these being the easier, meromorphic case).

Example: stationary probabilities in queuing model

A two-server queuing model moves from (r, s) to $(r-1, s)$ or $(r, s-1)$ with probabilities p and $1-p$ if $r > s$, reversed if $s > r$. There are boundary conditions on how the walk behaves from $(0, s)$ or $(r, 0)$. Let $\{p(r, s)\}$ be the stationary probabilities. Matching the boundary conditions in this kind of problem involves solving a Riemann-Hilbert problem. This is done by hand in [FM77, FH84]; later the problem was solved in general (for two variables) by [FIM99].

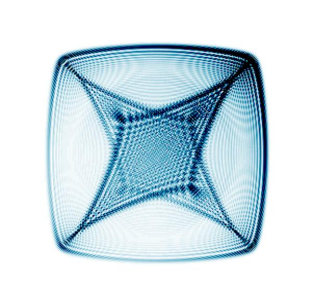
The resulting generating functions are transcendental but sometimes have properties resembling well known number-theoretic functions (theta functions, etc.).

Phenomena

To give an idea of the variety of behaviors that can be expressed even in the simplest case of a rational generating function, I will show a few pictures.

Example: quantum walk

Here $p(r, n)$ is the amplitude for a quantum walk to be at position r at time n. This satisfies a linear recursion over $\mathbb{C}$ that we will study in detail later. The picture shows, via an intensity plot, the probabilities (modulus squared of the amplitude) for the position of the particle at time 200.



Example: random tilings

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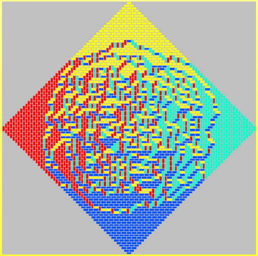
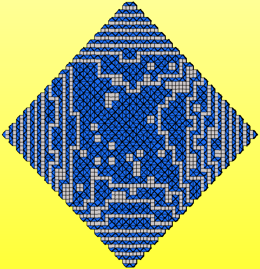
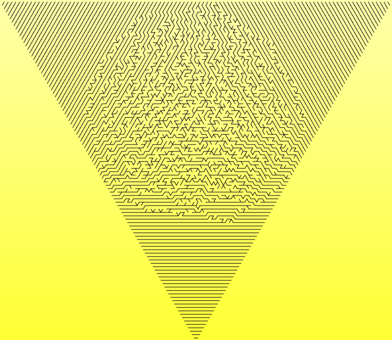
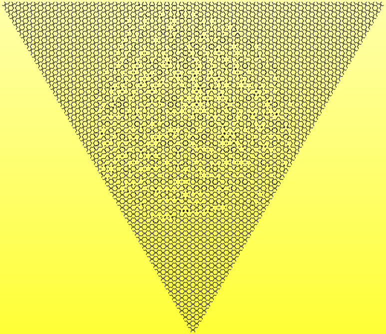




Left: Aztec diamond tiling; Right: fortress tiling.

More tilings





Left: order-100 cube grove; Right: order-50 double-dimer tiling (specializes to the Ising model on the triangular lattice)

Base case: smooth points

Smooth point formula

Let

$$F(z) = \sum_r a_r z^r = \frac{G(z)}{H(z)}$$

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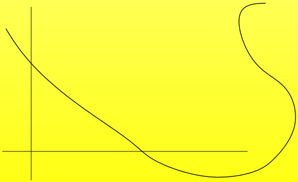
be a generating function with pole variety $\mathcal{V} := \{z : H(z) = 0\}$. For example, when $d = 2$, the set $\mathcal{V}$ is an algebraic curve in $\mathbb{C}^2$ (one complex dimension, two real dimensions). Illustrations usually only show the $\mathbb{R} \times \mathbb{R}$ slice.



Critical points

The logarithmic gradient is just the usual gradient, multiplied coordinatewise by $(z_1, \ldots, z_d)$. At the point $\mathit{1} = (1, \ldots, 1)$ the gradient and logarithmic gradient concide. We let $\hat{r} := r/|r|$ denote a unit vector parallel to r. Asymptotics "in the direction $\hat{r}_*$" refer to a_r as $r \to \infty$ with $\hat{r} \to \hat{r}_*$.

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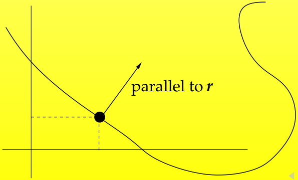
To compute asymptotics in the direction $\hat{r}$ we look for points z that lie on $\mathcal{V}$, and such that the logarithmic gradient to H at z is parallel to $\hat{r}$.



Critical point equations

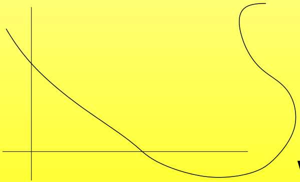
This means solving the **critical point equations**. These are d equations in d variables and typically describe a zero-dimensional ideal, i.e., a finite set of points; see [PW13, (8.3.1)-(8.3.2)].

$$\begin{aligned} H(z) &= 0 \\ r_d z_1 \frac{\partial H}{\partial z_1}(z) &= r_1 z_d \frac{\partial H}{\partial z_d}(z) \\ &\vdots \\ r_d z_{d-1} \frac{\partial H}{\partial z_{d-1}}(z) &= r_{d-1} z_d \frac{\partial H}{\partial z_d}(z)\,. \end{aligned}$$

Minimal points

Definition:

Say that $z \in \mathcal{V}$ is **minimal** *if $\mathcal{V}$ contains no other points w in the polydisk $\{w : |w_j| \leq |z_j|, 1 \leq j \leq d\}$.*



When the coefficients are nonnegative, the arc of real points of $\mathcal{V}$ bewteen the x- and y-axes consists of minimal points.

Smooth point theorem

Theorem (Smooth point asymptotics [PW13, Theorem 9.2.7])

Let $z(\hat{r})$ vary smoothly with $\hat{r}$ and be minimal. Then

$$a_r = (2\pi r_d)^{-(d-1)/2} z^{-r} R(z) \mathcal{H}(z)^{-1/2} + O\left(z^{-r} r_d^{-d/2}\right)$$

$$\text{where} \quad R(z) = \frac{G(z)}{z_d \partial H(z)/\partial z_d}$$

is the residue of F at z and $\mathcal{H}(z)$ is the Hessian matrix for the parametrization of $\mathcal{V}$ as a graph $z_d = h(z_1, \ldots, z_{d-1})$.

The remainder term is uniform as long as $\hat{r}$ remains in a compact set over which $z(r)$ varies smoothly and $\mathcal{H}(z(\hat{r})) \neq 0$.

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Think of $\{a_r\}$ as a function $a_{(\cdot)}$ from $\mathbb{Z}^3$ to the complex numbers. Its Fourier-Laplace transform (depending on whether u is real or imaginary) is given by

$$\hat{a}(u) = \sum_r \exp(u \cdot r) a_r \, .$$

Plugging in $z = \exp(u)$ coordinatewise, we see that $F(z) = \hat{a}(u)$.

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Generating functions are Fourier-Laplace transforms. To recover a_r from F we invert the transform. The inversion formula is none other than the multivariate Cauchy integral fomrula.

Cauchy integral

If $F(z) = \sum_r z^r$ and F is analytic on the polydisk bounded by a torus T then

$$a_r = (2\pi i)^{-d} \int_T z^{-r-1} F(z)\, dz\,.$$

We may push T arbitarily close to $z \in \mathcal{V}$ provided that z is minimal.

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We may push T arbitarily close to $z \in \mathcal{V}$ provided that z is minimal.

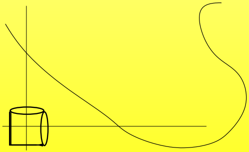

Figure: The torus T for the Cauchy integral and the singular variety of F

Dominating point: illustration

Pushing T to the dominating point $x \in \mathcal{V}$ and performing a simple residue computation proves the smooth point formula.

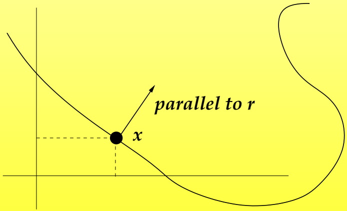

Figure: The dominating point, x

Application to CLT and large deviations

In the remainder of this lecture, I will illustrate how the smooth point formula may be applied to two classical limit theorems.

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Applications to quantum walks and random tilings (tomorrow's lecture) give results not subsumed by existing theory.

Random walk on $\mathbb{Z}^d$ with sub-exponential tails

Let μ be a probability measure on $\mathbb{Z}^{d-1}$ with probability generating function $g(z) = \sum_r \mu(r) z^r$.

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The spacetime generating function is a d-variate rational fuction:

$$\begin{aligned} F(z, y) &= \sum_{n \geq 0} \sum_r p_n(0, r) y^n \\ &= \sum_{n \geq 0} y^n g(z)^n \\ &= \frac{1}{1 - yg(z)} . \end{aligned}$$

F is meromorphic and its pole set is smooth

Assuming sub-exponential tails, the pole set $\mathcal{V}$ of F is an analytic variety $y = 1/g(z)$, as shown in the illustration.

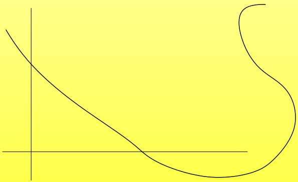

Figure: Pole is a complex analytic hypersurface; all that is shown here is the slice $(\mathbb{R}^+)^d \times \mathbb{R}^+$, depicted as $d = 1$.

Dominating point

The Cauchy integral becomes

$$p(r,n) = \int z^{-r-1} y^{-n-1} F(z,y)\, dy\, dz\,.$$

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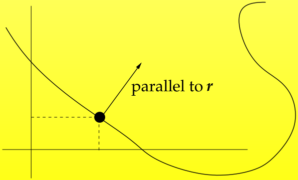
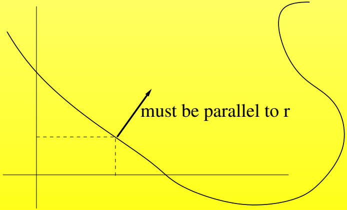
The dominating point is the point $(z, 1/g(z))$ on $\mathcal{V}$ where the lognormal to $\mathcal{V}$ is parallel to $\hat{r}$.



Tilted distribution

In our case, that's the point $(\lambda, 1/g(\lambda))$ where the **tilted distribution** μ_λ has mean r, where

$$\mu_\lambda(s) = \frac{1}{g(\lambda)} \lambda^r \mu(s) \, .$$



Resulting formula

The resulting formula is

$$p(r,n) \sim (2\pi n)^{-d/2} \, R(\lambda) \, \lambda^{-r} g(\lambda)^n \det \mathcal{H}(r)^{-1/2}$$

where $\mathcal{H}(r)$ is the Hessian determinant of $1/g(\lambda)$ at the point $\lambda(r)$.

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The Hessian matrix is the covariance matrix for the tilted distribution at mean r.

Local large deviation formula

To summarize: $p(r, n)$ is asymptotically estimated by

$$Ce^{\beta n}(2\pi n)^{-d/2}$$

where

$$\beta = \beta(\hat{r}) = g(\lambda(\hat{r})) - \hat{r} \cdot \log \lambda(\hat{r})$$

is the large deviation rate function.

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is the large deviation rate function.

The Hessian matrix $\mathcal{H}$ is the covariance matrix for the tilted distribution μ_λ, making it natural for its $-1/2$ power to appear in the normalizing constant C.

Central limit

The expression for $p(r, n)$ is uniform in r. We may expand near $r = m$, where m is the untilted mean. This always results in $(x, y) = (1, \ldots, 1)$ and $x^{-r} g(x)^n = 1$.

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where B is the quadratic form inverse to $\mathcal{H}$.

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$$x^{-r} \sim \exp\left[-B(r-m)/n\right]$$

where B is the quadratic form inverse to $\mathcal{H}$. This leaves

$$p(n,r) \sim (2\pi n)^{-1/2} |\mathcal{H}(m)|^{-1/2} e^{-B(r-m)/n}$$

which is the multivariate normal $N(m, \mathcal{H}(m))$.

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- For lattice distributions with small tails, the local CLT and local LD are a consequence of a general formula for the Taylor coefficients of a rational function in the smooth, minimal case.

Details may be found in [PW13, Section 9.6].

END PART I

II: Rates of Exponential Growth and Decay

Robin Pemantle

ICERM tutorial, 13-15 November, 2012

Lecture II outline

(i) Amoebas

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(ii) Upper bounds on rate functions via Legendre transforms

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(ii) Upper bounds on rate functions via Legendre transforms
(iii) Sharpness of rate functions via tangent and normal cones
(iv) Limit shapes via dual surfaces

Amoebas

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The **Amoeba** of H is the set

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In other words, amoeba(H) is the projection to $\mathbb{R}^d$ of the variety $\mathcal{V} \subseteq \mathbb{C}^d$ via the coordinatewise log-modulus map.

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In other words, amoeba(H) is the projection to $\mathbb{R}^d$ of the variety $\mathcal{V} \subseteq \mathbb{C}^d$ via the coordinatewise log-modulus map.

Our interest in these stems from the connection:

amoeba(H) $\rightarrow$ *domain of convergence of H* $\rightarrow$ *rate of growth of coefficients of G/H*

Properties of amoebas

The following properties of amoebas may be found in [GKZ94]; see also the summary in [PW13, Chapter 7].

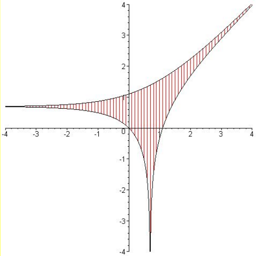
(i) Components of $\mathbb{R}^d \setminus \text{amoeba}(H)$ are open convex sets.

(ii) To each component B there is a Laurent expansion of $1/H$ convergent on the set

$$\exp(B) := \{\exp(x + iy) : x \in B, y \in \mathbb{R}^d\} .$$

Examples of amoebas

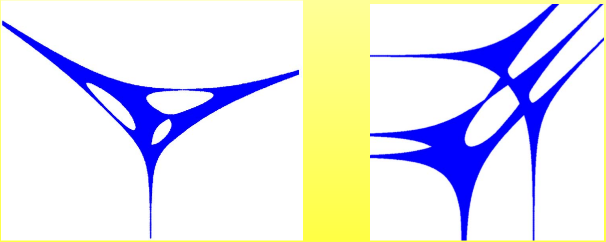
The amoeba of the polynomial $2 - x - y$ looks like this.



The complement has three components, all convex. The asymptotic directions of the arms form a tropical variety, though that will not be important to us.

More examples

Wikipedia has a number of other examples.



Left: $H = 1 + x + x^2 + x^3 + x^2y^3 + 10xy + 12x^2y + 10x^2y^2$
Right: H is a cubic of the form $A + Bx -$ other terms.

Upper bounds on the exponential rate

Components of the complement

Let us focus on one component B of the complement, namely the one closed under coordinatewise $\leq$; the Lauent series convergent in $\exp(B)$ is the ordinary power series (Taylor series).

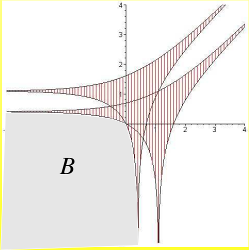


Figure: amoeba(H) for $H(x, y) = (3 - x - 2y)(3 - 2x - y)$

Exponential inequalities

Suppose $x \in B$. Convergence of the power series for $F(z)$ at $z = \exp(x)$ implies that $a_r \exp(x \cdot r) \to 0$ from which we take logs to deduce that all but finitely many r satisfy

$$\log |a_r| + r \cdot \log x \leq 0$$

whence

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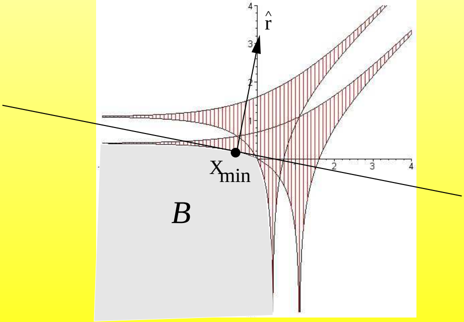
whence

$$\frac{\log |a_r|}{|r|} \leq -\hat{r} \cdot x \,.$$

To optimize in x for a given r_*, minimize $-r_* \cdot x$ over $\overline{B}$.

Optimal r

The point $x_{\min}$ will be the support point on ∂B to a hyperplane normal to $\hat{r}$.



Related to $z = \exp(x_{\min} + iy)$ with log-gradient parallel to r

Upper bound on the rate

Defining

$$\begin{aligned} x_{\min}(\hat{r}) &= \mathrm{Argmin}_{x \in B}(-\hat{r} \cdot x) \\ \beta_*(\hat{r}) &= \min_{x \in B}(-\hat{r} \cdot x) \\ \mathrm{rate}(\hat{r}_*) &= \limsup_{r \to \infty, \hat{r} \to \hat{r}_*} \frac{\log |a_r|}{|r|} . \end{aligned}$$

and optimizing the relation $\frac{\log |a_r|}{|r|} \leq -\hat{r} \cdot x$ over $x \in B$ yields

$$\mathrm{rate}(\hat{r}_*) \leq \beta_*(\hat{r}_*) .$$

Remarks

Remark 1: the theorem that $\text{rate}(\hat{r}_*) \leq \beta_*(\hat{r}_*)$ requires no assumptions. It is sometimes sharp.

The converse is more difficult.

Legendre transform

Remark 2: The function $\hat{r} \mapsto \beta_*(\hat{r})$ is a kind of Legendre transform. The usual Legendre transform arising in large deviation theory is of a function:

$$\mathcal{L}f(\lambda) := \sup_x \lambda \cdot x - f(x) .$$

The Legendre transform of the convex set B can be thought of as the Legendre transform of the convex function that is 1 on B and ∞ off of B.

Computing amoebas

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- Amoebas are effectively computable. This is a consequence of the computability of real semi-algebraic sets.
- Because the computations cannot be done within complex algebraic geometry, the computation can be messy and impractical.
- In two variables, more has been done to make this computation feasible; see [The02, Mik01].
- In the case of nonnegative coefficients, Pringsheim's Theorem has the following consequence: B is the coordinatewise log of the component B' of $(\mathbb{R}^+)^d \setminus \mathcal{V}$ containing the origin.

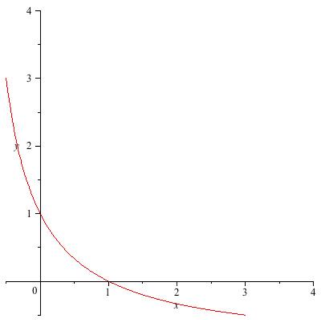
Example: Delannoy numbers

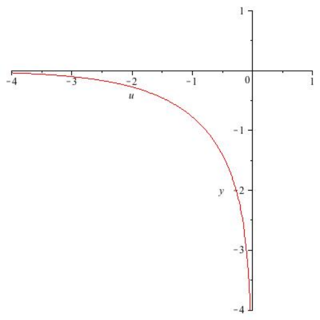
Let's see all of this in action. For a first example, consider the Delannoy numbers a_{rs} whose generating function was given by $1/(1 - x - y - xy)$.

This has nonnegative coefficients so we may invoke the Pringsheim result.

Note: generating functions of the form $1/(1 - P)$ where P has nonnegative coefficients will always themselves have nonnegative coefficients.

Real part of Delannoy variety





Left: $1 - x - y - xy = 0$; Right: logarithmic coordinates.

Delannoy critical point

Given a direction $(r, 1-r)$, the critical point equations are

$$\begin{aligned} 1 - x - y - xy &= 0 \\ (1-r)x(1-y) &= ry(1-x)\,. \end{aligned}$$

The solution is

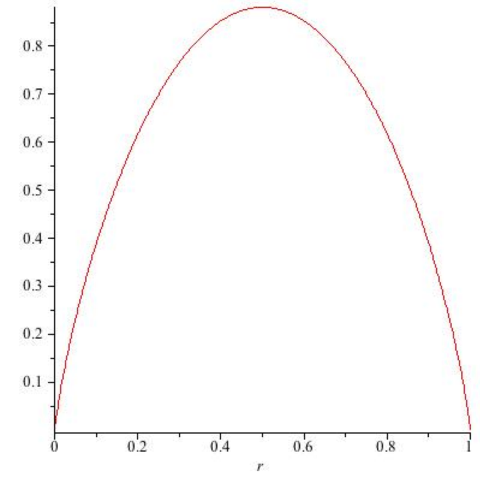
$$\begin{aligned} x(r) &= \frac{\sqrt{(1-r)^2 + r^2} - (1-r)}{r} \\ y(r) &= \frac{\sqrt{(1-r)^2 + r^2} - r}{1-r}\,. \end{aligned}$$

$x_{\min}$ at rate for the Delannoy numbers

Taking logs gives

$$\begin{aligned}
x_{\min} &= \log\left[\frac{\sqrt{(1-r)^2+r^2}-(1-r)}{r}\right] \\
y_{\min} &= \log\left[\frac{\sqrt{(1-r)^2+r^2}-r}{1-r}\right] \\
\beta_*(r) &= -r\log\left[\frac{\sqrt{(1-r)^2+r^2}-(1-r)}{r}\right] \\
&\quad -(1-r)\log\left[\frac{\sqrt{(1-r)^2+r^2}-r}{1-r}\right] .
\end{aligned}$$

Delannoy rate plot



Probability generating functions

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For a spacetime generating function, $\sum_r p(r, n) = 1$, thus

$$F(1, \ldots, 1) = \sum_n \sum_r p(r, n) = \sum_n 1 = \infty$$

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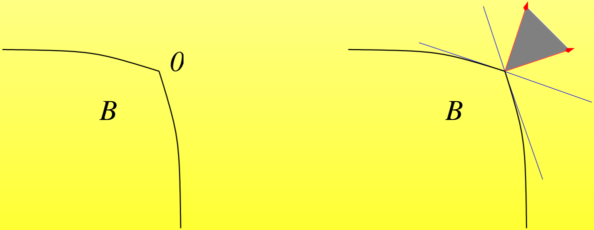
meaning $(1, \ldots, 1)$ is a pole of F and $(0, \ldots, 0) \notin B$.

Moreover, the point $(0, \ldots, 0)$ is often a special point of ∂B.

When $0 \in \partial B$

If $0 \in \partial B$ then $\beta_*(\hat{r}) \leq 0$ for all $\hat{r}$ because $\beta^*(\hat{r})$ is an infimum over a set that contains 0. In fact, $\beta(\hat{r}) = 0$ if and only if the hyperplane normal to $\hat{r}$ through the origin is a support hyperplane to B.



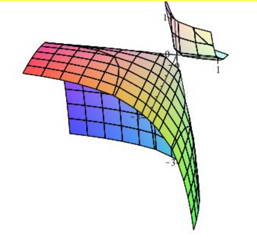
The set of r for which $\beta_*(\hat{r}) = 0$ is the **dual cone** to the **tangent cone** to B at the origin.

Example: cube groves

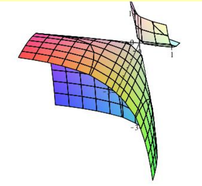
The cube grove generating function is

$$F(x,y,z) = \frac{1}{1 + xyz - (1/3)(x + y + z + xy + xz + yz)} .$$

Because of the combinatorial interpretation we know that the coefficients are nonnegative and again we can restrict our attention to the positive orthant, this time of $\mathbb{R}^3$. Taking logs gives:



Cube grove computation



$$3 + 3e^{u+v+w} - e^u - e^v - e^w - e^{u+v} - e^{u+w} - e^{v+w} = 0\,.$$

Plugging in zero for any two of the variables yields zero, thus the amoeba contains the x, y and z-axes. There appears to be a singularity at the origin. The nature of the singularity is easier to see in the original coordinates. Substituting $x = 1 + X$, $y = 1 + Y$, $z = 1 + Z$ to recenter at $(1, 1, 1)$ yields $2(XY + XZ + YZ) + 3XYZ$.

Feasible cone for cube groves

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The polynomial H is quadratic near $(1,1,1)$ with leading term $2(XY+XZ+YZ)$.

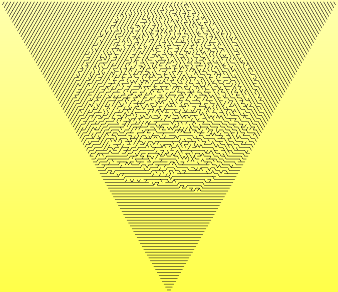
In symmetric coordinates, with $m := (X+Y+Z)/3$, the tangent cone is given by

$$(X-m)^2+(Y-m)^2+(Z-m)^2 = \frac{2}{3}m^2 \, .$$

The dual to a circular cone is a circular cone with complementary apex angle. In this case, the dual cone is given by

$$\{(r,s,t) : rs+rt+st \leq \frac{1}{2}(r^2+s^2+t^2)\} \, .$$

Feasible region for cube groves



Outside the circle, the probabilities are exponentially close to deterministic (just proved), while inside they converge to a nonzero function of rescaled position (remains to be proved).

Example: double-dimer configurations

In this simple case we used radial symmetry to conclude that the dual to a circular cone is circular. It is worth seeing how to compute the dual in the more general situation.

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We consider an example from [KP13]. Edge probabilities in a double-dimer configurations on a hexagonal lattice are shown to obey a set of four linear recurrences. Choosing periodic initial conditions simplifies the recurrence to one whose generating function $F = G/H$ is rational with

$$\begin{aligned} H &= 63x^2y^2z^2 - 62(x^2yz + xy^2z + xyz^2) - (x^2y^2 + x^2z^2 + y^2z^2) \\ &\quad +62(xy + xz + yz) + (x^2 + y^2 + z^2) - 63\,. \end{aligned}$$

Algebraic duals

Centering via $x = 1 + X, y = 1 + Y, z = 1 + Z$ and taking the leading homoegeneous term (the cubic term) produces the polynomial

$$\overline{H} = 62(X^2Y + XY^2 + X^2Z + XZ^2 + Y^2Z + YZ^2) + 132XYZ\,.$$

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A homogeneous polynomial in three variables is a projective polynomial in two variables. The dual of a projective curve may be computed by plugging in $Z = \alpha X + \beta Y$ and then solving for (α, β) such that $\partial\overline{H}/\partial X = \partial\overline{H}/\partial Y = 0$.

Gröbner basis computation of the dual

The Maple commands

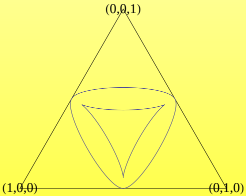
$$\begin{aligned} H1 &:= \texttt{subs}(Z = \alpha X + \beta Y, \overline{H}) \\ H2 &:= \texttt{diff}(\overline{H}, X) \\ H3 &:= \texttt{diff}(\overline{H}, Y) \\ gb &:= \texttt{Basis}([H1, H2, H3], \text{plex}(X, Y, \alpha, \beta))[1] \end{aligned}$$

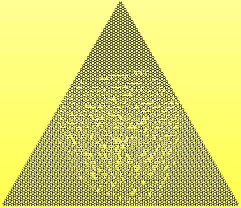
produce the polynomial

$$\begin{aligned} & 15759439 - 78914840\,\alpha^3 - 78914840\,\beta^3 + 34215444\,\alpha + 34215444\,\beta - 20624238\,\alpha^2 \\ - & 20624238\,\beta^2 + 117630120\,\alpha\,\beta + 84505896\,\alpha^2\beta + 84505896\,\beta^2\alpha - 20624238\,\alpha^4 \\ - & 20624238\,\beta^4 + 34215444\,\alpha^5 + 34215444\,\beta^5 - 64351116\,\beta^3\alpha - 64351116\,\alpha^3\beta + 167534388\,\beta^2\alpha^2 \\ - & 97424940\,\alpha\,\beta^4 - 15751503\,\alpha^2\beta^4 + 63075096\,\alpha^2\beta^3 + 64468220\,\alpha^3\beta^3 - 97424940\,\beta\,\alpha^4 + 63075096\,\beta^2\alpha^3 \\ + & 15759439\,\alpha^6 + 15759439\,\beta^6 - 32226174\,\beta\,\alpha^5 - 15751503\,\beta^2\alpha^4 - 32226174\,\beta^5\alpha\,. \end{aligned}$$

Illustration of the dual

The dual curve in the figure on the left is plotted in barycentric coordinates $\alpha = r/(r+s+t), \beta = s/(r+s+t)$.





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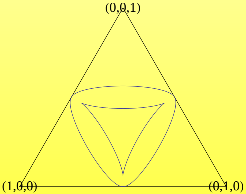

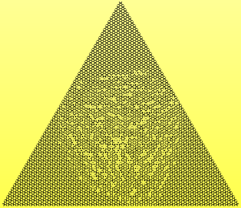

The outer branch of the dual curve is the phase boundary between the feasible region (nonzero limiting probabilities) and infeasible region (deterministic limit). Probabilities are constant inside the inner "facet".

Sharpness and the complex normal cone

Soft improvements to rate function

The normal cone in real space is a projection of a finer structure in complex space. Resolving into complex cones can sharpen the upper bound β_* on the rate function. This argument is still somewhat soft, as it avoids computing inverse Fourier transforms.

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$$H_1 = (3 - x - 2y)(3 - 2x - y)\,;$$
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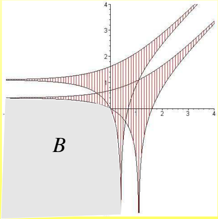
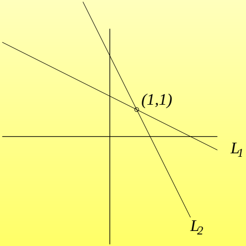
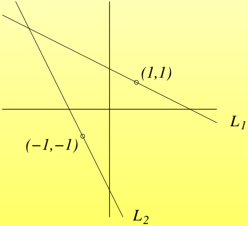
$$\begin{aligned} H_1 &= (3 - x - 2y)(3 - 2x - y)\,; \\ H_2 &= (3 - x - 2y)(3 + 2x + y)\,. \end{aligned}$$

The amoeba of a product is the union of the amoebas; pre-composing with $(x, y) \mapsto (e^{i\theta} x, e^{i\psi} y)$ does not change an amoeba; therefore H_1 and H_2 have the same amoebas.

$$H_1 = (3 - x - 2y)(3 - 2x - y); \quad H_2 = (3 - x - 2y)(3 + 2x + y)$$







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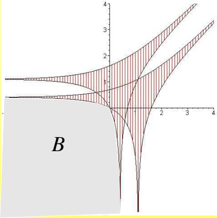

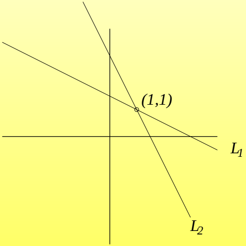

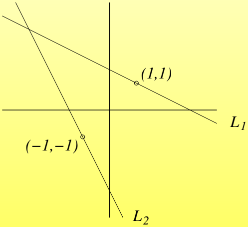

For H_1, above the point $(0, 0) \in \partial B$ lies the point $(1, 1) \in \mathcal{V}$ whose algebraic tangent cone is the union of lines of slopes -2 and $-1/2$.

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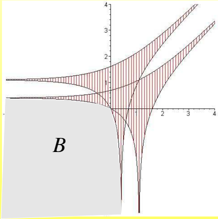

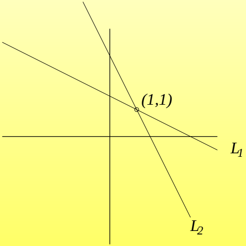

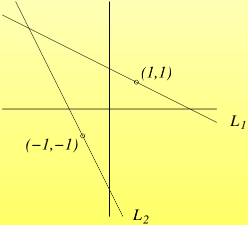

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For H_2, above the point $(0,0) \in \partial B$ lies a point $(1,1) \in \mathcal{V}$ whose algebraic tangent cone is a line of slopes $-1/2$ and another point $(-1,-1) \in \mathcal{V}$ whose algebraic tangent cone is a line of slopes -2.

Nonnegative coefficients

By Pringsheim's Theorem, if F has nonnegative coefficients then $(1,1,1)$ always "covers" $(0,0,0)$, that is, the algebraic tangent cone at $(1,1,1)$, maps onto the solid tangent cone K_0 to B at $(0,0,0)$ under the log-modulus map.

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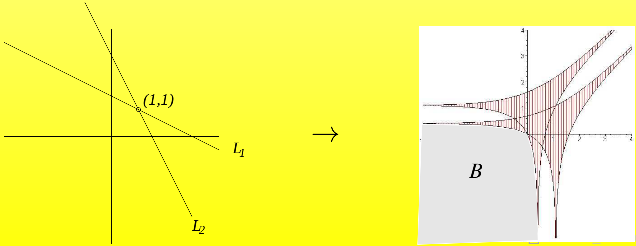
H_1 is an illustration of this.



Complexification

Given $x \in \partial B$, for each $z = \exp(x + iy)$, we need to define a piece of the algebraic tangent cone. Its dual will be the set of directions controlled by the point z.

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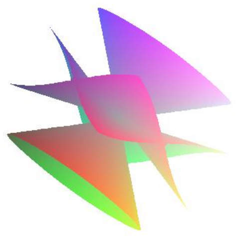

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(i) *Let* x *be a point on the boundary of* amoeba(H)*. Then as* $z = \exp(x + iy)$ *varies, there are cones* $K(z)$ *all containing the solid tangent cone* K_0 *and varying semi-continuously with* z*.*

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(ii) *The contribution to asymptotics from* z *in direction* $\hat{r}$ *is zero unless* $\hat{r} \in K(z)^*$.

Strengthened upper bound

Corollary

If $(0,0,0) \in \partial B$ *then asymptotics in direction* $\hat{r}$ *decay exponentially unless* $\hat{r} \in \overline{N}$ *where* N *is the union of* $K(z)^*$ *over all* z *in the unit torus, where* $K(z)^*(z) := \emptyset$ *if* $z \notin \mathcal{V}$*.*

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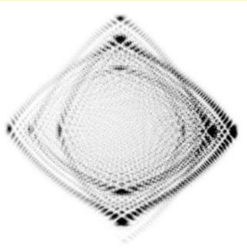
Example

For $H = (3 - x - 2y)(3 + 2x + y)$ there are two points $(1,1)$ and $(-1,-1)$ on the unit torus in $\mathcal{V}$. In each case, the dual cone is a the outward normal ray. In the directions of these two rays, the asymptotics do not decay exponentially, but in all other directions they do.

Worked example: quantum walk

I will end this lecure with a more interesting example. The spacetime generating function for a particular quantum walk in three dimensions is a rational function with denominator

$$H := 2\left(x^2y^2 + y^2 - x^2 - 1 + 2xyz^2\right)z^2 - 4xy$$
$$-z\left(xy^2 - x^2y - y - x + z^2\left(xy^2 + x^2y + y - x\right)\right) .$$



Intensity plot of quantum walk at time 200.

Note that the feasible region is not convex.

The normal cone (dual to the solid tangent cone of B) is always convex, so the feasible region is a proper subset. We can identify this by computing the union N of the normal cones $K(z)^*$.

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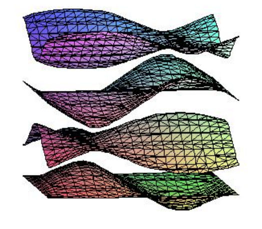
Let $(\alpha, \beta, \gamma) \in (\mathbb{R}/2\pi)^3$ be a triple in the flat unit torus. Simplifying by hand, we find that $(e^{i\alpha}, e^{i\beta}, e^{i\gamma}) \in \mathcal{V}$ if and only if

$$\left(1 - \cos^2 \gamma\right) (4 \cos \gamma - \cos \alpha)^2 = \left(1 - \cos^2 \beta\right) (\cos \gamma - 2 \cos \alpha) \, r \, .$$

The projection of $T^3 \cap \mathcal{V}$ to T^2 is a 4-fold cover, meaning that (α, β) parametrize $\mathcal{V} \cap T^3$ with four solutions for each (α, β).



Each of these points is smooth, therefore determines asymptotics along a single ray.

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The direction associated with (α, β, γ) is

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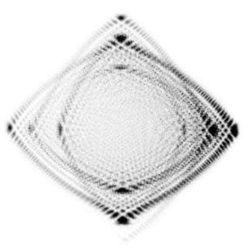
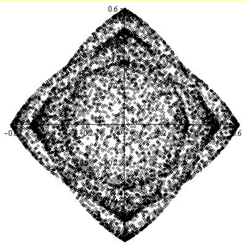
Plotting this direction for values of (α, β) filling out the 2-torus gives the plot on the right (compare to the actual intensity plot on the left).





END PART II

III: Inverse Fourier Transforms

Robin Pemantle

ICERM tutorial, 13-15 November, 2012

Lecture III outline

(i) Cauchy's integral theorem in d variables

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(iv) Self-intersections: stratified Morse theory

(v) Cone points: homogeneous expansion and the inverse Fourier transform

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