Stat 531 / Math 547, Spring 2007, Homework Set III

1. Durrett Chapter 4, Exercise 7.7

2. (a) Suppose $\{X_n : n = 0, 1, 2, \ldots\}$ is a time-homogeneous Markov chain on a finite state space $\mathcal{S}$. Let A be any subset of $\mathcal{S}$ and let $\tau_0, \tau_1, \tau_2, \ldots$ be the return times to A defined by $\tau_0 = \inf\{n : X_n \in A\}$ and $\tau_{k+1} = \inf\{n > \tau_k : X_n \in A\}$. Let $Y_n = X_{\tau_n}$. Prove or give a counterexample: $\{Y_n\}$ is a Markov chain.

   (b) Is $\{\tau_1, \tau_2, \tau_3, \ldots\}$ a Markov Chain?

   (c) Suppose $\{X_n : n = 0, 1, 2, \ldots\}$ is a time-homogeneous Markov chain on a finite state space $\mathcal{S}$. Let f be a map from $\mathcal{S}$ to another finite set. Let $Y_n = f(X_n)$. Prove or give a counterexample: $\{Y_n\}$ is a Markov chain.

3. Consider the Markov chain with three states and transition matrix

$$M := \begin{pmatrix} .1 & .4 & .5 \\ .7 & .2 & .1 \\ .4 & .4 & .2 \end{pmatrix} .$$

Compute the stationary distribution π. Then determine with proof the least constants C and β such that for all starting states x,

$$||M^n(x, \cdot) - \pi||_{TV} := \frac{1}{2} \sum_j |M^n(x, j) - \pi(j)| \leq (C + o(1))\beta^j .$$

4. Let $\{X_n : n \geq 0\}$ be the random walk on the symmetric group $\mathcal{S}_k$ obtained by taking $X_{n+1} = X_n \sigma_{n+1}$ where $\{\sigma_n\}$ are IID uniform on cycles $\pi_j := (1, 2, \ldots, j)$ that pull the j^{th} card out of the deck and place it on top; thus $\mathbf{P}(\sigma_1 = \pi_j) = 1/k$ for $1 \leq j \leq k$, where π_1 is the identity. Find a strong uniform time and use it to obtain an estimate on $||X_n - U||_{TV}$ when $n = k(\log k + c)$.

There is one more problem on the next page...

5. (i) Compute the probability, starting at A, of returning to A before hitting the vertex B in the graph pictured here. (ii) Compute the expected signed number of transits from v to w for a random walk on this graph started at A and stopped when it hits B.

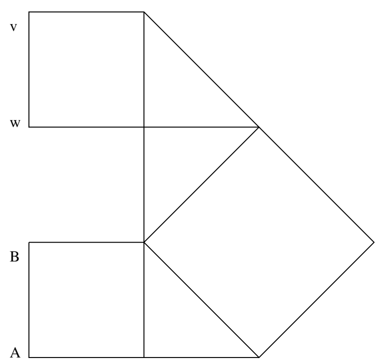

Figure 1: Each edge has unit conductance