

### Solutions for Homework Set III

1. The answer for the first question is No. For example, let  $X_n$  and  $Y_n$  are both i.i.d. standard normal distribution, and  $X_n$  is independent of  $Y_n$  for any  $n$ , let  $X = Y$  be a standard normal distribution, then  $X_n \rightarrow X$  in distribution and  $Y_n \rightarrow Y$  in distribution. But  $(X_n, Y_n)$  does not converges to  $(X, Y)$  in distribution.

The answer for second question is Yes.

**Proof:** For any  $\epsilon > 0$

$$P(\|(X_n, Y_n) - (X, Y)\|^2 > \epsilon^2) < P(|X_n - X| > \epsilon) + P(|Y_n - Y| > \epsilon) \rightarrow 0.$$

2. (a) Let  $\Omega_1 = \{X_n \rightarrow X\}$  and  $\Omega_2 = \{N_n \rightarrow \infty\}$ . By assumption we know  $P(\Omega_1) = 1$  and  $P(\Omega_2) = 1$ . And it is easy to see that  $P(\Omega_1 \cap \Omega_2) = 1$ . For any  $\omega \in \Omega_1 \cap \Omega_2$ ,  $\forall \epsilon > 0, \exists K$  such that  $\forall n > K$ ,

$$d(X_n(\omega) - X(\omega)) < \epsilon.$$

Since  $N_n(\omega) \rightarrow \infty$ , for this  $K \exists K'$  such that  $\forall n > K', N_n(\omega) > K$ . Thus for any  $\forall \epsilon > 0, \exists K'$  such that  $\forall n > K'$ ,

$$d(X_{N_n}(\omega) - X(\omega)) < \epsilon.$$

(b) The example goes like this, let  $(\Omega, \mathcal{F}, P) = ([0, 1], \beta[0, 1], \mathcal{L})$ , let

$$X_1 = I_{[0,1)},$$

$$X_2 = I_{[0,1/2)}, X_3 = I_{[1/2,1)},$$

$$X_4 = I_{[0,1/4)}, X_5 = I_{[1/4,1/2)}, X_6 = I_{[1/2,3/4)}, X_7 = I_{[3/4,1)}, \dots$$

i.e,

$$X_i = I_{\left[\frac{i-2^{\lfloor \log_2 i \rfloor}}{2^{\lfloor \log_2 i \rfloor}}, \frac{i+1-2^{\lfloor \log_2 i \rfloor}}{2^{\lfloor \log_2 i \rfloor}}\right)},$$

where  $\lfloor x \rfloor$  stands for the largest integer less than or equal to  $x$ .

Now  $P(|X_i| > \epsilon) = P(X_i = 1) = 1/2^{\lfloor \log_2 i \rfloor} \rightarrow 0$ . Also it is easy to see that for any  $\omega$  there exists a sequence of natural numbers  $\{a_1(\omega), a_2(\omega), \dots\}$  such that  $X_{a_n(\omega)}(\omega) = 1, \forall a_n$ . Let  $N_n(\omega) = a_n(\omega), \forall \omega$  finishes the example.

3. Durrett chapter 1 exercise 7.4

(i) First, I need to assume that  $EV_i > 0$ .  $W_n = \prod_{i=1}^{n-1} (ap + (1-p)V_i)W_0$  therefore,

$$\frac{\log W_n}{n} = \frac{\sum_{i=1}^{n-1} \log(ap + (1-p)V_i) + \log W_0}{n} = \frac{\sum_{i=1}^{n-1} \log(ap + (1-p)V_i)}{n-1} \frac{n-1}{n} + \frac{\log W_0}{n}.$$

Now since  $\log$  is a concave function, by Jensen's inequality, we know that

$$-\infty < E \log ap < E \log(ap + (1-p)V_i) < \log E(ap + (1-p)V_i) < \infty.$$

Hence by SLLN, we can conclude that  $\frac{\log W_n}{n}$  converges to  $E \log(ap + (1-p)V_i) = c(p)$ .

(ii).

$$c'(p) = E \frac{a - V_i}{ap + (1-p)V_i}, c''(p) = E \frac{-(a - V_i)^2}{(ap + (1-p)V_i)^2} \leq 0.$$

(iii). Since

$$c'(0)c'(1) = E\left(\frac{a}{V_i} - 1\right)E\left(1 - \frac{V_i}{a}\right).$$

By (ii) we know  $c(p)$  is concave, therefore, the condition for an interior optimal choice is  $c'(0) > 0$  and  $c'(1) < 0$  which is equivalent to  $EV_i > a$  and  $E(1/V) > 1/a$ .

(iv) In this case,

$$c(p) = \frac{1}{2} \ln(ap + 1 - p) + \frac{1}{2} \ln(ap + 4(1 - p)).$$

We want a  $p$  which solves the  $c'(p) = 0$ , i.e.,

$$\frac{a - 1}{ap + (1 - p)} = \frac{4 - a}{ap + 4(1 - p)}.$$

Solving this equation, we can conclude that

$$p = \frac{8 - 5a}{2a^2 - 10a + 8}$$

on the interval  $[8/5, 5/2]$ . When the bond rate is greater than  $5/2$  we invest only in bonds and when it below  $8/5$  we invest only in stocks. FYI: plugging the optimal value into the equation defining  $c$  in our example yields, for  $8/5 \leq a \leq 5/2$ , an optimal value

$$c_{\text{opt}} = \ln(3) - \ln(2) + \ln\left(\frac{a}{\sqrt{(4-a)(a-1)}}\right).$$

4. First notice that  $P(\cup A_n) = 1$  implies  $P(\cap A_n^c) = 0$ . Since  $A_n$  are independent, we have  $P(\prod_{n=1}^{\infty} A_n^c) = 0$ . By assumption  $P(A_n) < 1, \forall n$ , thus  $\prod_{n=m}^{\infty} A_n^c = 0, \forall m$ , which is equivalent to  $P(\cup_{n=m}^{\infty} A_n) = 1, \forall m$ . Since  $\cup_{n=m}^{\infty} A_n \downarrow \overline{\lim} A_n$  it follows that  $P(\overline{\lim} A_n) = 1$ . ■

5a.

Since  $X_i$  has density  $p(1-p)^{k-1}$   $k=1,2,\dots$ , we can get that  $EX_i = \frac{1}{p} = 2$ , thus  $p=0.5$ . Let  $\phi(\lambda) := Ee^{\lambda X} = \frac{e^\lambda}{2-e^\lambda}$ . By solving  $3 = \phi'(\lambda)/\phi(\lambda)$ , we can get that  $\lambda = \log \frac{4}{3}$ . By theory of large deviations, we can get that:

$$\frac{1}{100} \log P(S_{100} \geq 300) \approx -3 \times \log \frac{4}{3} + \log 2$$

Thus,  $P(S_{100} \geq 300) \approx (27/32)^{100} \approx 4 \times 10^{-8}$

5b.

Let  $\mu$  be the geometric of mean 2. For any  $\lambda > 0$  for which  $\phi(\lambda) := Ee^{\lambda X}$  is finite, we may tilt to obtain  $\mu_\lambda$  where

$$\mu_\lambda(k) = (1/\phi(\lambda))e^{\lambda k}\mu(k).$$

Large deviation theory tells us that setting  $\lambda$  so that  $\mu_\lambda$  has mean 3 will give us the likely distribution of the  $X_k$ 's conditional on  $S_n > 3n$ . Our problem is just the case with  $n=100$ . Therefore, the empirical distribution is a geometric of mean 3.

5c.

Since geometrics are an exponential family, all the tilted distributions  $\mu_\lambda$  also have geometric distribution.

6a. No. First notice the question is equivalent to the following question  $\forall Y$ , is it always possible to find  $X_1, X_2$  i.i.d. such that  $Y = X_1 + X_2$  in distribution?

Here is a counter example. Define a random variable  $Y : \Omega \rightarrow \{0, 1\}$  such that  $P(Y = 1) = P(Y = 0) = 1/2$ . First it is easy to see that  $X_1, X_2$  can not have continuous distribution. Now  $X_1, X_2$  must be discrete, it is also not possible. For if  $X_1, X_2$  concentrate on one point, they must concentrate on the same point, contradiction. If  $X_1, X_2$  concentrate on more than one point, then  $X_1 + X_2$  will have positive probability on at least three points, contradiction again.

6b.

$$\begin{aligned} Ee^{itS} &= E[E(e^{itS}|N)] = \sum_{m=0}^{\infty} e^{-\lambda} \frac{\lambda^m}{m!} E(e^{itS}|N=m) \\ &= \sum_{m=0}^{\infty} e^{-\lambda} \frac{\lambda^m}{m!} \phi^m = e^{\lambda(\phi-1)}. \end{aligned}$$