Solutions for Homework Set IV

1. (a) a_n, b_n do not exist. To see this, note that T_n converges with no normalization to a Poisson with mean 1. Therefore, if $b_n \to \infty$, the limit is degenerate, while if b_n does not go to infinity then any limit will be supported on distributions where F is constant on intervals of size $1/b_n$, hence not a normal or any continuous distribution.

(b) $a_n = ne^{-1}, b_n = \sqrt{ne^{-1}}$. This converges to a Poisson with mean a_n since the boxes become asymptotically independent as $n \to \infty$ Thus, with the above normalization, we have convergence to a normal.

(c) $a_n = \sqrt{n}, b_n = \sqrt{\sqrt{n}(1 - \frac{1}{\sqrt{n}})}$.
We can verify it by using Lindeberg-Feller theorem, let $X_i = 1$ if the ith ball is in the first box and $X_i = 0$ otherwise. And let $X_{n,m} = (X_m - \frac{1}{\sqrt{n}})/b_n$. Now $\sum_{m=1}^{n} EX_{n,m}^2 = 1$. To check the second condition, notice that both $|X_{n,m}| \to 0$ as $n \to \infty$. Thus we can pick n large enough so that $|X_{n,m}| < \epsilon, \forall m = 1, \cdots, n$. Hence the desired result follows.

(d) For the limit to be tight, we need b_n to be of order n (argument to follow shortly). Assuming that, let $\alpha := \overline{\lim}\, b_n/n$; then $T_n/b_n \in [0, \alpha^{-1}]$. A limit of random variables in $[0, C]$ cannot be normal for any fixed C. The fact that b_n must be of order n follows if we show that $P(T_n < n/3)$ and $P(T_n > 2n/3)$ are both bounded from below. Certainly

$$P(T_n > 2n/3) \geq P(S_{n/3} > \sqrt{n}, \inf_{n/3<k\leq n} S_k - S_{n/3} > -\sqrt{n}).$$

Independence of S_j and $\{S_{j+r} - S_j : r = 1, 2, 3, \ldots\}$ implies this last probability is the product of $P(S_{n/3} > \sqrt{n})$ and $P(\inf_{n/3<k\leq n} S_k - S_{n/3} > -\sqrt{n})$. The first is converges to $1 - \Phi(\sqrt{3})$ (by the CLT) and the second is at least $1 - 2\Phi(-\sqrt{3})$ by Kolmogorov's Maximal inequality (8.2).

(e) Let $T_n = Y_1 + \cdots + Y_n$ where Y_i is the ith decimal digit of X_n. So Y_i are i.i.d. with uniform distribution on $\{0, 1, 2, \cdots, 9\}$. Therefore, $a_n = 4.5n, b_n = \sqrt{8.25n}$.

(f) Notice that

$$T_n - 1 = \frac{X_1}{n} + \cdots + \frac{X_n}{n} + o_p(\frac{1}{n}),$$

where $o_p(\frac{1}{n})$ means that $\frac{1}{n}R_n$ and $R_n \to 0$. Let $S_n = (\frac{X_1}{n} + \cdots + \frac{X_n}{n})\sqrt{n}$ then by CLT $S_n \Rightarrow z$. Since $o_p(\frac{1}{n})\sqrt{n}$. Therefore, $(T_n - 1)\sqrt{n} \Rightarrow z$. From this we can conclude that $a_n = 1, b_n = \frac{1}{\sqrt{n}}$.

(g) Since T_n is completely deterministic, a_n, b_n do not exist.

(h) Z_n converges in distribution to a random variable Z taking values in $\{1, \ldots, 9\}$ with $P(Z = k) = \log_{10}((k+1)/k)$. To see this, note that

$$P(Z_n = k) = P\left(\log_{10} k \le \{nX_n \log_{10} 2\} < \log_{10}(k+1)\right),$$

where $\{u\}$ denotes the fractional part of u, that is, $u - \lfloor u \rfloor$; since $nX_n \log_{10} 2$ is uniform on a large interval, its fractional part is converging to uniform on $[0, 1]$. Now a CLT follows for T_n with $a_n = \mu n$ and $b_n = \sigma\sqrt{n}$, with μ and σ^2 being the mean and variance of the law of Z.

2. (Chapter 2, Exercise 4.13 Durrett)

(i) If $\beta > 1$ then $\sum_j P(X_j \neq 0) < \infty$, by Borel Cantelli implies $P(X_j \neq 0, i.o.) = 0$ therefore, $\sum_j X_j$ exists.

(ii) To prove this, we will use the Lindeberg-Feller theorem. Let $X_{n,m} = \frac{X_m}{\sqrt{n^{3-\beta}}}$. Notice that

$$\sum_{j=1}^{n} (j-1)^{2-\beta} \le \int_0^n x^{2-\beta} dx \le \sum_{j=1}^{n} j^{2-\beta},$$

therefore

$$\lim_{n\to\infty} \frac{\sum_{j=1}^n j^{2-\beta}}{\frac{n^{3-\beta}}{3-\beta}} = 1.$$

Thus,

$$\sum_{m=1}^{n} EX_{n,m}^2 = \frac{\sum_{m=1}^n m^{2-\beta}}{n^{3-\beta}} = \frac{1}{3-\beta}.$$

To check the second condition, for any given $\epsilon > 0$ we can pick n large enough so that $|X_{n,m}| < \epsilon, \forall m = 1, \cdots, n$. Thus, the second condition is verified. Thus by Lindeberg-Feller theorem, $S_n/n^{(3-\beta)/2} \Rightarrow cz$ with $c = \sqrt{\frac{1}{3-\beta}}$.

(iii) When $\beta = 1$,

$$Eexp(it\frac{S_n}{n}) = \prod_{j=1}^{n} \left(1 + \frac{1}{n}\frac{cos(\frac{jt}{n}) - 1}{j/n}\right).$$

Since the Riemann sums $\sum_{j=1}^n \frac{1}{n}\frac{cos(\frac{jt}{n}-1)}{j/n} \to \int_0^1 \frac{cos(xt)-1)}{x}dx$. Therefore, the desired results follows from Durrett Exercise 1.1 in Chapter 2.

3. (Chapter 2, Exercise 4.5 Durrett) First by CLT $\sum_{m=1}^n \frac{X_m}{\sqrt{n}\sigma} \Rightarrow z$ and by WLLN $\sum_{m=1}^n \frac{X_m^2}{n\sigma^2} \to$ 1 in probability.

By Slutsky's theorem which states that for univariate random variables Y_n and Z_n with $Y_n \Rightarrow Y$ and $Z_n \to c$ in probability, then $Y_n Z_n \Rightarrow cY$. Now since $f(x) = \sqrt{x}$ is a continuous function, the continuous mapping theorem shows that $(\sum_{m=1}^n X_m^2/n\sigma^2)^{1/2} \to 1$ in probability. Then the desired result follows. ∎

4. Sequence W_n satisfies a Poisson limit theorem, because:
let Y_α be the indicator random variable for the event that $X_{\alpha+1}$=0, $X_{\alpha+2}$=1,...,$X_{\alpha+k} = 1$. We choose B_α as the set of β's such that $|\alpha - \beta| \leq k-1$. $I = 0, ..., n-k$. We can compute $b_1, b_2, b3$:

$$b_1 = \sum_{\alpha \in I} \sum_{\beta \in B_\alpha} \leq \frac{2(k-1)(2^k - k + 1)}{2^{2k}} \to 0$$

Since $Y_\alpha Y_\beta = 0$ for $\alpha \in I, \alpha \neq \beta \in B_\alpha$, it is easy to see that

$$b_2 = 0$$

Since Y_α, Y_β are independent if β does not belong to B_α, it is easy to see that

$$b_3 = 0$$

Thus, from Arratia-Goldstein-Gordon, we have a Poisson limit theorem for W_n with mean 1.

For sequence V_n, b_1, b_3 will be the same as those of W_n, but b_2 will not converge to 0. Intuitively, the reason for this is that if $Y_\alpha = 1$ then we expect an interval of ones, $Y_j = 1$ for $\alpha \leq j \leq \alpha + N - 1$, where N is geometric of mean 2. The expected number of α wuch that $Y_\alpha = 1$ is of course still 1, so that means the expected number of clumps must be 1/2. This means that V_n should behave like a random variable that has a Poisson (1/2) number of clumps, each clump being geometrically distributed with mean 2. Hence, $V_n \to V$ where V is this compound Poisson. We compare the distributions of W_n and V_n. Both have mean 1 but the variances are respectively 1 and 3 and the first few probabilities are as follows.

W :	0.368	0.368	0.184	0.061	0.015
V :	0.607	0.151	0.094	0.058	0.036