MULTIVARIATE GENERATING FUNCTIONS: NON-GENERIC DIRECTIONS AND REGIME CHANGE

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Theory work joint with Yuliy Baryshnikov and Robin Pemantle

Computer algebra work joint with Éric Schost and Kevin Hyun

Basics of Analytic Combinatorics

There are deep links between **analytic properties** of a generating function and **asymptotics** of its coefficients.

If $F(z) = \sum_{n \geq 0} f_n z^n$ is analytic at the origin, then CIF implies

$$f_n = \frac{1}{2\pi i} \int_C \frac{F(z)}{z^{n+1}} dz$$

where C is a sufficiently small circle around the origin

There are uniform treatments for functions satisfying (algebraic, differential, ...) equations of different forms. Can be linked to different combinatorial behaviours.

Multivariate Rational Diagonals

Idea: Use a multivariate rational function $F(\mathbf{z}) = G(\mathbf{z})/H(\mathbf{z})$ to encode sequences

$$F(\mathbf{z}) = \sum_{(i_1,\ldots,i_d)\in\mathbb{N}^d} f_{i_1,\ldots,i_d} z_1^{i_1} \cdots z_d^{i_d} = \sum_{\mathbf{i}\in\mathbb{N}^d} f_{\mathbf{i}} \mathbf{z}^{\mathbf{i}}$$

Example (Main Diagonal)

The main diagonal sequence consists of the terms $f_{n,n,\ldots,n}$

$$F(x,y) = \frac{1}{1-x-y}$$

$$= 1 + x + y + 2xy + x^2 + y^2 + x^3 + 3x^2y + 3xy^2 + y^3 + 6x^2y^2 + \cdots$$

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Example (Apéry)

$$F(w,x,y,z) = \frac{1}{1-z(1+w)(1+x)(1+y)(wxy+xy+x+y+1)}$$

Here $(f_{n,n,n,n})_{n\geq 0}$ determines Apéry's sequence, related to his celebrated proof of the irrationality of $\zeta(3)$.

Multivariate Rational Diagonals

In general, the ***r**-diagonal* of F forms the coefficient sequence of

$$(\Delta_{\mathbf{r}} F)(t) = \sum_{n \geq 0} f_{nr_1,\ldots,nr_d} z_1^{nr_1} \cdots z_d^{nr_d} = \sum_{n \geq 0} f_{n\mathbf{r}} \mathbf{z}^{n\mathbf{r}}$$

A priori, the coefficient $f_{n\mathbf{r}}$ is only nonzero if $n\mathbf{r} \in \mathbb{N}^d$

In particular, this sequence is only non-trivial when $\mathbf{r} \in \mathbb{Q}^d_{\geq 0}$

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In particular, this sequence is only non-trivial when $\mathbf{r} \in \mathbb{Q}^d_{\geq 0}$
The CIF has a (somewhat) natural generalization

$$f_{n\mathbf{r}} = \frac{1}{(2\pi i)^d} \int_{\mathcal{C}} F(\mathbf{z}) \frac{d\mathbf{z}}{\mathbf{z}^{n\mathbf{r}+\mathbf{1}}}$$

The field of *analytic combinatorics in several variables (ACSV)* uses this expression and singularity analysis to determine asymptotics

Analytic Combinatorics in Several Variables

Singularities of $F(\mathbf{z}) = G(\mathbf{z})/H(\mathbf{z})$ form algebraic set $\mathbb{V}(H)$

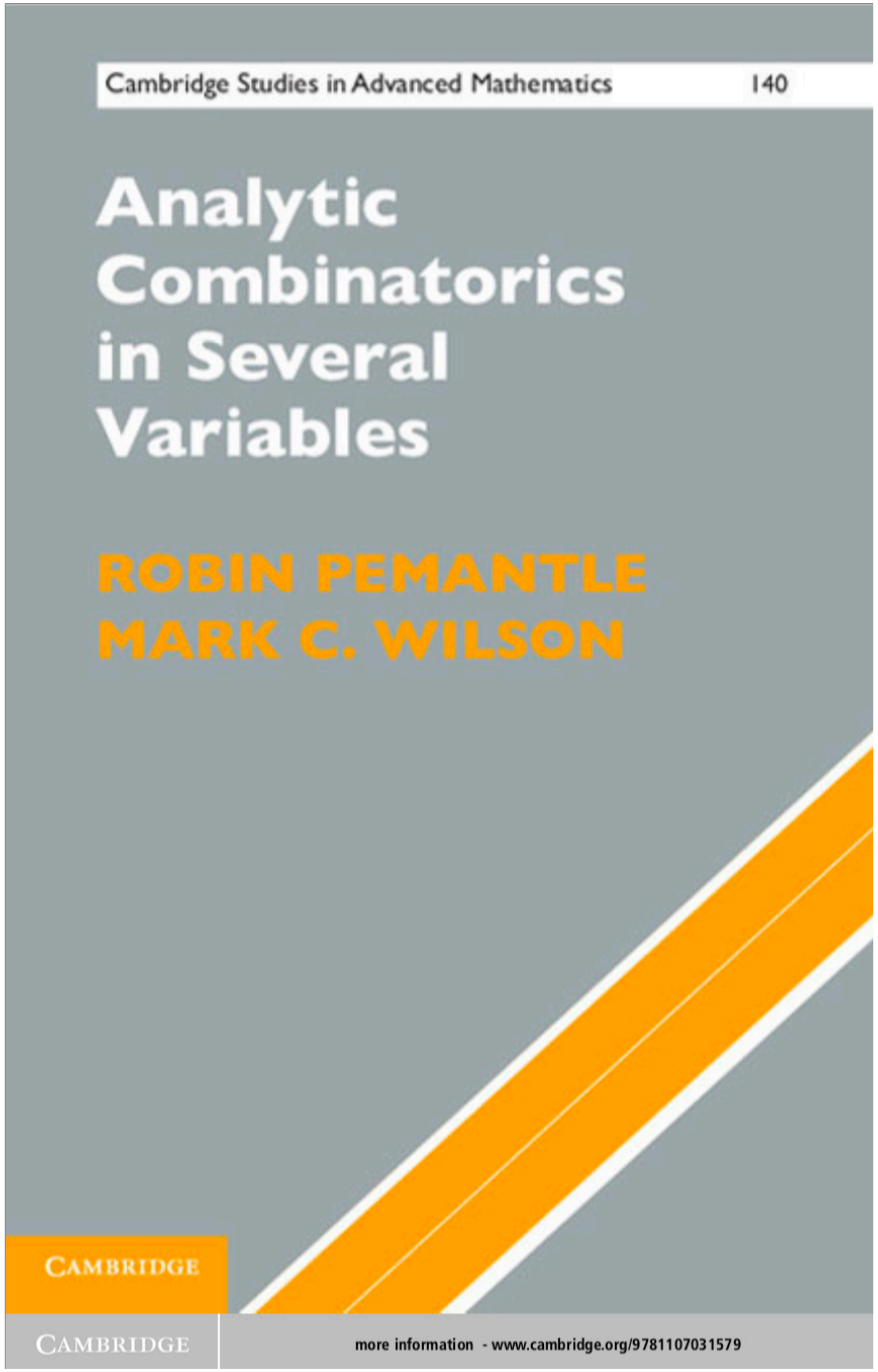
Easiest cases:

- A finite set of singularities determines asymptotics of the $\mathbf{r}$-diagonal
- A local analysis of F at these points can be automated, and effective methods have been developed

Difficulties:

- An infinite number of singularities to consider
- Geometry of singular set determines type of singularity
- Singularities of multivariate functions can be *very* complicated

Analytic Combinatorics in Several Variables



arXiv.org > math > arXiv:1709.05051

Mathematics > Combinatorics

Analytic Combinatorics in Several Variables: Effective Asymptotics and Lattice Path Enumeration

Stephen Melczer

Comments: PhD thesis, University of Waterloo and ENS Lyon – 259 pages
Subjects: **Combinatorics (math.CO)**; Symbolic Computation (cs.SC)
Cite as: arXiv:1709.05051 **[math.CO]**

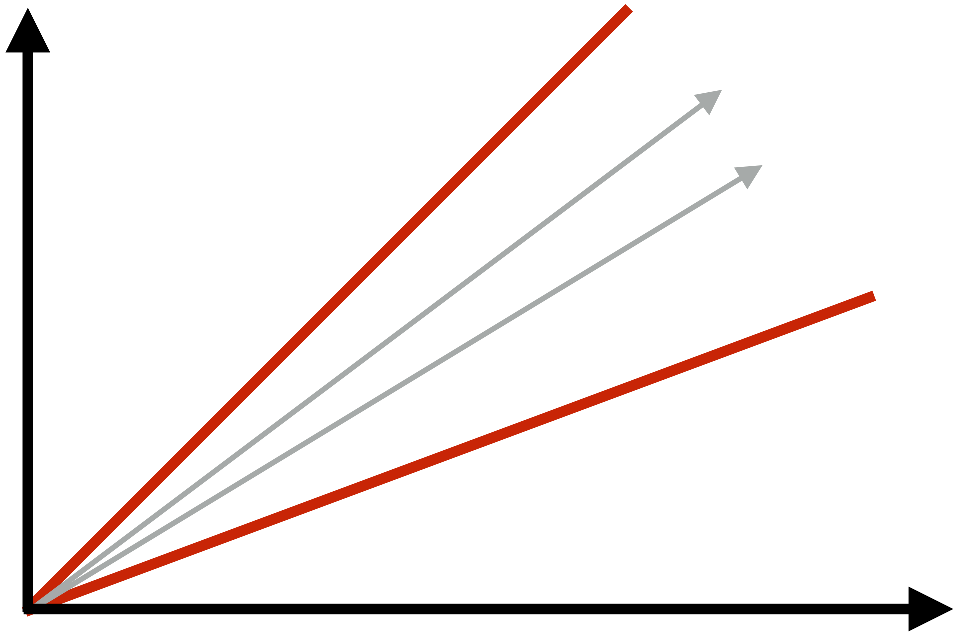
Theory developing rapidly

Generic Asymptotics

For "generic" directions $\mathbf{r}$ asymptotics have a uniform expression varying smoothly with $\mathbf{r}$ staying in fixed cones of $\mathbb{R}^d$

Thus, one can define asymptotics for any (generic) direction $\mathbf{r} \in \mathbb{R}^d_{\geq 0}$ as a limit!

$$f_{n\mathbf{r}} \to \lim_{\substack{\mathbf{s}\to\mathbf{r} \\ \mathbf{s}\in\mathbb{Q}^d}} \left(\lim_{n\to\infty} f_{n\mathbf{s}} \right)$$

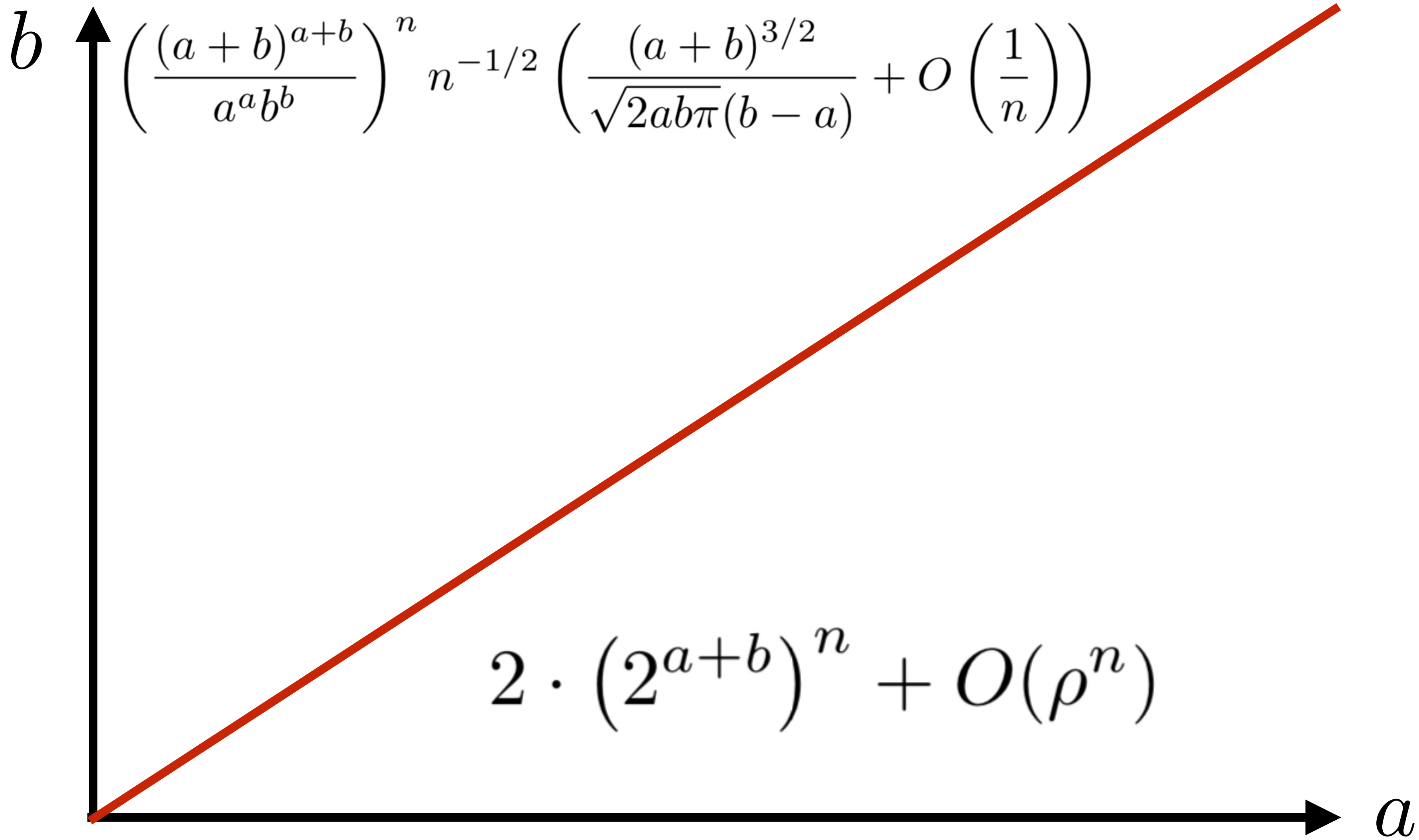


2D Example

Let

$$F(x,y) = \frac{1}{(1-x-y)(1-2x)}$$

Then $[x^{an}y^{bn}]F(x,y)$ satisfies



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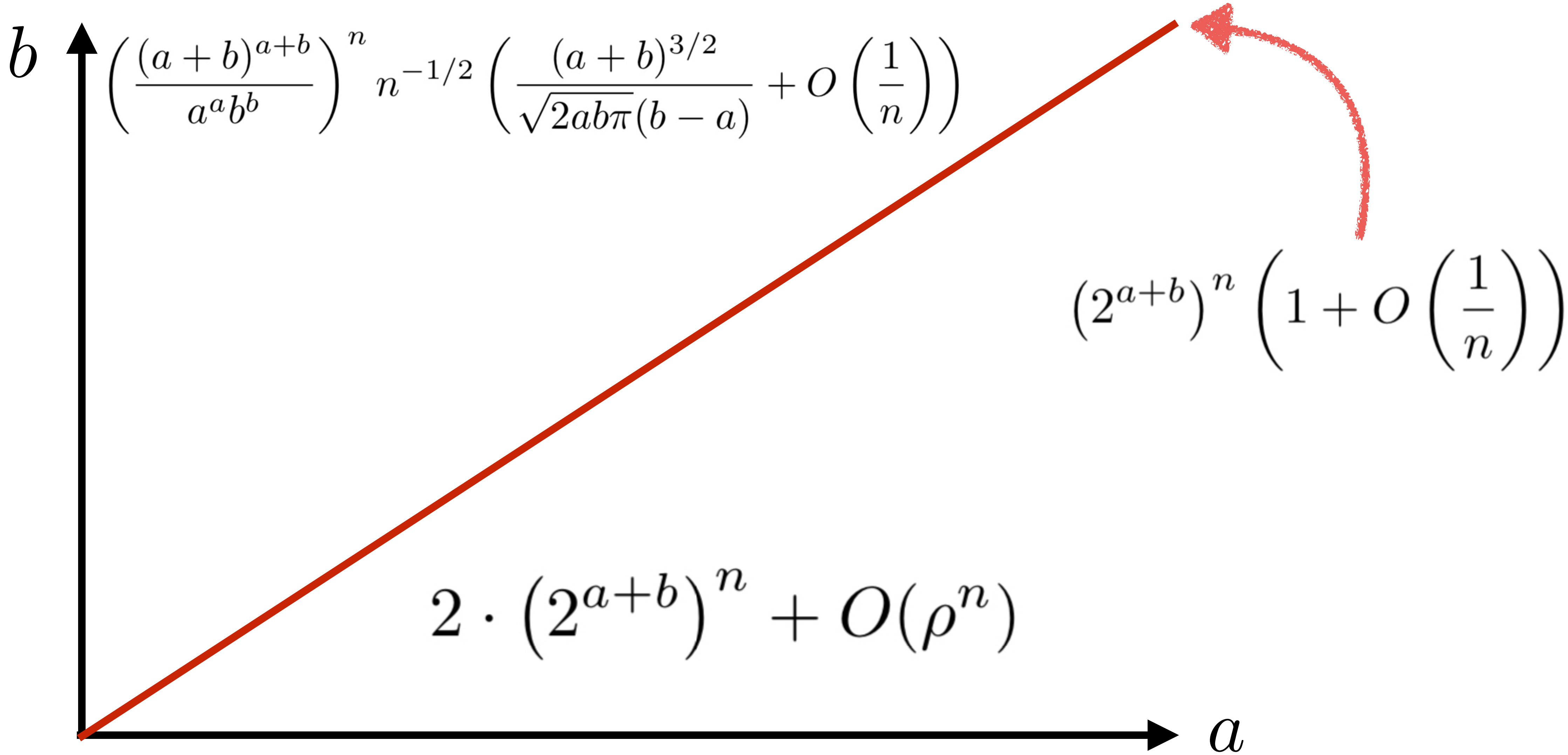
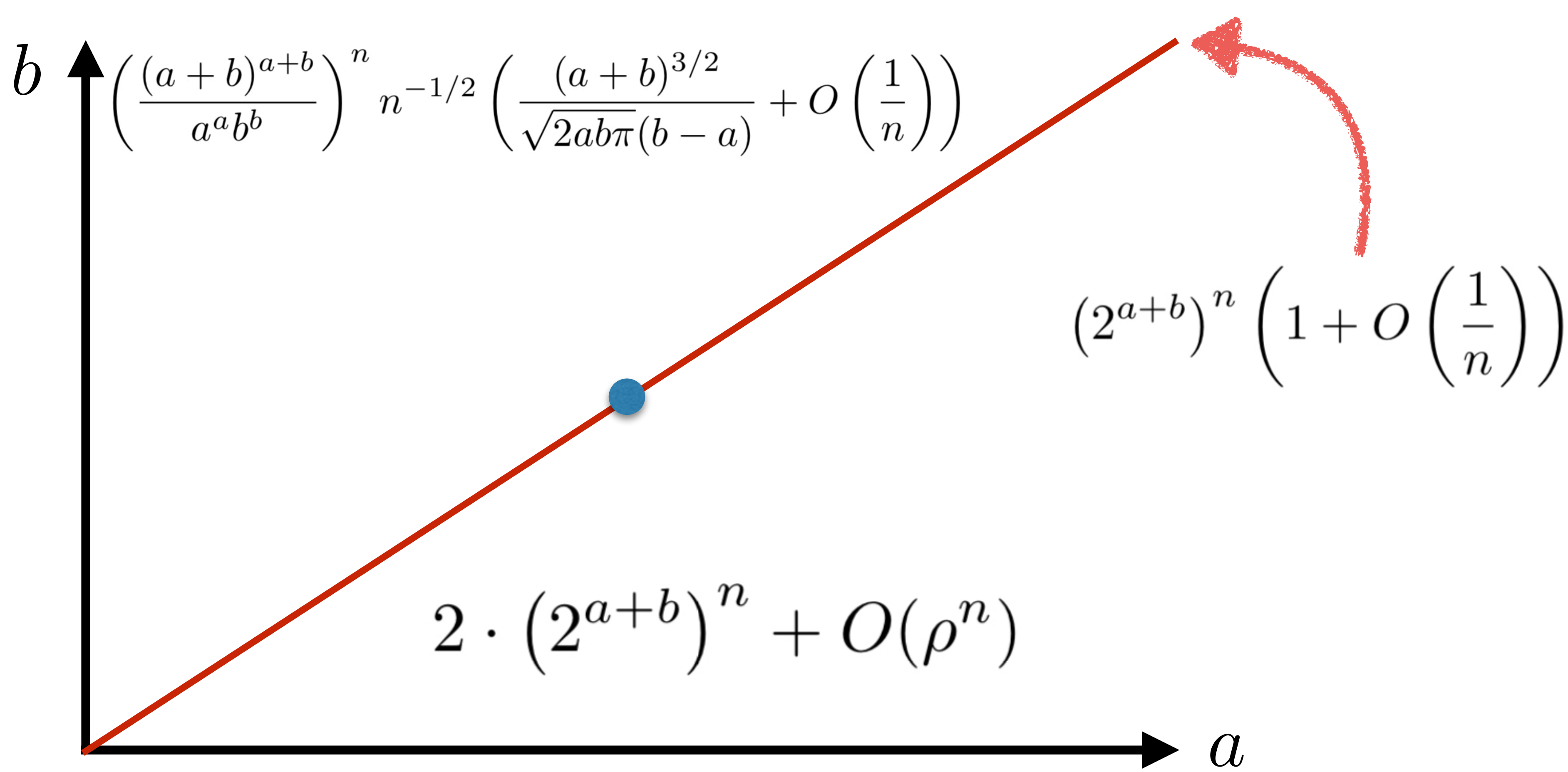


Asymptotic Regime Change

The exponential growth of $[x^{an}y^{bn}]F(x,y)$ varies smoothly with (a,b), so scale by the exponential growth.

For our example, around $\mathbf{r}=(1,1)$ the remaining terms go from decaying as $n^{-1/2}$ to being the constant 2.



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How does this transition occur?
It makes sense to look at the transition on the square-root scale

$$[x^{n+t\sqrt{n}}y^n]F(x,y) \quad \text{for} \quad t = O(n^c) \text{ with } 0 < c < 1/2$$

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First step: Get data for our example!

Experimental Data

How do we usually generate $f_{n\mathbf{r}}$ for large n?

Theorem (Christol, Lipshitz): The sequence $f_{n\mathbf{r}}$ satisfies a linear recurrence relation with polynomial coefficients.

There are effective algorithms (Lairez / Bostan, Lairez, Salvy) for determining such a recurrence and practical implementations (**Best**: Lairez's MAGMA package, **Also Good**: Koutschan's Mathematica package)

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Problem #1: Singly exponential complexity which increases with the numer/denom of $\mathbf{r}$'s coordinates

Problem #2: We need truly multidimensional data

Computing Coefficients

With Kevin Hyun and Éric Schost:
Efficient algorithm for generating terms of multivariate rational function (right now only in *bivariate case*)

Idea: Each *section* $\alpha_j(x) = \sum_{n \geq 0} f_{n,j} x^n$ is a rational function $\frac{P_j(x)}{H(x,0)^j}$

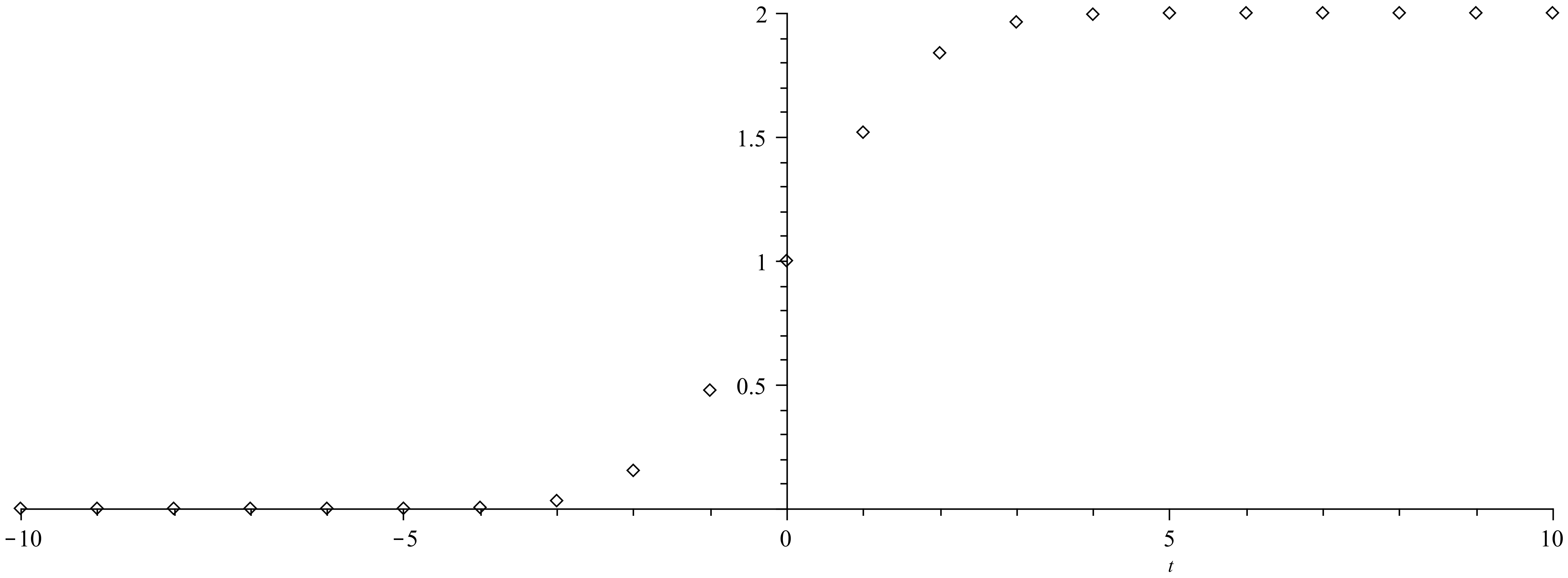
Can find P_j using fast interpolation procedures

Since denominator is a power of a fixed polynomial, can find terms in good complexity using work of Hyun, M., Schost, and St-Pierre

Very efficient implementation in C++ using Shoup's NTL library

```
void bivariate_lin_seq::find_row_geometric(zz_pX &num, zz_pX &den, const long &D){
    long degree = (D+1) * d1;
    zz_pX x;
    SetCoeff(x,1,1);

    zz_p x_0;
    random(x_0);
    zz_pX_Multipoint_Geometric eval(x_0, x_0, degree);

    Vec<zz_p> pointsX, pointsY;
    pointsX.SetLength(degree);
    pointsY.SetLength(degree);
    eval.evaluate(pointsX, x); // grabs all the points used for evaluation

    Vec<zz_pX> polX_num, polX_den;
    create_poly(polX_num, num_coeffs);
    create_poly(polX_den, den_coeffs);

    for (long i = 0; i < degree; i++){
        zz_pX eval_num, eval_den;
        eval_x(eval_num, pointsX[i], polX_num);
        eval_x(eval_den, pointsX[i], polX_den);

        Vec<zz_p> init = get_init(d2, eval_num, eval_den);
        auto rp = get_elem(D,reverse(eval_den), init);
        auto p_pow = power(ConstTerm(eval_den), D+1);

        pointsY[i] = (rp*p_pow);
    }
    eval.interpolate(num, pointsY);
    power(den, polX_den[0], D+1);
}

void bivariate_lin_seq::get_entry_sq_ZZ
(Vec<ZZ> &entries_num,
 Vec<ZZ> &entries_den,
```

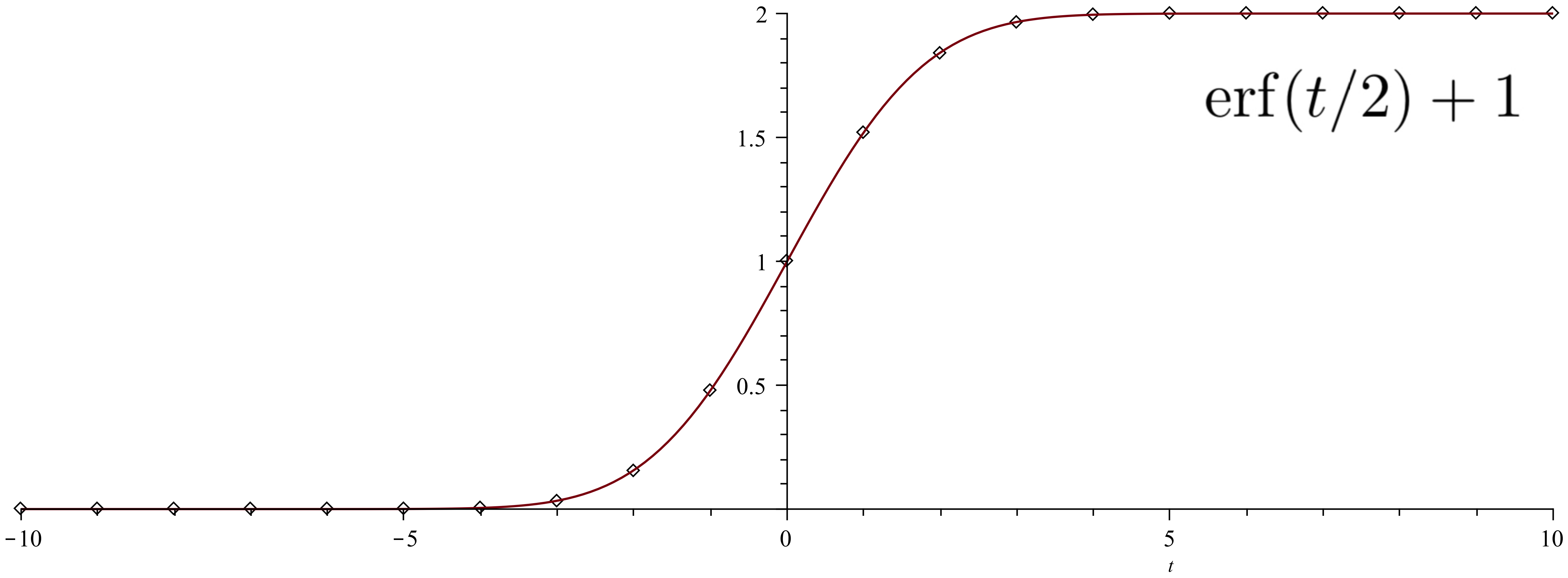
Asymptotic Transition For Our Example



$$4^{-2\cdot 50^2 - t50} \cdot [x^{50^2+t50} y^{50^2}] F(x,y) \text{ for } t = -10 \ldots 10$$

A Gaussian error curve!

$$\mathrm{erf}(x) = \frac{1}{\sqrt{\pi}} \int_{-x}^{x} e^{-y^2}\, dy$$



$$4^{-2\cdot 50^2 - t50} \cdot [x^{50^2+t50} y^{50^2}] F(x,y) \ \text{ for } \ t = -10 \ldots 10$$

Final term calculated (5501 bits)

9247112633865973228836926990252927536356128705864994391723960554842197828011919474188031
1840050067111278111780191338963196100213646176384616576895324325774311651633061291743511
1528172307641969079370616908774932526257748200792620808754002776970859314141249780545077
8103255913168249620154652817830950635794229671872993810041692625728133745324643626841293
0259564647442319740147252362804562844434857835125458940592134491474970770607230655221867
5366230681922963259368342680997668526477479402147170142640019971630836873779496410564406
5906486259309487970100334323892438718399499179010927377682177528243724037074218571133372
5542774057540268752388779449398881580396831894698931952530172625133010565323295147885324
9981002946718644699833713280981651736705195798719880743558954453380941098600643926040411
4496539256860182158422734589455124276305689168482910661467600355604435267838066675355087
9311733057968744439375914536704720736701280856507092158687171417876146691374315589264408
9749686947951486155583039909969190414112626413695581796272088309197088870117259664085189
7628170182782844835742032533698459985431963124199119073986596954833469830341670440503081
4142884824014900626562588911196406528928198509499728155987916438342256979170118456640402
7939362451483545842365315802379461162277246402661979338172430393316433538350972283167985
5945250295071620153743584846519241968287635621625773570912765784809250497309984552598716
2260107070515687329791339969156814011616512253084076327937423777720247529424544504161301
8998699781303328086317552377901540356213863616459034770127913986510273876354130346015132
130022875206194551835993328855937212541423908519982433862931456214453776

Transition in this Example

Integral manipulations show

$$2^{-2n-t\sqrt{n}} \cdot [x^{n+t\sqrt{n}}y^n]F(x,y) \sim I(t) = \frac{1}{\pi i}\int_{\mathbb{R}-i\epsilon} \frac{e^{-4nz^2+2i\sqrt{n}tz}}{z} dz$$

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$$(\partial I/\partial t)(t) = \frac{2\sqrt{n}}{\pi} \int_{\mathbb{R}-i\epsilon} e^{-4nz^2+2i\sqrt{n}tz} dt \quad = \frac{e^{-t^2/4}}{\sqrt{\pi}}$$

$$I(0) = \frac{1}{\pi i} \int_{\mathbb{R}-i\epsilon} \frac{e^{-nz^2}}{z} dz = 1$$

General (Linear) 2D Transition

Theorem (Baryshnikov, M., Pemantle): This error function appears more generally. For instance, suppose

$$F(x,y) = \frac{G(x,y)}{\ell_1(x,y)\ell_2(x,y)}$$

For "non-generic" directions where asymptotics are determined by a singularity $\boldsymbol{\sigma}$ there exist explicit constants $A, B \in \mathbb{R}$ and $\mathbf{v} \in \mathbb{R}^2$ such that

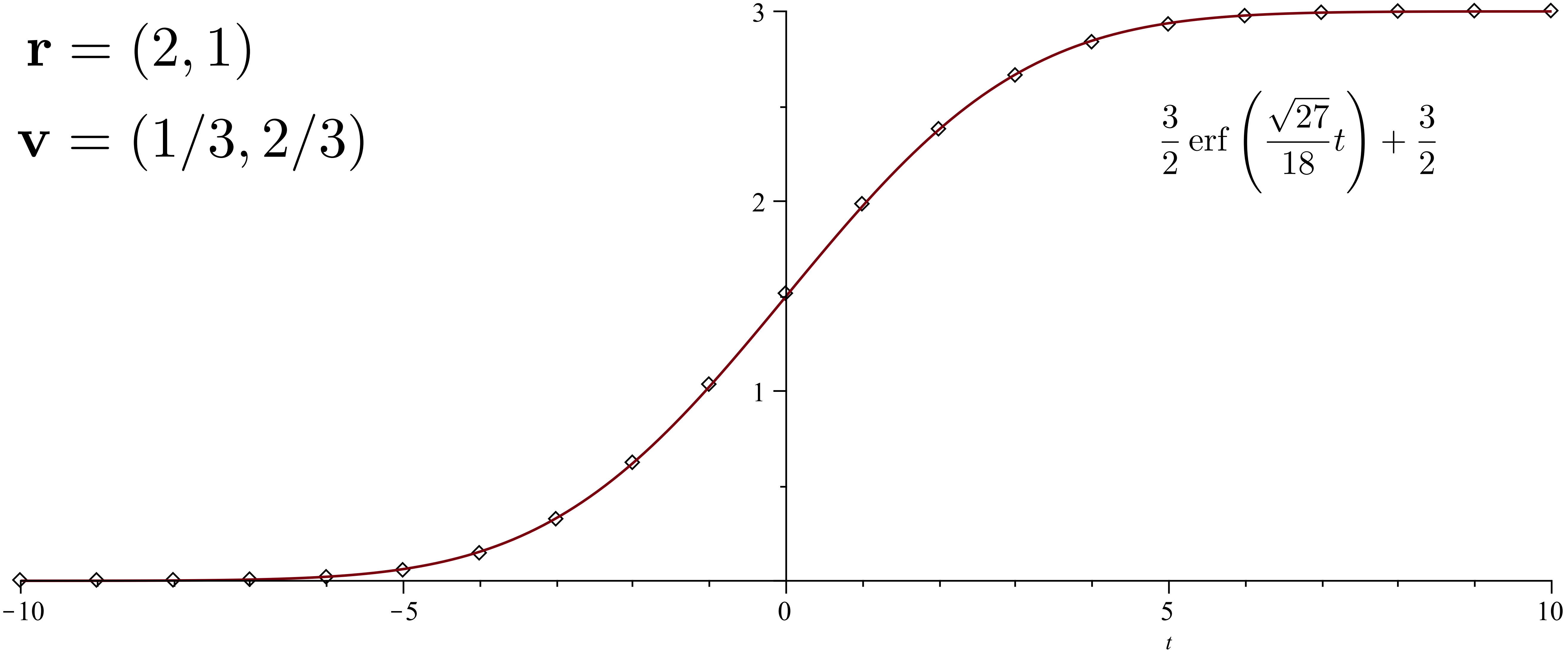
$$\boldsymbol{\sigma}^{n\mathbf{r}+t\sqrt{n}\mathbf{v}} \cdot \left[\mathbf{z}^{n\mathbf{r}+t\sqrt{n}\mathbf{v}}\right] F(\mathbf{z}) \sim A \cdot \operatorname{erf}(Bt) + A$$

Example #2

$$F(x,y) = \frac{1}{(1-2x-y)(1-x-2y)}$$

$$\mathbf{r} = (2,1)$$

$$\mathbf{v} = (1/3, 2/3)$$



$$9^{-3\cdot 30^2-30t}\left[x^{2\cdot 30^2+10t}y^{30^2+20t}\right]F(x,y)$$

CONCLUSION

Conclusion

- ACSV developing rapidly
- Diagonals are data structures for univariate sequences, but ACSV also allows for treatment of truly multivariate questions
- Now that “generic” behaviour is starting to be figured out, time to branch out to more pathological cases
- Perhaps most interesting, we can examine how behaviour transitions between different uniform regimes
- **Still many ways to generalize, and lots more to come!**

THANK YOU!

Asymptotics of multivariate sequences IV: generating functions with poles on a hyperplane arrangement.
Y. Baryshnikov, S. Melczer, and R. Pemantle.
In preparation.

Please contact me if interested in knowing more!

Asymptotics in Generic Directions

After introducing negligible error terms, some residue computations reduce dominant asymptotics to finding asymptotics of a *Fourier-Laplace* integral

$$\int_{\mathbb{R}^r} \boldsymbol{\theta}^{\mathbf{m}} e^{-n\left(\boldsymbol{\theta}^T \mathcal{H} \boldsymbol{\theta}\right)} \, d\boldsymbol{\theta} \qquad (r < d)$$

where $\mathbf{m} \in \mathbb{N}^r$ and $\mathcal{H}$ is a symmetric positive definite matrix

Terms in such an asymptotic expansion are known **explicitly**.

Asymptotics in Non-Generic Directions

In "non-generic" directions, one is not allowed to do all the necessary residue computations needed to reduce to a Fourier-Laplace integral, while still having acceptable error bounds

One ultimately obtains a modified expression of the form

$$\int_{\mathbb{R}^r + i(\epsilon,\ldots,\epsilon)} \boldsymbol{\theta}^{\mathbf{m}} e^{-n\left(\boldsymbol{\theta}^T \mathcal{H} \boldsymbol{\theta}\right)} \, d\boldsymbol{\theta} \qquad (r < d)$$

where $\mathbf{m} \in \mathbb{Z}^r$.

These "negative Gaussian moments" seem to be much less studied (one dimension is easy, otherwise ad hoc using e.g. int. by parts)