COUNTING PARTITIONS INSIDE A RECTANGLE

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Joint work with Greta Panova and Robin Pemantle

Inside cover illustration from
Euler's Introductio in analysin infinitorum

What are Partitions?

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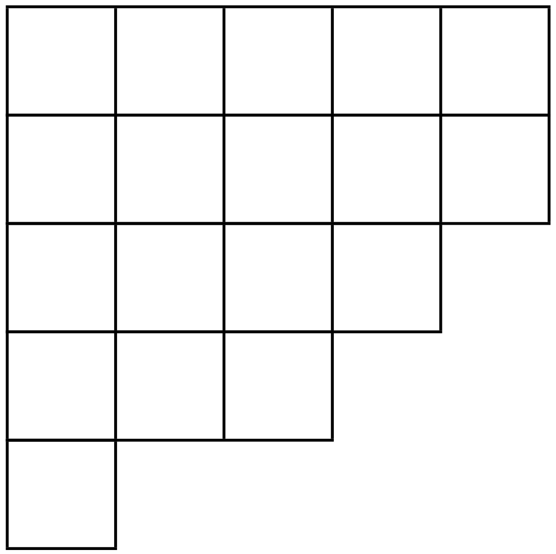
A partition $\lambda \vdash n$ is a sequence of nonnegative integers

$$\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots)$$

with

$$n = |\lambda| = \lambda_1 + \lambda_2 + \cdots$$

$$N_n = \# \text{ partitions of } n$$



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A Constructive theory of Partitions, arranged in three Acts, an Interact and an Exodion.

By J. J. Sylvester, *with Insertions by* Dr. F. Franklin.

(2) The most obvious mode of graphically representing a partition is by means of a network or web formed by two systems of parallel lines or filaments.

American Journal of Mathematics, Vol. 5, No. 1 (1882), pp. 251-330

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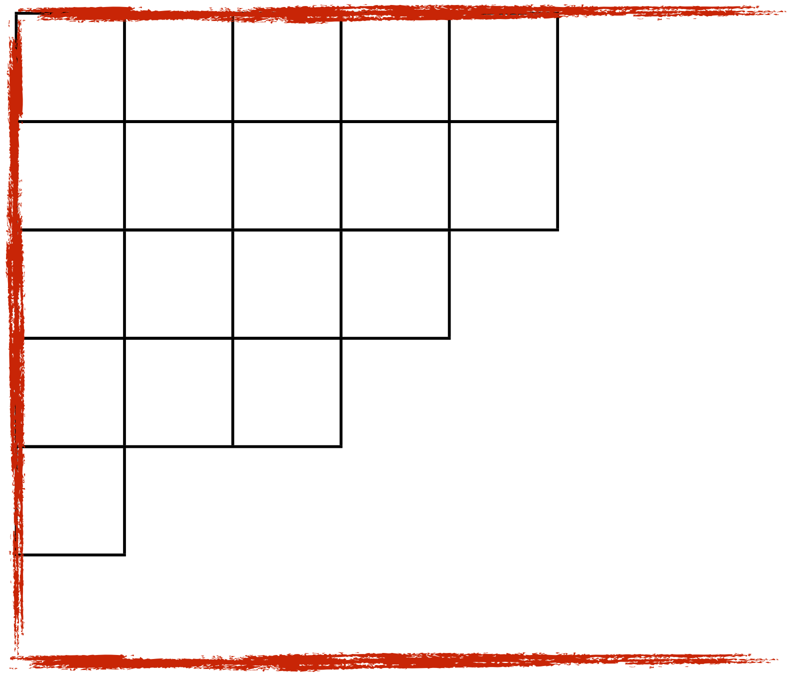
$$N_n(m) = \#\text{ partitions of } n \text{ with at most } m \text{ parts}$$



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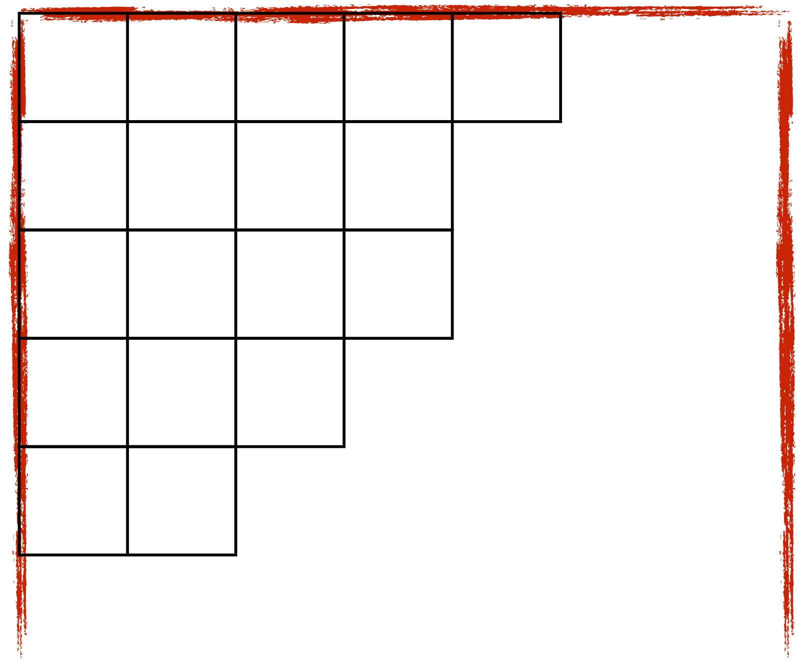

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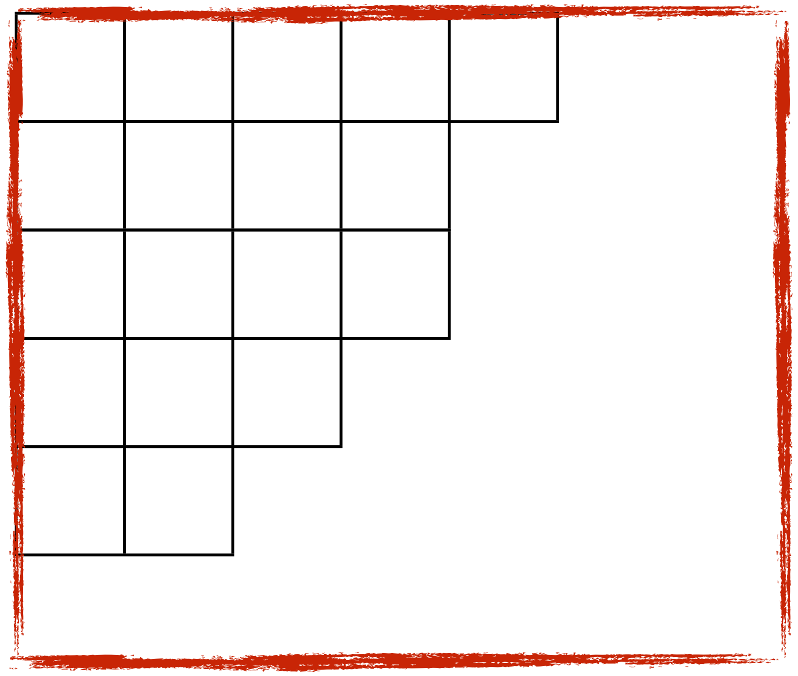
$$\begin{aligned} N_n &= \#\text{ partitions of } n \\ N_n(m) &= \#\text{ partitions of } n \text{ with at most } m \text{ parts} \\ N_n(\ell, m) &= \#\text{ partitions of } n \text{ with at most } m \text{ parts of size } \ell \end{aligned}$$



Why Partitions?

- Index conjugacy classes and irreducible representations of S_n
- Signatures of irreducible polynomial representations of GL_n
- Basis for the ring of symmetric functions
- Connections to Lie algebra identities
- Arise in physics
  (ex: Baxter's solution of the hard hexagon model)

q-Binomial coefficients

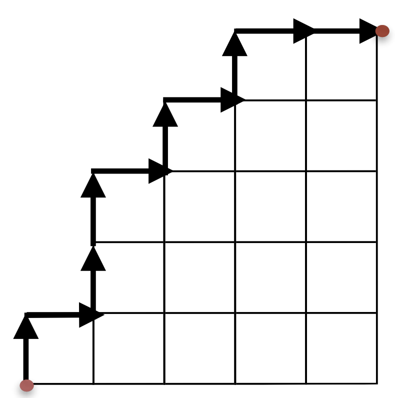
$N_n(\ell, m) = \#$ partitions of n with at most m parts of size ℓ

This is also the q-binomial coefficient

$$\binom{\ell+m}{m}_q = \frac{\prod_{i=1}^{\ell+m}(1-q^i)}{\prod_{i=1}^{\ell}(1-q^i)\prod_{i=1}^{m}(1-q^i)} = \sum_{n=0}^{\ell m} N_n(\ell, m) q^n$$

They

- Count ℓ-dimensional subspaces of $\mathbb{F}_q^{\ell+m}$
- Count lattice paths taking fixed $\#$ of north and east steps
- Appear in statistical tests (Wilcoxon rank sum test)



SVMMATIO QVARVMDAM SERIERVM SINGVLARIVM

AVCTORE

CAROLO FRIDERICO GAVSS.

EXHIBITA SOCIETATI

D. XXIV. AVGVST. CIƆIƆCCCVIII

Petita est demonstratio nostra e consideratione generis singularis progressionum, quarum termini pendent ab expressionibus talibus

$$\frac{(1-x^m)(1-x^{m-1})(1-x^{m-2})\ldots(1-x^{m-\mu+1})}{(1-x)(1-xx)(1-x^3)\ldots\ldots(1-x^\mu)}$$

History of Partitions

Partitions w/ restricted parts and sizes studied at least as far back as Bishop Wibold of Cambrai (c. 965) in the context of dice

Leibniz appears to be first interested explicitly in partitions (“divulsions”)

LEIBNITII AD BERNOULLIUM.

An unquam conſideraſti numerum diſcerptionum vel divulſionum numeri dati, quot ſcilicet modis poſſit divelli in partes duas, tres, &c. Videtur mihi ejus determinatio non facilis, & tamen digna quæ habeatur.

Dabam *Hanoveræ* 28. Julii 1699.

Deditiſſimus

G. G. LEIBNITIUS.

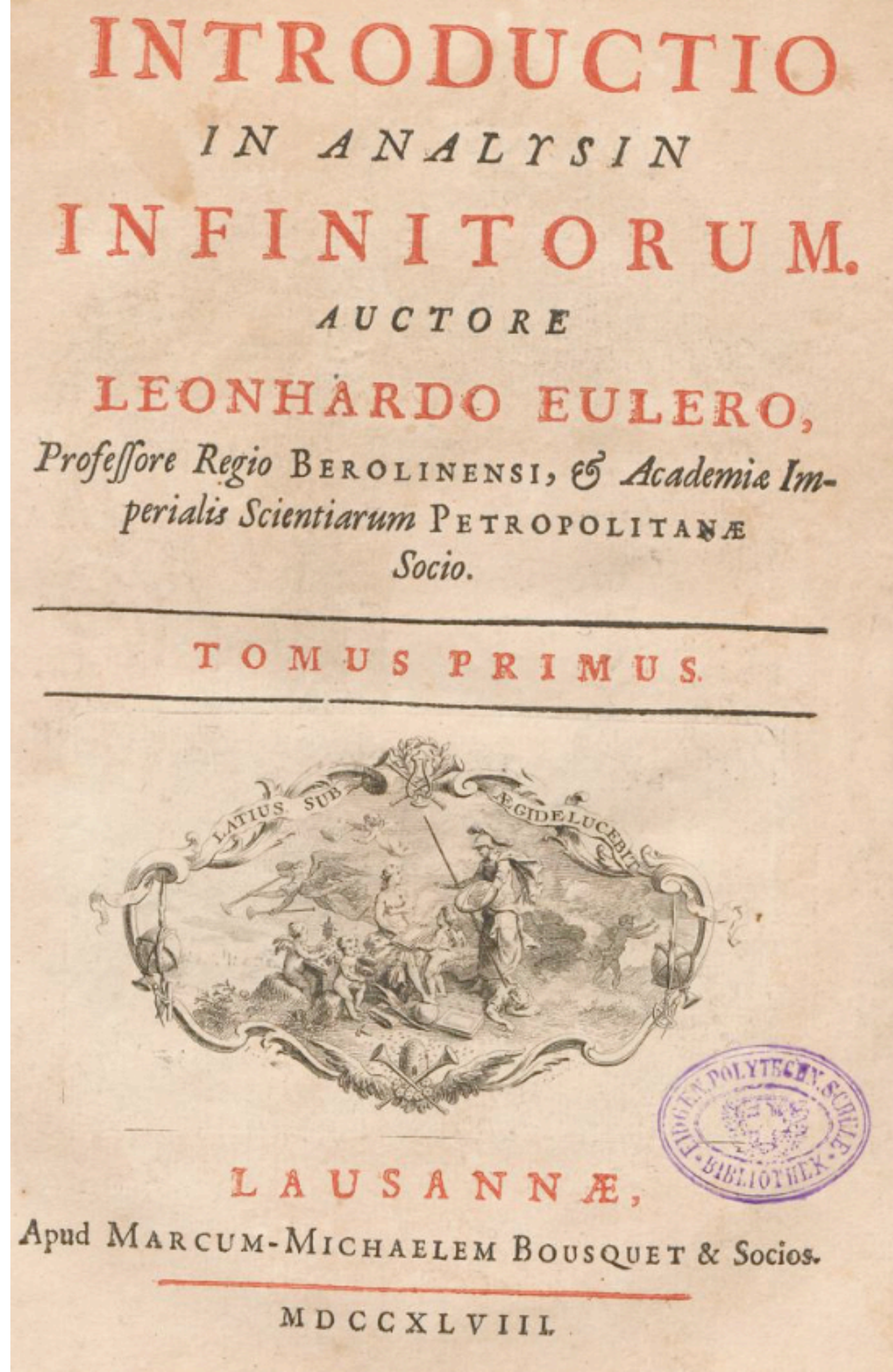
(Excerpt from 1745 printing of Leibniz-Bernoulli letters)

Generating Function

First major results by Euler in 1748, using generating function

$$\sum_{n=0}^{\infty} N_n q^n = \prod_{I=1}^{\infty} \frac{1}{1-q^i}$$

Euler's use of generating functions was the most important innovation in the entire history of partitions. Almost every discovery in partitions owes something to Euler's beginnings. - George Andrews



INTRODUCTIO
IN ANALYSIN
INFINITORUM.
AUCTORE
LEONHARDO EULERO,
Professore Regio BEROLINENSI, & *Academiæ Imperialis Scientiarum* PETROPOLITANÆ *Socio.*

TOMUS PRIMUS.

LATIUS SUB ÆGIDE LUCEBIT

LAUSANNÆ,
Apud MARCUM-MICHAELEM BOUSQUET & Socios.

MDCCXLVIII.

CAPUT XVI.

De Partitione numerorum.

305. Si ponatur $z = 1$, atque similes Potestates ipsius x conjunctim exprimantur, hæc expressio

$$\frac{1}{(1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)(1-x^6)\ \&c.},$$

evolvetur in hanc Seriem

$$1 + x + 2x^2 + 3x^3 + 5x^4 + 7x^5 + 11x^6 + 15x^7 + 22x^8\ \&c.;$$

in qua quilibet coëfficiens indicat, quot variis modis Exponens Potestatis adjunctæ per additionem produci queat ex numeris integris, sive æqualibus sive inæqualibus. Scilicet ex termino

History

In 1856, Cayley conjectured that for fixed ℓ, m the sequence $N_n(\ell, m)$ is *unimodal*:

$$1 = N_0(\ell, m) \leq N_1(\ell, m) \leq \cdots \leq N_{\lfloor m\ell/2 \rfloor} \geq \cdots \geq N_{m\ell}(\ell, m) = 1$$

Proven by Sylvester via representation theory of sl_2

Several modern proofs, none asymptotic - none with good bounds

XXV. ***Proof of the hitherto undemonstrated Fundamental Theorem of Invariants.*** ***By*** **J. J. SYLVESTER,** ***Professor of Mathematics at the Johns Hopkins University, Baltimore**.**

I AM about to demonstrate a theorem which has been waiting proof for the last quarter of a century and upwards.

At the moment of completing a memoir, to appear in Borchardt's Journal, demonstrating my quarter-of-a-century-old theorem for enabling Invariants to procreate their species, as well by an act of self-fertilization as by conjugation of arbitrarily paired forms, the unhoped and unsought-for prize fell into my lap, and I accomplished with scarcely an effort a task which I had believed lay outside the range of human power.

November 13, 1877.

History

(Pak and Panova 2013)

q-binomial coefficients are *strictly* unimodal

Authors later showed

$$N_n(\ell, m) - N_{n-1}(\ell, m) \geq 0.004 \frac{2^{\sqrt{s}}}{s^{9/4}}, \qquad s = \min\{2n, \ell^2, m^2\}$$

Use that

$$N_n(\ell, m) - N_{n-1}(\ell, m) = g\big((\ell m - n, n), m^\ell, m^\ell\big)$$

Kronecker coefficient (describes decomposition of tensor product of two reps of Sn).
Geometric complexity theory relies on (conjectured) ability to show positivity in poly time.

Asymptotics

Statement of the main theorem.

THEOREM. *Suppose that*

$$\phi_q(n) = \frac{\sqrt{q}}{2\pi\sqrt{2}} \frac{d}{dn} \left(\frac{e^{C\lambda_n/q}}{\lambda_n} \right), \tag{1·71}$$

where C and λ_n are defined by the equations (1·53), for all positive integral values of q; that p is a positive integer less than and prime to q; that $\omega_{p,q}$ is a $24q$-th root of unity, defined when p is odd by the formula

(1·721)

$$\omega_{p,q} = \left(\frac{-q}{p}\right) \exp\left[-\left\{ \tfrac{1}{4}(2 - pq - p) + \tfrac{1}{12}\left(q - \frac{1}{q}\right)(2p - p' + p^2 p') \right\} \pi i \right],$$

and when q is odd by the formula

(1·722)

$$\omega_{p,q} = \left(\frac{-p}{q}\right) \exp\left[-\left\{ \tfrac{1}{4}(q - 1) + \tfrac{1}{12}\left(q - \frac{1}{q}\right)(2p - p' + p^2 p') \right\} \pi i \right],$$

where (a/b) is the symbol of Legendre and Jacobi†, and p' is any positive integer such that $1 + pp'$ is divisible by q; that

$$A_q(n) = \sum_{(p)} \omega_{p,q} e^{-2np\pi i/q}; \tag{1·73}$$

and that α is any positive constant, and ν the integral part of $\alpha\sqrt{n}$.

Then

$$p(n) = \sum_1^{\nu} A_q \phi_q + O(n^{-\frac{1}{4}}), \tag{1·74}$$

so that $p(n)$ is, for all sufficiently large values of n, the integer nearest to

$$\sum_1^{\nu} A_q \phi_q. \tag{1·75}$$

TABLE IV*: $p(n)$.

1...	1	51...	239943	101...	214481126	151...	45060624582
2...	2	52...	281589	102...	241265379	152...	49686288421
3...	3	53...	329931	103...	271248950	153...	54770336324
4...	5	54...	386155	104...	304801365	154...	60356673280
5...	7	55...	451276	105...	342325709	155...	66493182097
6...	11	56...	526823	106...	384276336	156...	73232243759
7...	15	57...	614154	107...	431149389	157...	80630964769
8...	22	58...	715220	108...	483502844	158...	88751778802
9...	30	59...	831820	109...	541946240	159...	97662728555
10...	42	60...	966467	110...	607163746	160...	107438159466
11...	56	61...	1121505	111...	679903203	161...	118159068427
12...	77	62...	1300156	112...	761002156	162...	129913904637
13...	101	63...	1505499	113...	851376628	163...	142798995930
14...	135	64...	1741630	114...	952050665	164...	156919475295
15...	176	65...	2012558	115...	1064144451	165...	172389800255
16...	231	66...	2323520	116...	1188908248	166...	189334822579
17...	297	67...	2679689	117...	1327710076	167...	207890420102
18...	385	68...	3087735	118...	1482074143	168...	228204732751
19...	490	69...	3554345	119...	1653668665	169...	250438925115
20...	627	70...	4087968	120...	1844349560	170...	274768617130
21...	792	71...	4697205	121...	2056148051	171...	301384802048
22...	1002	72...	5392783	122...	2291320912	172...	330495499613
23...	1255	73...	6185689	123...	2552338241	173...	362326859895
24...	1575	74...	7089500	124...	2841940500	174...	397125074750
25...	1958	75...	8118264	125...	3163127352	175...	435157697830
26...	2436	76...	9289091	126...	3519222692	176...	476715857290
27...	3010	77...	10619863	127...	3913864295	177...	522115831195
28...	3718	78...	12132164	128...	4351078600	178...	571701605655
29...	4565	79...	13848650	129...	4835271870	179...	625846753120
30...	5604	80...	15796476	130...	5371315400	180...	684957390936
31...	6842	81...	18004327	131...	5964539504	181...	749474411781
32...	8349	82...	20506255	132...	6620830889	182...	819876908323
33...	10143	83...	23338469	133...	7346629512	183...	896684817527
34...	12310	84...	26543660	134...	8149040695	184...	980462880430
35...	14883	85...	30167357	135...	9035836076	185...	1071823774337
36...	17977	86...	34262962	136...	10015581680	186...	1171432692373
37...	21637	87...	38887673	137...	11097645016	187...	1280011042268
38...	26015	88...	44108109	138...	12292341831	188...	1398341745571
39...	31185	89...	49995925	139...	13610949895	189...	1527273599625
40...	37338	90...	56634173	140...	15065878135	190...	1667727404093
41...	44583	91...	64112359	141...	16670689208	191...	1820701100652
42...	53174	92...	72533807	142...	18440293320	192...	1987276856363
43...	63261	93...	82010177	143...	20390982757	193...	2168627105469
44...	75175	94...	92669720	144...	22540654445	194...	2366022741845
45...	89134	95...	104651419	145...	24908858009	195...	2580840212973
46...	105558	96...	118114304	146...	27517052599	196...	2814570987591
47...	124754	97...	133230930	147...	30388671978	197...	3068829878530
48...	147273	98...	150198136	148...	33549419497	198...	3345365983698
49...	173525	99...	169229875	149...	37027355200	199...	3646072432125
50...	204226	100...	190569292	150...	40853235313	200...	3972999029388

Asymptotics

Herschel (1818), Cayley (1855), Sylvester (1882)

Asymptotics of $N_n(m)$ for small fixed m

Easy using partial fraction decomposition

Hardy and Ramanujan (1918)

Asymptotics of N_n (w/ error tending to 0)

$$p(n) \sim \frac{1}{4n\sqrt{3}} \exp\left\{\pi\sqrt{\left(\frac{2n}{3}\right)}\right\}$$

Rademacher (1937)

Convergent series expansion of N_n

Asymptotics of $N_n(m)$

Erdös and Lehner (1941)

$$N_n(m) \sim \frac{n^{m-1}}{m!(m-1)!} \text{ for } m = o(n^{1/3})$$

Szekeres (1953)

THEOREM 1. *Let n/k^2 be bounded, $n \leqslant c_1 k^2$, and let β, v be determined from*

$$v\beta = k, \qquad \beta^2 \int_0^v \frac{t}{e^t-1}\,dt + \tfrac{1}{2}\beta\left(\frac{v}{e^v-1}-1\right) + \frac{1}{12}\left(\frac{1}{2}+\frac{1}{e^v-1}-\frac{ve^v}{(e^v-1)^2}\right) = n. \tag{1}$$

Then, uniformly in n and k,

$$P(n,k) = \frac{1}{2\pi} B_0^{-\frac{1}{2}}\beta^{-2}\exp\left\{2\beta\int_0^v \frac{t}{e^t-1}\,dt - (v\beta+\tfrac{1}{2})\log(1-e^{-v}) + \right.$$
$$\left. +\frac{1}{2}\left(\frac{v}{e^v-1}-1\right)\right\}[1+B_1(v)\beta^{-1}+\ldots+B_{m-1}(v)\beta^{-m+1}+O(\beta^{-m})] \tag{2}$$

for any given $m > 0$, where

$$B_0 = \int_0^v \frac{t^2 e^t}{(e^t-1)^2}\,dt = 2\int_0^v \frac{t}{e^t-1}\,dt - \frac{v^2}{e^v-1} \tag{3}$$

Asymptotics of $N_n(l,m)$

Mann and Whitney (1947)

Size of a uniform random partition in a rectangle satisfies a normal distribution

Takács (1986)

$$N_n(\ell, m) \sim \frac{1}{\sigma_{\ell,m}\sqrt{2\pi}} \binom{\ell+m}{\ell} \exp\left[-\frac{1}{2}\left(\frac{n-\ell m/2}{\sigma_{\ell,m}}\right)^2\right], \qquad \sigma_{\ell,m} = \sqrt{lm(\ell+m+1)/12}$$

when

$$|n - \ell m/2| < K\sigma_{\ell,m} = O\left(\sqrt{\ell m(\ell+m)}\right)$$

Asymptotics of $N_n(l,m)$

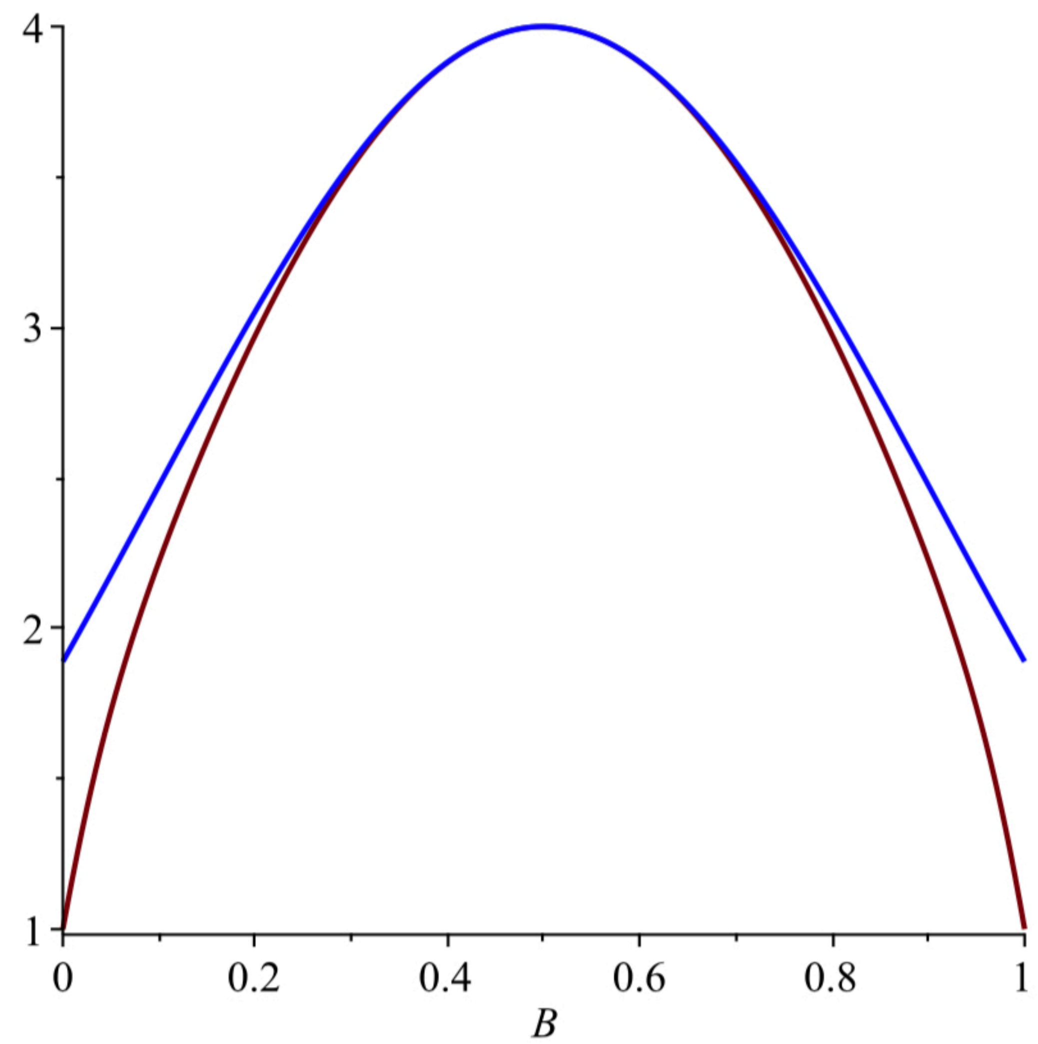


Figure 1: Exponential growth of $N_{Bm^2}(m,m)$ predicted by Takács' formula (blue, above) compared to the actual exponential growth

Our Results

We give asymptotics in all cases where a limit shape exists

$$\ell/m \to A \quad \text{and} \quad n/m^2 \to B$$

Given A and B, let c and d be defined from

$$A = \int_0^1 \frac{1}{1 - e^{-c-dt}} dt - 1$$

$$B = \int_0^1 \frac{t}{1 - e^{-c-dt}} dt - \frac{1}{2}$$

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$$A = \int_0^1 \frac{1}{1 - e^{-c-dt}} dt - 1 = \frac{1}{d} \log\left(\frac{e^{c+d} - 1}{e^c - 1}\right)$$

$$B = \int_0^1 \frac{t}{1 - e^{-c-dt}} dt - \frac{1}{2} = \frac{d \log(1 - e^{-c-d}) + \text{dilog}(1 - e^{-c}) - \text{dilog}(1 - e^{-c-d})}{d^2}$$

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and set

$$\Delta := \frac{2Be^c(e^d-1) + 2A(e^c-1) - 1}{d^2(e^{d+c}-1)(e^c-1)} - \frac{A^2}{d^2}$$

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and set

Sufficient to consider

$$A \geq 2B$$

$A = \ell/m$

$B = n/m^2$

Our Results

Theorem (M., Panova, Pemantle 2018)

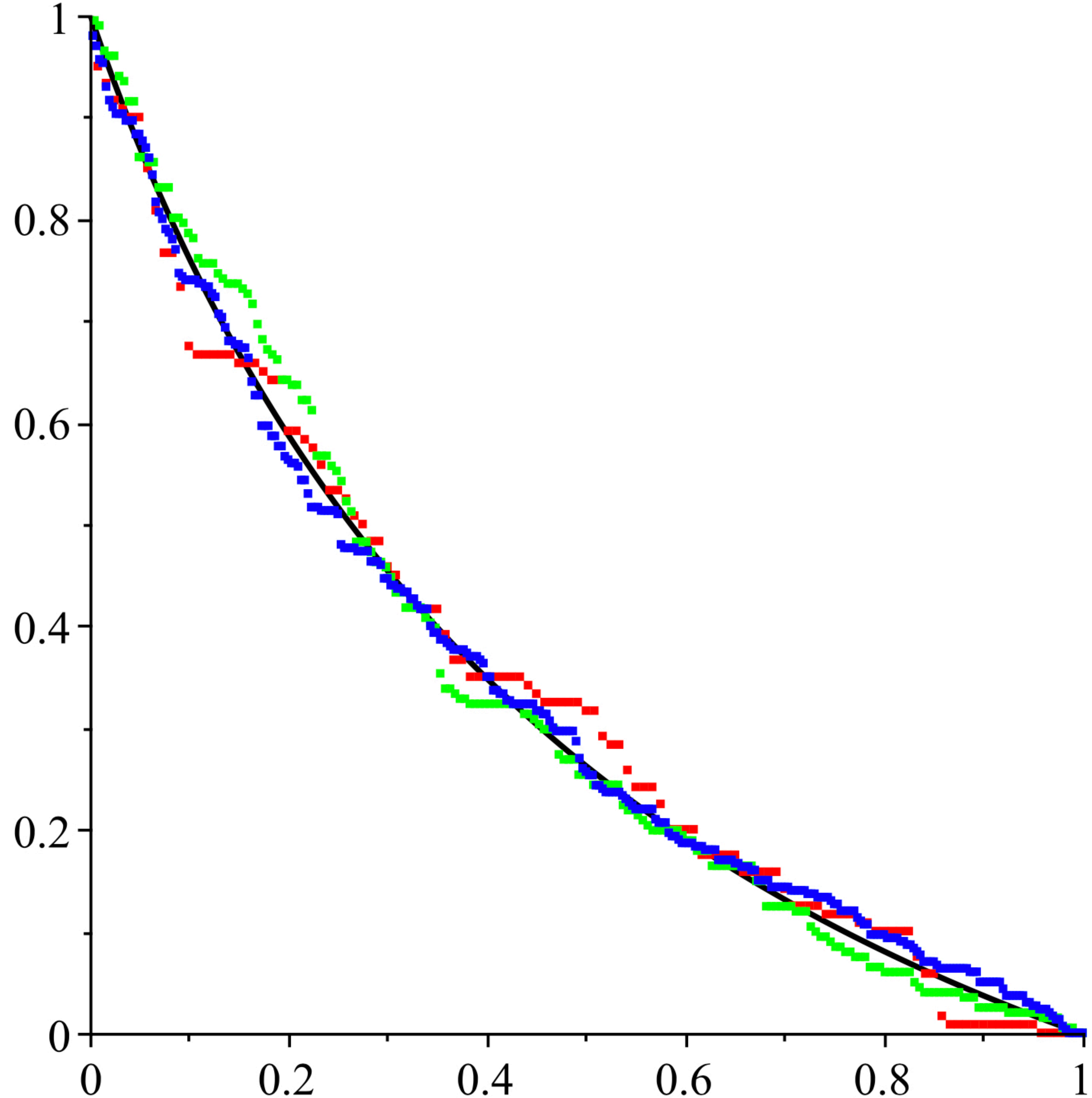
Let K be a compact subset of $\{(x,y) : x > 2y > 0\}$

As $m \to \infty$ and l and n vary so that (A,B) remains in K,

$$N_n(\ell, m) \sim \frac{e^{m\left[cA + 2dB - \log(1-e^{-c-d})\right]}}{2\pi m^2 \sqrt{\Delta\,(1-e^{-c})\,(1-e^{-c-d})}}$$

where c and d vary in a Lipshitz manner with (A,B)

Our methods allows us to determine the expected limit curve



Limit curve of $(A, B) = (1, 1/3)$ and random partitions of size 120, 201 and 300.

$A = \ell / m$

$B = n / m^2$

Our Results

Theorem (M., Panova, Pemantle 2018)

Let K be a compact subset of $\{(x, y) : x > 2y > 0\}$

As $m \to \infty$ and l and n vary so that (A,B) remains in K,

$$N_{n+1}(\ell, m) - N_n(\ell, m) \sim \frac{d}{m} N_n(\ell, m)$$

This gives a significant asymptotic generalization of Sylvester's unimodality theorem

RANDOM GENERATION AND LOCAL LIMIT THEOREMS

Encoding by Gaps

Fix partition $\lambda = (\lambda_1, \ldots, \lambda_m)$ and define $\lambda_0 := \ell, \quad \lambda_{m+1} := 0$

A partition is uniquely determined by its gaps

$$x_j := \lambda_j - \lambda_{j+1} \geq 0$$

Encoding by Gaps

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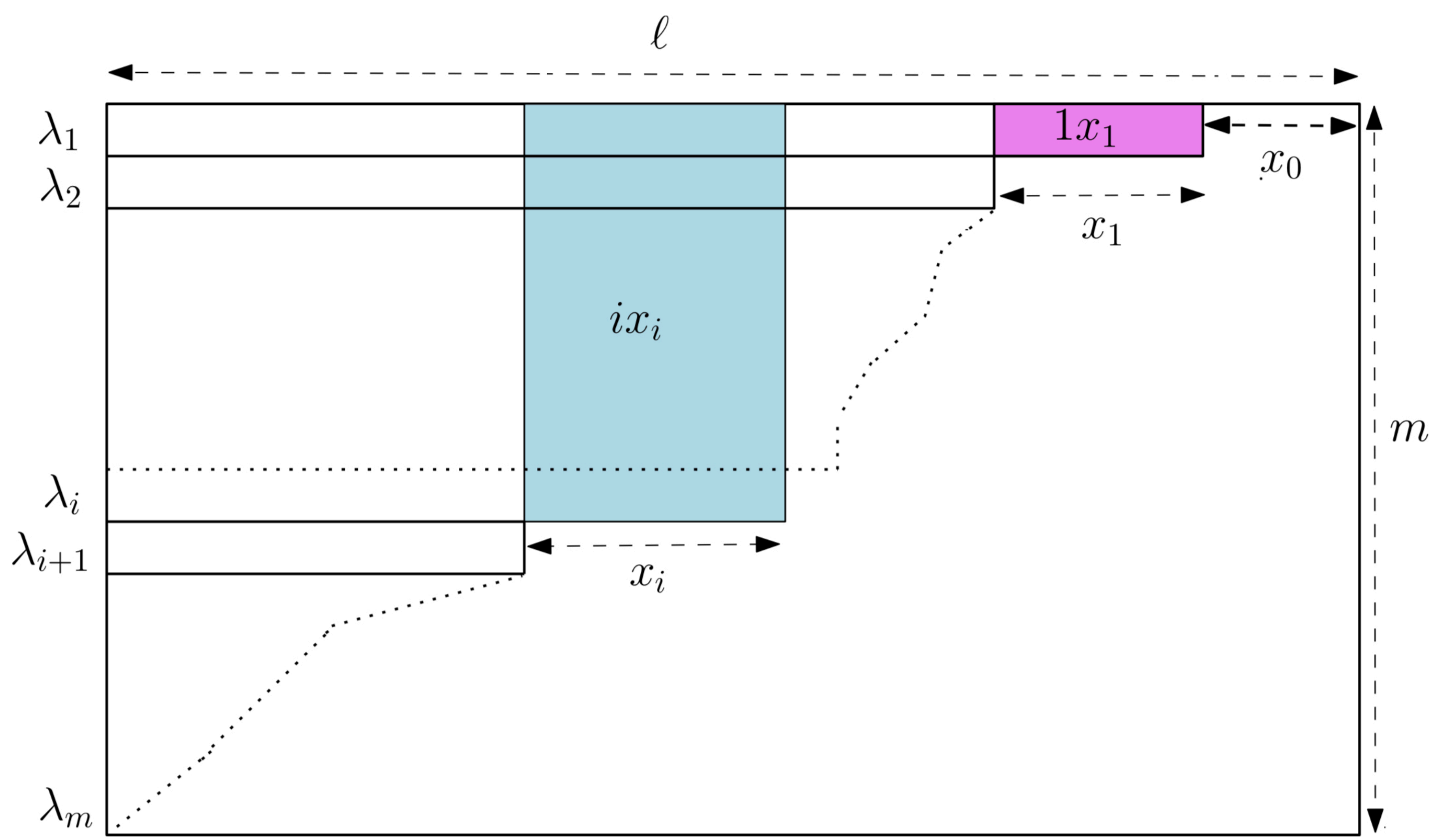
A partition is uniquely determined by its gaps

$$x_j := \lambda_j - \lambda_{j+1} \geq 0$$

Being in the rectangle corresponds to

$$\sum_{j=0}^{m} x_j = \ell$$

$$\sum_{j=0}^{m} j x_j = n$$



Encoding by Gaps

This is a bijection: given $x_0, \ldots, x_m \geq 0$ with

$$\sum_{j=0}^{m} x_j = \ell \qquad \sum_{j=0}^{m} j x_j = n \qquad (\star)$$

the partition with $\lambda_j = \ell - x_0 - \cdots - x_{j-1}$ is in the rectangle.

Suppose we want to generate a partition uniformly at random

Generate a non-negative tuple subject to $(\star)$

Encoding by Gaps

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$$\sum_{j=0}^{m} x_j = \ell \qquad \sum_{j=0}^{m} j x_j = n \qquad (\star)$$

the partition with $\lambda_j = \ell - x_0 - \cdots - x_{j-1}$ is in the rectangle.

Suppose we want to generate a partition uniformly at random

Generate a non-negative tuple subject to $(\star)$

Random var X has *geometric distribution with parameter* p if

$$\mathbb{P}(X = k) = p \cdot (1 - p)^k, \quad k = 0, 1, \ldots$$

Rejection Sampling

Suppose $\mathbf{x} = (x_0, \ldots, x_m)$ satisfies $(\star)$

$\mathbf{X} = (X_0, \ldots, X_m)$ RV geometrics with parameters $p_0, \ldots, p_m$

Then

$$\mathbb{P}(\mathbf{X} = \mathbf{x}) = (p_0 \cdots p_m)(1 - p_0)^{x_0} \cdots (1 - p_m)^{x_m}$$

Rejection Sampling

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$\mathbf{X} = (X_0, \ldots, X_m)$ RV geometrics with parameters $p_0, \ldots, p_m$

Then

$$\mathbb{P}(\mathbf{X} = \mathbf{x}) = (p_0 \cdots p_m)(1 - p_0)^{x_0} \cdots (1 - p_m)^{x_m}$$

If $1 - p_j = e^{-\alpha - \beta j}$,

$$\mathbb{P}(\mathbf{X} = \mathbf{x}) = (p_0 \cdots p_m) e^{-\alpha \sum x_j - \beta \sum j x_j} = (p_0 \cdots p_m) e^{-\alpha \ell - \beta n}$$

Independent of $\mathbf{x}$!

Rejection Sampling

Thus, to randomly sample a partition in a box we can sample the RVs $\mathbf{X}$ until we get a sequence satisfying $(\star)$

But which distribution should we use?

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$$S_m := \sum_{i=0}^{m} X_i \qquad\qquad T_m := \sum_{i=0}^{m} iX_i$$

It makes sense to take $\alpha = c, \quad \beta = d/m$ so that

$$\ell = \mathbb{E}\left[S_m\right] \qquad\qquad n = \mathbb{E}\left[T_m\right]$$

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$$\ell = m \sum_{j=0}^{m} \frac{1/m}{1 - e^{-c-dj/m}} - (m+1) \qquad n = m^2 \sum_{j=0}^{m} \frac{j/m^2}{1 - e^{-c-dj/m}} - \frac{m(m+1)}{2}$$

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$$\ell = m\left(\int_0^1 \frac{1}{1-e^{-c-dt}}dt - 1\right) + O(1) \qquad n = m^2\left(\int_0^1 \frac{t}{1-e^{-c-dt}}dt - \frac{1}{2}\right) + O(m)$$

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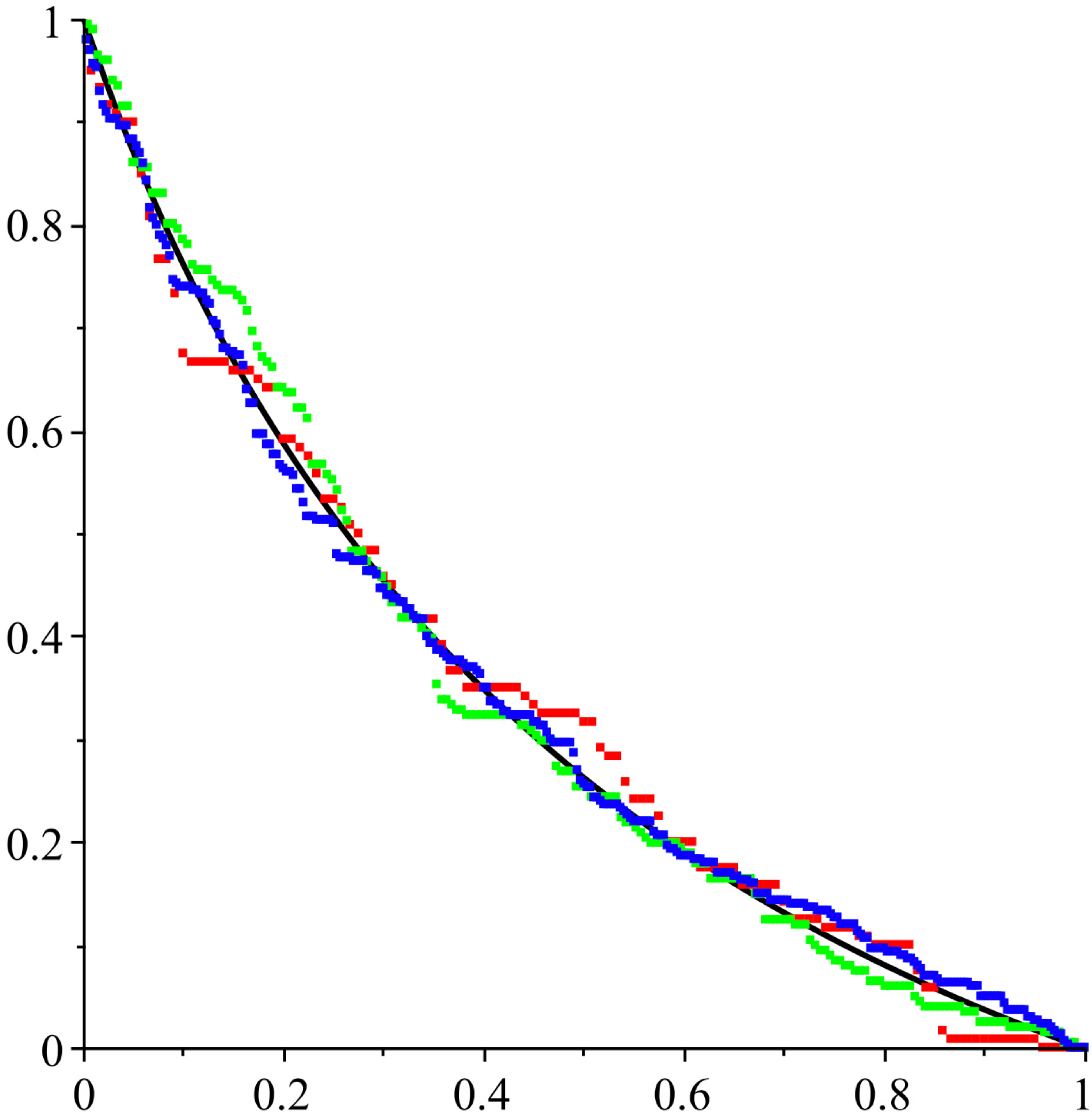
$$\ell = Am + O(1) \qquad n = Bm^2 + O(m)$$



1
0.8
0.6
0.4
0.2
0
0
0.2
0.4
0.6
0.8
1

To Counting

If $\mathbf{x}$ satisfies $(\star)$ then $\mathbb{P}(\mathbf{X} = \mathbf{x})$ is constant

Thus,

$$N_n(\ell, m) \cdot \mathbb{P}(\mathbf{X} = \mathbf{x}) = \mathbb{P}\left[(S_m, T_m) = (\ell, n)\right]$$

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where $\quad L_m = \sum_{j=0}^{m} \log p_j$

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Local Central Limit Theorem

Let

$$M = \text{covariance matrix for } (S_m, T_m)$$
$$\mu = \mathbb{E}[S_m] \qquad \nu = \mathbb{E}[T_m]$$

$$p(a,b) = \mathbb{P}[(S_m, T_m) = (a,b)]$$
$$\mathcal{N}(a,b) = \frac{1}{2\pi(\det M)^{1/2}} e^{-\frac{1}{2}(a-\mu, b-\nu)M^{-1}(a-\mu, b-\nu)^T}$$

Then

$$\sup_{a,b\in\mathbb{Z}} |p(a,b) - \mathcal{N}(a,b)| = O(m^{-5/2})$$

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Then

$$\sup_{a,b\in\mathbb{Z}} \left| p(a,b+1) - p(a,b) - \big(\mathcal{N}(a,b+1) - \mathcal{N}(a,b)\big) \right| = O(m^{-4})$$

Conclusion

Partitions are classical objects, appearing all over mathematics

We give the first asymptotics of partitions in a rectangle for the general regime where a limit shape exists

Can we use these methods to derive new results on other kinds of partitions?

THANK YOU!

Counting partitions in a rectangle
S. Melczer, G. Panova and R. Pemantle
Submitted May 2018
arxiv.org/abs/1805.08375

FILM REVIEW

The Man Who Knew Infinity: A Report on the Movie

by George E. Andrews

The pioneering combinatory analyst, Major P. A. MacMahon, has an important part in the movie. Since I edited MacMahon's Collected Papers for the MIT Press [4], I watched this role with great interest. Actually I was delighted by the first seemingly implausible interaction between MacMahon and Ramanujan. MacMahon challenges Ramanujan to give the square root of a quite large integer. Ramanujan responds correctly after some hesitation and has to correct his result with a few added decimal places. Ramanujan then asks MacMahon to square the original number which he does with lightning speed. MacMahon is triumphant at having won the contest.

Surely you are wondering why this story would please me. After all, this must be pure fantasy and unlike any interaction of serious mathematicians. In fact, this is a fairly accurate account of history. According to Gian-Carlo Rota in his introduction to Volume I of MacMahon's Collected Papers: "It would have been fascinating to be present at one of the battles of arithmetical wits at Trinity College, when MacMahon would regularly trounce Ramanujan by the display of superior ability for fast mental calculation (as reported by D. C. Spencer, who heard it from G. H. Hardy). The written accounts of the lives of these characters, however, omit any mention of this episode, since it clashes against our prejudices."

Shtetl-Optimized

The Blog of Scott Aaronson

"Largely just men doing sums": My review of the excellent Ramanujan film

Audiences might even have *liked* some more T&A (theorems and asymptotic bounds).

Apparently, Brown struggled for an entire decade to attract funding for a film about a turn-of-the-century South Indian mathematician visiting Trinity College, Cambridge, whose work had no commercial or military value whatsoever. At one point, Brown was actually told that he could get the movie funded, *if he'd agree to make Ramanujan fall in love with a white nurse*, so that a British starlet who would sell tickets could be cast as his love interest. One can only imagine what a battle it must have been to get a correct explanation of the partition function onto the screen.