COLORING A $d \geq 3$ DIMENSIONAL LATTICE WITH TWO INDEPENDENT RANDOM WALKS

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To Susan, Mom, and Dad.

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1 Introduction

Simple random walk has been studied extensively in both the mathematics and statistical physics literature. In [1], Júnior, *et al.* consider two independent simple random walks coloring a circle of points, $\mathbf{Z}/n\mathbf{Z}$. One walk colors points red and the other colors points blue. Once colored, a point does not change colors, *i.e.* the color of a point is determined by the first random walk to reach it. Throughout the introduction, we ignore ties where the red walk and the blue walk reach a given point at the same time.

Rather than coloring the circle, here we consider two independent simple random walks coloring the lattice torus, $\mathbf{Z}^d/n\mathbf{Z}^d$, $d \geq 3$, in the same manner as [1]. Eventually, all points of the torus will be colored and the question we face, regarding the coloring, is: what does the torus look like? Our approach is to investigate the number of red points minus the number of blue points on the lattice torus. Understanding the distribution of this quantity is a step towards answering the posed question. The main results presented here are nontrivial upper and lower bounds on the variance of the number of red points minus the number of blue points on the colored lattice torus.

Júnior, *et al.* do not directly address the difference in number of red points and number of blue points in their paper [1], however they consider a related problem. Using generating functions, Júnior *et al.* obtain an approximation to the probability that a given site is colored red when the two walks start opposite each other on the circle. If the red walk starts from 0, the blue walk starts from $n/2$, and m is some other point on the circle, then

$$\mathbf{P}\{m \text{ is colored red}\}$$

tends to a nonlinear but, according to experimental evidence, close to linear function of $\lambda = m/n$ as $n \to \infty$ and λ remains fixed.

Other quantities might reasonably be studied to help us better understand what the torus looks like after it is colored. In [1], Júnior *et al.* offer empirical results on the average number of red-blue interfaces when $d = 1$. (A red-blue interface is a pair of neighboring points with one point colored red and the other colored blue.) Their simulations indicate that when the two random walks start opposite each other on the circle the average number of red-blue interfaces is 2.5. Moreover, Júnior *et al.* suggest that the number of red-blue interfaces scales logarithmically with the ratio n/a, where a is the distance between the initial position of the red walk and the initial position of the blue walk, and that this relationship is nearly independent of n. Júnior *et al.* give no analytical results regarding red-blue interfaces, citing that the expressions involved are "cumbersome." We will not address red-blue interfaces any further in what follows.

1.1 Recoloring

In a different setting, we have two independent simple random walks coloring the lattice torus red and blue, but each point is recolored whenever it is visited by a random walk (contrast with the above, where, once colored, a point is never recolored). With recoloring, the current color of a point is determined by the most recent random walk to have visited it. Furthermore, with recoloring, the colors of the torus will never reach a final state and the question "what does the torus look like?" is more ambiguous than above. The colors of the torus together with the position of the random walks is a Markov chain on a finite state space and, thus, has a stationary distribution. (See [3], for instance, for the elementary facts about stationary distributions of Markov chains we shall require.) Assuming that points are recolored, an appropriate question regarding the coloring of the torus is: what does the torus look like under the stationary distribution?

At stationary, we may consider the random walks on the torus with recoloring process as a bi-infinite Markov chain, that is, we may consider the process at times $-\infty < j < \infty$. Under the stationary distribution, at time $j = 0$ (or at any fixed time) the two random walks are uniformly and independently distributed on $\mathbf{Z}^d/n\mathbf{Z}^d$ and the color of $x \in \mathbf{Z}^d/n\mathbf{Z}^d$ is red if it was most recently visited by the red walk and blue otherwise. Equivalently, x is red at time $j = 0$ if and only if the time reversal of the red walk on $-\infty < j \leq 0$ reaches x before the time reversal of the blue walk on $-\infty < j \leq 0$. Since the time reversal of the red and blue simple random walks on $-\infty < j \leq 0$ are independent simple random walks, indexed in time by the nonnegative integers, with initial positions uniformly and independently distributed on $\mathbf{Z}^d/n\mathbf{Z}^d$, we may take the view that x is red at time $j = 0$ if and only if it is reached by a red simple random walk before it is reached by an independent blue walk. But this is the same way that the final color of x is determined when the torus is colored without recoloring. It follows that the stationary distribution of colors on the torus when points are recolored is the same as the final distribution of colors when there is no recoloring and the two random walks' initial positions are independent and uniformly distributed on the torus. In the sequel, we will consider only the process without recoloring.

1.2 Organization

The paper is organized as follows. First, an overview of simple random walk (SRW) on $\mathbf{Z}^d$ is provided. Our ultimate objective is to better understand two independent SRWs on $\mathbf{Z}^d/n\mathbf{Z}^d$, however, we take the point of view that SRW is occurring in $\mathbf{Z}^d$ and all events of interest can be described with reference to the cosets $x + n\mathbf{Z}^d$.

When dealing with more than one SRW, parity is a concern. For instance, on the circle $\mathbf{Z}/2n\mathbf{Z}$, two independent SRWs started at points with different parity will never meet. To avoid this issue, we work with SRWs that take steps in continuous time, according to independent exponential random variables. After explaining some of the basics of discrete time SRWs, we prove a couple of results that are useful in converting results for discrete time random walk into analogous results for continuous time random walk.

Next, we introduce two independent continuous time SRWs and, finally, our bounds on the variance of the number of red sites minus the number of blue sites on the colored lattice torus are established. Finding an upper bound on the variance is (easily) reduced to bounding the probability that two points $x, y \in \mathbf{Z}^d/n\mathbf{Z}^d$ are the same color minus the probability that x, y are different colors. At time 0, the red and blue walks are started and some time later one of x, y is colored, say, for instance, that x is colored red (and y remains uncolored). Then x, y are the same color if and only if the red walk, starting from x, reaches y before the blue walk reaches y. Initially, the red walk may have the advantage in reaching y before the blue walk. However, after some length of time the red and the blue walks become "lost" and, if y has not been colored before this time, then the two walks are equally likely to reach y first. Making this idea precise and then using techniques found in Chapter 1 of [2] lead to the desired upper bound.

To find a lower bound on the variance of the number of red sites minus the number of blue sites, we isolate contributions to the variance that arise whenever one of the two random walks neighbors an uncolored site of $\mathbf{Z}^d/n\mathbf{Z}^d$. To prove the lower bound we use a result proved in the course of obtaining the upper bound, namely, that there is a significant probability that a walk neighboring an uncolored point gets lost before reaching that point. Note that the upper and lower bounds proved in this paper do not match.

In the discussion at the end of the paper, we mention how the arguments given here can be modified to obtain an upper bound on the variance when $d = 2$. Also in the discussion, we briefly mention the conjectured rate that the variance of the number of red sites minus the number of blue sites follows as n tends to infinity (Pemantle, Peres and Revelle).

2 Notation

For $x = (x^1, ..., x^d) \in \mathbf{R}^d$ a superscript is used to denote the coordinates of x. Let $|x| = ||x||_2$ denote the Euclidean norm of $x \in \mathbf{R}^d$, $||x||_1 = |x^1| + \cdots + |x^d|$, and

$$||x||_\infty = \max_{1 \leq j \leq d} |x^j|.$$

Following the standard practice, if $x, y \in \mathbf{R}$, then $x \wedge y := \min\{x, y\}$ and $x \vee y := \max\{x, y\}$. For $x \in \mathbf{Z}^d$, let $[x] \in \mathbf{Z}^d/n\mathbf{Z}^d$ denote the image of x in $\mathbf{Z}^d/n\mathbf{Z}^d$, *i.e.* $[x]$ is the coset $x + n\mathbf{Z}^d$.

We will frequently make use of standard asymptotic notation. For functions $f, g : \mathbf{Z}^d \to \mathbf{R}$, we write $f(x) \sim g(x)$ if

$$\frac{f(x)}{g(x)} \to 1 \text{ as } |x| \to \infty.$$

We write $f(x) = O(g(x))$ if there is a constant $C > 0$ such that

$$|f(x)| \leq C|g(x)|$$

for all $x \in \mathbf{Z}^d$ and we write $f(x) \asymp g(x)$ if there are constants $c_1, c_2 > 0$ such that

$$c_1|g(x)| \leq |f(x)| \leq c_2|g(x)|$$

for all $x \in \mathbf{Z}^d$. Throughout we allow constants to depend on the dimension, d.

The setting is a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ that is assumed to carry all of the random variables and processes introduced below. All of the random variables and processes considered here are standard and there are no significant existence issues. We will sometimes use superscripts, for instance $\mathbf{P}^x$ or $\mathbf{P}^{r,b}$, to indicate the initial point of a random walk or pair of random walks. It should be clear from the context which processes the superscripts refer to and this convention will be further explained as the need arises. The superscript may also appear when taking expectations as $\mathbf{E}^x$ or $\mathbf{E}^{r,b}$, *etc.*

On one occasion we need the concept of the law of a stochastic process. We say that two stochastic processes $X(t)$, $Y(t)$, $0 \le t < \infty$, are equal in law, denoted $X(t) \stackrel{\mathscr{L}}{=} Y(t)$, if, for each finite sequence of numbers $0 \le t_1, ..., t_k < \infty$ and Borel subsets $B_1, ..., B_k \subset \mathbf{R}$, we have

$$\mathbf{P}\{X(t_1) \in B_1, ..., X(t_k) \in B_k\} = \mathbf{P}\{Y(t_1) \in B_1, ..., Y(t_k) \in B_k\}.$$

For further details, see [4].

3 Simple random walk

Let $X_1, X_2, ...$ be iid with

$$\mathbf{P}\{X_j = e\} = \frac{1}{2d}$$

provided $e \in \mathbf{Z}^d$ and $|e| = 1$. Define the simple random walk $S_k = (S_k^1, ..., S_k^d) = X_0 + X_1 + \cdots + X_k$ where $X_0 = x \in \mathbf{Z}^d$. A superscript in $\mathbf{P}^x$ and $\mathbf{E}^x$ will indicate $X_0 = x$; take the absence of a superscript to mean $X_0 = 0$. Observe that

$$\mathbf{E}|S_k|^2 = k\mathbf{E}|X_1|^2 = k/d.$$

This simple calculation suggests that after k steps, the SRW S_k is displaced order $k^{1/2}$ units from its initial position. Refinements of this idea are relied on repeatedly in the sequel.

3.1 Large deviations

Following notation in [2], define $C_m = \{z \in \mathbf{Z}^d;\ |z| < m\}$ and for subsets $A \subset \mathbf{Z}^d$ define the boundary of A,

$$\partial A = \{z \in A^c \cap \mathbf{Z}^d;\ |z - x| = 1 \text{ for some } x \in A\}.$$

Since

$$\begin{aligned}
\mathbf{E}\left[|S_{k+1}|^2 \middle| X_1, ..., X_k\right] &= \sum_{j=1}^{d} \mathbf{E}\left[(X_{k+1}^j + S_k^j)^2 \middle| X_1, ..., X_k\right] \\
&= |S_k|^2 + 1 + 2\sum_{j=1}^{d} S_k^j \mathbf{E}\left[X_{k+1}^j \middle| X_1, ..., X_k\right] \\
&= |S_k|^2 + 1,
\end{aligned}$$

$|S_k|^2 - k$ is a martingale with respect to $\mathcal{F}_k = \sigma(X_1, ..., X_k)$, the σ-field generated by $X_1, ..., X_k$.

Proposition 3.1. *Let* $\tau_m = \inf\{j \geq 0;\ S_j \in \partial C_m\}$*. Then*

$$m^2 \leq \mathbf{E}\tau_m < (m+1)^2.$$

If $d = 1$*, then* $\mathbf{E}\tau_m = m^2$*.*

Proof. Since $|S_k|^2 - k$ is a martingale, so is $|S_{k\wedge\tau_m}|^2 - (k \wedge \tau_m)$ (see, for instance, [4]). Thus,

$$\mathbf{E}(k \wedge \tau_m) = \mathbf{E}|S_{k\wedge\tau_m}|^2.$$

Applying the monotone convergence theorem to the left side and the dominated convergence theorem to the right side above, $\mathbf{E}\tau_m = \mathbf{E}|S_{\tau_m}|^2$. Since $m^2 \leq |S_{\tau_m}|^2 < (m+1)^2$,

$$m^2 \leq \mathbf{E}\tau_m < (m+1)^2.$$

Note that when $d = 1$, $|S_{\tau_m}|^2 = m^2$ and the inequality on the left above is an equality. □

The main goal of this subsection is to use Chebyshev's inequality to obtain large deviations results for τ_m, defined in Proposition 3.1.

Lemma 3.2. *Suppose* $d = 1$*. If* $x > 0$ *is an integer, then*

$$\mathbf{P}^x\{\tau_m > a\} \leq \mathbf{P}^{x-1}\{\tau_m > a\}.$$

Proof. The proof is by induction on x. For $x = 1$,

$$\begin{aligned}\mathbf{P}^0\{\tau_m > a\} &= \frac{1}{2}\mathbf{P}^1\{\tau_m > a-1\} + \frac{1}{2}\mathbf{P}^{-1}\{\tau_m > a-1\} \\ &= \mathbf{P}^1\{\tau_m > a-1\} \\ &\geq \mathbf{P}^1\{\tau_m > a\}.\end{aligned}$$

For $x > 1$,

$$\begin{aligned}\mathbf{P}^{x-1}\{\tau_m > a\} &= \frac{1}{2}\mathbf{P}^x\{\tau_m > a-1\} + \frac{1}{2}\mathbf{P}^{x-2}\{\tau_m > a-1\} \\ &\geq \frac{1}{2}\mathbf{P}^x\{\tau_m > a-1\} + \frac{1}{2}\mathbf{P}^{x-1}\{\tau_m > a-1\} \\ &\geq \frac{1}{2}\mathbf{P}^x\{\tau_m > a\} + \frac{1}{2}\mathbf{P}^{x-1}\{\tau_m > a\},\end{aligned}$$

which leads to

$$\mathbf{P}^x\{\tau_m > a\} \leq \mathbf{P}^{x-1}\{\tau_m > a\}.$$

This completes the proof. □

Corollary 3.3. *Suppose $d = 1$. If $x \in \mathbf{Z}$, then*

$$\mathbf{P}^x\{\tau_m > a\} \leq \mathbf{P}\{\tau_m > a\}.$$

□

Assuming $d = 1$, Chebyshev's inequality and Proposition 3.1 imply

$$\mathbf{P}\{\tau_m > 2m^2\} \leq \frac{1}{2m^2}\mathbf{E}\tau_m = \frac{1}{2}.$$

If $a \geq 2$, then by Corollary 3.3,

$$\begin{aligned}\mathbf{P}\{\tau_m > am^2\} &= \sum_{z \in \mathbf{Z}} \mathbf{P}\{\tau_m > (a-2)m^2,\ S_{(a-2)m^2} = z\}\mathbf{P}^z\{\tau_m > 2m^2\} \\ &\leq \mathbf{P}\{\tau_m > (a-2)m^2\}\mathbf{P}\{\tau_m > 2m^2\} \\ &\leq \frac{1}{2}\mathbf{P}\{\tau_m > (a-2)m^2\} \\ &\vdots \\ &\leq 2^{-\lfloor a/2 \rfloor}. \end{aligned} \tag{3.1}$$

Proposition 3.4. *There is a constant $c > 0$ such that*

$$\mathbf{P}\{\tau_m > am^2\} \leq e^{-ca}$$

for all integers a.

Proof. The inequality (3.1) gives the result for $d = 1$ immediately. For $d > 1$, observe that if $S_k \in C_m$, then $S_k^j \in (-m, m)$, $j = 1, ..., d$. Moreover, $S_k^1 + \cdots + S_k^d$ is a one-dimensional SRW. Let $\tau_m' = \inf\{j \geq 0;\ S_j^1 + \cdots + S_j^d \notin (-m, m)\}$. By the result for $d = 1$,

$$\mathbf{P}\{\tau_m > am^2\} \leq \mathbf{P}\{\tau_{dm}' > am^2\} \leq e^{-ca/d^2}.$$

□

Proposition 3.4 bounds the probability that a random walk remains in a ball of radius m for am^2 steps. In [2], Lemma 1.5.1, Lawler gives an upper bound on the probability that a random walk leaves a ball of radius am in am^2 steps, *i.e.* a bound on $\mathbf{P}\{\tau_{am} \leq am^2\}$. The proof of this upper bound relies on the reflection principle ([2], Exercise 1.3.4) which is stated and proved below.

Proposition 3.5 (reflection principle)**.**

$$\mathbf{P}\{\tau_m \le k\} \le 2\mathbf{P}\{|S_k| \ge m\}.$$

Proof. By the Markov property,

$$\mathbf{P}\{\tau_m \le k,\ |S_a| < m\} = \sum_{j=0}^{k} \sum_{x \in \partial C_m} \mathbf{P}\{\tau_m = j, S_{\tau_m} = x\}\mathbf{P}\{|x + S_{k-j}| < m\}.$$

Essentially by convexity,

$$\mathbf{P}\{|x + S_{k-j}| < m\} \le \mathbf{P}\{|x + S_{k-j}| \ge m\}. \tag{3.2}$$

Indeed, suppose that $x, y \in \mathbf{R}^d$, $|x| \ge m$, and $|x + y| < m$. If $|x - y| < m$, then convexity of the open ball $\{z \in \mathbf{R}^d;\ |z| < m\}$ implies that $|x| < m$, a contradiction. So we must have $|x - y| \ge m$. Hence, $|x + S_{k-j}| < m$ implies $|x - S_{k-j}| \ge m$ and (3.2) follows.

Combining (3.1) and (3.2) gives

$$\mathbf{P}\{\tau_m \le k,\ |S_k| < m\} \le \mathbf{P}\{\tau_m \le k,\ |S_k| \ge m\}.$$

Thus,

$$\begin{aligned}
\mathbf{P}\{\tau_m \le k\} &= \mathbf{P}\{\tau_m \le k,\ |S_k| < m\} + \mathbf{P}\{\tau_m \le k,\ |S_k| \ge m\} \\
&\le 2\mathbf{P}\{\tau_m \le k,\ |S_k| \ge m\} \\
&\le 2\mathbf{P}\{|S_k| \ge m\}.
\end{aligned}$$

□

A straightforward application of Chebyshev's inequality now gives the bound mentioned above (details can be found in [2]).

Proposition 3.6. *There is a constant $c > 0$ such that*

$$\mathbf{P}\{\tau_{am} \le am^2\} \le 2\mathbf{P}\{|S_{am^2}| \ge am\} \le 2e^{-ca^{1/2}}.$$

□

3.2 Local results

This subsection contains two significant, well-known results about SRW: the local central limit theorem and asymptotics of the Green's function for SRW, $d \ge 3$. These and other related results can be found in [2], Chapter 1. Note that all arguments in this section are due to Lawler, [2].

The parity of a point $x \in \mathbf{Z}^d$ is defined to be the parity of $||x||_1$. For $x \in \mathbf{Z}^{d_1}$ and $y \in \mathbf{Z}^{d_2}$, we write $x \leftrightarrow y$ if x and y have the same parity. Notice that $\mathbf{P}\{S_k = x\} = 0$ if $x \not\leftrightarrow k$.

Theorem 3.7 (local central limit theorem)**.** *If $x \leftrightarrow k$, then*

$$\left| \mathbf{P}\{S_k = x\} - 2\left(\frac{d}{2\pi k}\right)^{d/2} e^{-\frac{d|x|^2}{2k}} \right| \le O(k^{-(d+2)/2}).$$

Heuristic. By the central limit theorem, if $A \subset \mathbf{R}^d$ is an open ball, then

$$\lim_{k\to\infty} \left| \mathbf{P}\left\{S_k/\sqrt{k} \in A\right\} - \int_A \left(\frac{d}{2\pi}\right)^{d/2} e^{-\frac{d|u|^2}{2}}\, du \right| = 0.$$

Since S_k takes value in $\mathbf{Z}^d$, if $S_k/\sqrt{k} \in A$, then, in fact, $S_k/\sqrt{k} \in A_k = A \cap k^{-1/2}\mathbf{Z}^d$. Observe that A_k contains roughly $k^{d/2}m(A)$ points where $m(A)$ is the Lebesgue measure of the open ball A. Moreover, since S_k can only occupy points with the same parity as k, and about half the points of $k^{1/2}A_k$ have the same parity as k, $S_k/\sqrt{k} \in A$ implies that $S_k/\sqrt{k}$ occupies one of roughly $\frac{1}{2}k^{d/2}m(A)$ possible points in A_k. If we assume that k is so large that $S_k/\sqrt{k}$ is close to evenly distributed over these points and if we take A to be small enough that $e^{-\frac{d|u|^2}{2}}$ is roughly constant over $u \in A$, then it is reasonable to suppose that

$$\mathbf{P}\left\{S_k/\sqrt{k} = x/\sqrt{k}\right\} \approx 2\left(\frac{d}{2\pi k}\right)^{d/2} e^{-\frac{d|x|^2}{2k}}$$

for $x/\sqrt{k} \in A_k$ and $x \leftrightarrow k$. □

The above heuristic is appealing, but proving the local central limit theorem requires considerably more work than the heuristic suggests. The proof involves a careful analysis of the characteristic function of S_k. A sketch of the proof is provided below.

Sketch proof of Theorem 3.7. For $\theta = (\theta^1, ..., \theta^d) \in \mathbf{R}^d$, define

$$\phi(\theta) = \mathbf{E}e^{iX_1 \cdot \theta} = \frac{1}{d}\sum_{j=1}^{d} \cos\theta^j,$$

the characteristic function of X_1. Then $\phi(\theta)^k$ is the characteristic function of S_k. Multiplying both sides of the equation

$$\sum_{z \in \mathbf{Z}^d} \mathbf{P}\{S_k = z\}e^{iz\cdot\theta} = \phi(\theta)^k$$

by $e^{-ix\cdot\theta}$ and integrating θ over $[-\pi,\pi]^d$ gives,

$$\mathbf{P}\{S_k = x\} = (2\pi)^{-d} \int_{[-\pi,\pi]^d} e^{-ix\cdot\theta}\phi(\theta)^k \, d\theta. \tag{3.3}$$

Note that if $\theta \in \{\pm(\pi,...,\pi),0\}$, then $|\phi(\theta)| = 1$ and if θ is away from the three points $\{\pm(\pi,...,\pi),0\}$, then $\phi(\theta)^k$ decreases exponentially in k. Hence, the main contribution in (3.3) comes from integrating near $\pm(\pi,...,\pi)$ and 0. By symmetry, we need only consider the integral near 0 (and not $\pm(\pi,...,\pi)$). Indeed, let $A = [-\pi/2,\pi/2]\times[-\pi,\pi]^{d-1}$. Since $k \leftrightarrow x$,

$$e^{-ix\cdot\theta}\phi(\theta)^k = e^{-ix\cdot(\theta+(\pi,...,\pi))}\phi(\theta+(\pi,...,\pi)).$$

Using this fact, one easily checks that

$$\int_{[-\pi,\pi]^d} e^{-ix\cdot\theta}\phi(\theta)^k \, d\theta = 2\int_A e^{-ix\cdot\theta}\phi(\theta)^k \, d\theta.$$

Choosing $r \in (0,\pi/2)$ judiciously, we find $\rho < 1$ such that

$$\begin{aligned}\int_A e^{-ix\cdot\theta}\phi(\theta)^k \, d\theta &= \int_{|\theta|\le r} e^{-ix\cdot\theta}\phi(\theta)^k \, d\theta + O(\rho^k) \\ &= k^{-d/2}\int_{|\alpha|\le r\sqrt{k}} \exp\left(-\frac{ix\cdot\alpha}{\sqrt{k}}\right)\phi(\alpha/\sqrt{k})^k \, d\alpha + O(\rho^k)\end{aligned}$$

where the second equality follows after making the substitution $\alpha = \sqrt{k}\theta$. Thus,

$$\mathbf{P}\{S_k = x\} = 2(2\pi\sqrt{k})^{-d}\int_{|\alpha|\le r\sqrt{k}} \exp\left(-\frac{ix\cdot\alpha}{\sqrt{k}}\right)\phi(\alpha/\sqrt{k})^k \, d\alpha + O(\rho^k). \tag{3.4}$$

The integral above can be decomposed as follows:

$$\int_{|\alpha|\le r\sqrt{k}} \exp\left(-\frac{ix\cdot\alpha}{\sqrt{k}}\right)\phi(\alpha/\sqrt{k})^k \, d\alpha = \sum_{j=0}^{3} I_j(k,x), \tag{3.5}$$

where

$$\begin{aligned} I_0(k,x) &= \int_{\mathbf{R}^d} \exp\left(-\frac{ix\cdot\alpha}{\sqrt{k}}\right)\exp\left(-\frac{|\alpha|^2}{2d}\right)\,d\alpha \;=\; (2\pi d)^{d/2}e^{-\frac{d|x|^2}{2k}}, \\ I_1(k,x) &= \int_{|\alpha|\le k^{1/4}} \left[\phi(\alpha/\sqrt{k})^k - \exp\left(-\frac{|\alpha|^2}{2d}\right)\right]\exp\left(-\frac{ix\cdot\alpha}{\sqrt{k}}\right)\,d\alpha, \\ I_2(k,x) &= -\int_{|\alpha|\ge k^{1/4}} \exp\left(-\frac{ix\cdot\alpha}{\sqrt{k}}\right)\exp\left(-\frac{|\alpha|^2}{2d}\right)\,d\alpha, \\ I_3(k,x) &= \int_{k^{1/4}\le|a|\le rk^{1/2}} \exp\left(-\frac{ix\cdot\alpha}{\sqrt{k}}\right)\phi(\alpha/\sqrt{k})^k\,d\alpha. \end{aligned}$$

Though it requires some care, it is straightforward to show that

$$I_1(k,x) = O(k^{-1})$$

and that there is a constant $c > 0$ such that $I_2(k,x), I_3(k,x) = O(\exp(-ck^{1/2}))$. Combining this with (3.4) and (3.5) gives the Theorem. □

When $d \ge 3$, SRW is transient and the Green's function

$$G(x,y) = \mathbf{E}^x\left[\sum_{k=0}^{\infty} I\{S_k = y\}\right] = \sum_{k=0}^{\infty}\mathbf{P}^x\{S_k = y\}$$

is finite. To simplify notation, we sometimes write $G(x) = G(0,x)$. The local central limit theorem leads to approximations for $G(x)$ and related quantities that will be useful in the sequel.

Theorem 3.8. *Suppose $d \ge 3$. For $x \in \mathbf{Z}^d$,*

$$G(x) \sim a_d|x|^{2-d}$$

where $a_d = \frac{d}{2}\Gamma\left(\frac{d}{2}-1\right)\pi^{-d/2} = \frac{2}{(d-2)\omega_d}$ and ω_d is the volume of the d-dimensional $||\cdot||_2$-unit ball.

Sketch proof. Assume $x \leftrightarrow 0$ and fix some small positive ϵ. Using Proposition 3.6 to approximate terms of $G(x)$ with $k \leq |x|^{2-\epsilon}$ and the local central limit theorem to approximate terms with $k > |x|^{2-\epsilon}$, one finds

$$G(x) = \sum_{k=0}^{\infty} \mathbf{P}\{S_{2k} = x\} \sim \sum_{k=0}^{\infty} 2\left(\frac{d}{4\pi k}\right)^{d/2} e^{-\frac{d|x|^2}{4k}}.$$

Hence, making the change of variables $v = \frac{d|x|^2}{4u}$ in the integral below,

$$G(x) \sim \sum_{k=0}^{\infty} 2\left(\frac{d}{4\pi k}\right)^{d/2} e^{-\frac{d|x|^2}{4k}} \sim \int_0^{\infty} 2\left(\frac{d}{4\pi u}\right)^{d/2} e^{-\frac{d|x|^2}{4u}}\, du = a_d|x|^{2-d}.$$

To handle the case $x \not\leftrightarrow 0$, first note that when $x \neq 0$,

$$G(x) = \sum_{k=0}^{\infty} \mathbf{P}\{S_k = x\} = \sum_{k=1}^{\infty} \frac{1}{2d} \sum_{|e|=1} \mathbf{P}\{S_{k-1} = x\} = \frac{1}{2d} \sum_{|e|=1} G(x+e)$$

(we say that G is harmonic on $\mathbf{Z}^d \setminus \{0\}$). If $x \not\leftrightarrow 0$, then harmonicity of G and the result for $x \leftrightarrow 0$ imply

$$G(x) = \frac{1}{2d} \sum_{|e|=1} G(x+e) \sim a_d|x|^{2-d}.$$

□

Proposition 3.9. *Suppose $d \geq 3$ and $x \in C_m$. Define $\tau_m^0 = \inf\{j \geq 0;\ S_j \in \partial C_m \cup \{0\}\}$. Then*

$$\mathbf{P}^x\{S_{\tau_m^0} = 0\} = \frac{G(x)}{G(0)} + O(m^{2-d}).$$

Proof. Let $\tau^0 = \inf\{j \geq 0;\ S_j = 0\}$. Using the fact that G is harmonic on $\mathbf{Z}^d \setminus \{0\}$,

$$\begin{aligned}
\mathbf{E}^x[G(S_{(k+1)\wedge\tau^0})|\mathcal{F}_k] &= I\{\tau^0 \leq k\}G(0) + \mathbf{E}^x[G(S_{k+1})I\{\tau^0 > k\}|\mathcal{F}_k] \\
&= I\{\tau^0 \leq k\}G(0) + I\{\tau^0 > k\}\mathbf{E}^{S_k}[G(S_1)] \\
&= I\{\tau^0 \leq k\}G(0) + I\{\tau^0 > k\}\frac{1}{2d}\sum_{|e|=1} G(S_k + e) \\
&= I\{\tau^0 \leq k\}G(0) + I\{\tau^0 > k\}G(S_k) \\
&= G(S_{k\wedge\tau^0}).
\end{aligned}$$

Thus, $G(S_{k\wedge\tau^0})$ is a bounded martingale and, consequently, $M_k = G(S_{k\wedge\tau^0_m})$ is also a bounded martingale.

Since $G(S_{k\wedge\tau^0_m})$ is a martingale, $G(x) = E^x[G(S_{k\wedge\tau^0_m})]$. Taking $k \to \infty$ gives

$$G(x) = \mathbf{E}^x[G(S_{\tau^0_m})] = G(0)\mathbf{P}^x\{S_{\tau^0_m} = 0\} + \mathbf{E}^x[G(S_{\tau^0_m})|S_{\tau^0_m} \in \partial C_m]\mathbf{P}^x\{S_{\tau^0_m} \in \partial C_m\}.$$

Rearranging above and using Theorem 3.8,

$$\mathbf{P}^x\{S_{\tau^0_m} = 0\} = \frac{G(x) - \mathbf{E}^x[G(S_{\tau^0_m})|S_{\tau^0_m} \in \partial C_m]}{G(0) - \mathbf{E}^x[G(S_{\tau^0_m})|S_{\tau^0_m} \in \partial C_m]} = \frac{G(x)}{G(0)} + O(m^{2-d}).$$

□

3.3 SRW in continuous time

In the previous subsection we saw that parity is an issue for SRW on $\mathbf{Z}^d$. To avoid this issue in the sequel, we consider simple random walks in continuous time.

Let $X^{1,\pm}(t), ..., X^{d,\pm}(t)$ be $2d$ independent rate $\lambda/2d$ Poisson processes. Let $e_j \in \mathbf{R}^d$ be the unit vector with a one in the j-th coordinate, $j = 1, ..., d$. Define the rate

λ continuous time SRW

$$S(t) = X(0) + \sum_{j=1}^{d} \left[X^{j,+}(t) - X^{j,-}(t) \right] e_j$$

where $X(0) = x \in \mathbf{Z}^d$. Note that $S(t)$ is a rate λ compound Poisson process.

Dealing with continuous time SRW allows us to sidestep some problems with parity, however we need to develop some tools that will allow us relate results for discrete time SRW to continuous time SRW. First recall the fact about Poisson processes:

Proposition 3.10. *Let $N(t)$ be a rate λ Poisson process. Define $\xi_0 = 0$ and for $j \geq 1$ define $\xi_j = \inf\{t \geq \xi_{j-1};\ N(t) \neq N(\xi_{j-1})\}$. Then $\xi_j - \xi_{j-1},\ j = 1, 2, ...$ are independent mean $1/\lambda$ exponential random variables ($\xi_j - \xi_{j-1}$ is called the j-th inter-arrival time).*

□

Our primary tool is a large deviations result for sums of independent exponential random variables.

Lemma 3.11. *Suppose that Z_j is a sum of $j \geq 1$ independent mean $1/\lambda$ exponential random variables and that $z > 0$. If $\lambda > 1$, then*

$$\mathbf{P}\{Z_j > z\} \leq \left(\frac{\lambda}{\lambda - 1} \right)^j e^{-z}.$$

If $\lambda > 0$, then

$$\mathbf{P}\{Z_j < z\} \leq \left(\frac{\lambda}{\lambda + 1} \right)^j e^{z}.$$

Proof. The random variable Z_j has a gamma distribution with density

$$f_j(t) = \frac{\lambda^j}{(j-1)!} t^{j-1} e^{-\lambda t}, \ t > 0.$$

By Chebyshev's inequality,

$$\mathbf{P}\{Z_j > z\} \leq e^{-z} \mathbf{E} e^{Z_j} = \left(\frac{\lambda}{\lambda - 1}\right)^j e^{-z}.$$

This is the first inequality. The second inequality is obtained similarly:

$$\mathbf{P}\{Z_j < z\} \leq e^{z} \mathbf{E} e^{-Z_j} = \left(\frac{\lambda}{\lambda + 1}\right)^j e^{z}.$$

□

Proposition 3.12. *Let Z_j be as in Lemma 3.11. Let $\epsilon > 0$. There is $c = c(\epsilon) > 0$ such that*

$$\mathbf{P}\{|Z_j - j/\lambda| > \epsilon j/\lambda\} \leq e^{-cj}.$$

Proof. Note that $\lambda Z_j/\mu$ is a sum of j independent mean $1/\mu$ random variables. From Lemma 3.11,

$$\mathbf{P}\{Z_j > (1+\epsilon)j/\lambda\} = \mathbf{P}\{\lambda Z_j/\mu > (1+\epsilon)j/\mu\} \leq \left(\frac{\mu}{\mu - 1}\right)^j e^{-(1+\epsilon)j/\mu}$$

and

$$\mathbf{P}\{Z_j < (1-\epsilon)j/\lambda\} = \mathbf{P}\{\lambda Z_j/\mu < (1-\epsilon)j/\mu\} \leq \left(\frac{\mu}{\mu + 1}\right)^j e^{(1-\epsilon)j/\mu}.$$

The proposition is a consequence of the fact that $(1+\epsilon)/\mu - \log \frac{\mu}{\mu-1} > 0$ and $(1-\epsilon)/\mu + \log \frac{\mu}{\mu+1} < 0$ for μ sufficiently large. □

4 Two SRWs

We consider two independent continuous time SRWs coloring points on the integer torus $\mathbf{Z}^d/n\mathbf{Z}^d$. To begin, let $R(t)$, $B(t)$ be independent rate λ continuous time simple random walks on $\mathbf{Z}^d$. Let $R^j(t)$, $B^j(t)$ denote the j-th coordinate of $R(t)$, $B(t)$, respectively, $j = 1, ..., d$. The superscripts in $\mathbf{P}^{r,b}$ and $\mathbf{E}^{r,b}$ indicate that $R(0) = r$, $B(0) = b$. Define $S(t) = (S^1(t), ..., S^d(t)) = R(t) - B(t)$ and notice that $S(t)$ is a rate 2λ continuous time SRW.

By considering the image of each walk $R(t)$, $B(t)$ in $\mathbf{Z}^d/n\mathbf{Z}^d$ we can view $R(t)$, $B(t)$ as simple random walks on $\mathbf{Z}^d/n\mathbf{Z}^d$. We imagine that $R(t)$ colors sites in $\mathbf{Z}^d/n\mathbf{Z}^d$ red, $B(t)$ colors sites in $\mathbf{Z}^d/n\mathbf{Z}^d$ blue, and, once colored, a site can not change colors. To make this more precise, for $x \in \mathbf{Z}^d$, define

$$\rho_x = \inf\{t \geq 0;\ R(t) \in [x]\} \text{ and } \beta_x = \inf\{t \geq 0;\ B(t) \in [x]\}.$$

(Recall that $[x] \in \mathbf{Z}^d/n\mathbf{Z}^d$ is the coset $x + n\mathbf{Z}^d$.) Let

$$\mathscr{R}(t) = \{[x] \in \mathbf{Z}^d/n\mathbf{Z}^d;\ \rho_x \leq \beta_x,\ \rho_x \leq t\}$$

be the collection of points in $\mathbf{Z}^d/n\mathbf{Z}^d$ that are red at time t and let

$$\mathscr{B}(t) = \{[x] \in \mathbf{Z}^d/n\mathbf{Z}^d;\ \beta_x \leq \rho_x,\ \beta_x \leq t\}$$

be the collection of points that are blue at time t. Set $\mathscr{R} = \mathscr{R}(\infty)$ and $\mathscr{B} = \mathscr{B}(\infty)$. Clearly, $\mathscr{R} \cup \mathscr{B} = \mathbf{Z}^d/n\mathbf{Z}^d$ and $\mathscr{R} \cap \mathscr{B} = \emptyset$ almost surely.

We are primarily interested in the random variable

$$\Delta = \#\mathscr{R} - \#\mathscr{B}.$$

Using symmetry, the expectation of Δ is easily computed:

$$\begin{aligned}
\mathbf{E}^{r,b}\Delta &= \sum_{[x]\in\mathbf{Z}^d/n\mathbf{Z}^d} \mathbf{P}^{r,b}\left\{\rho_x \le \beta_x\right\} - \mathbf{P}^{r,b}\left\{\beta_x \le \rho_x\right\} \\
&= \sum_{[x]\in\mathbf{Z}^d/n\mathbf{Z}^d} \mathbf{P}^{r,b}\left\{\rho_{-x+r+b} \le \beta_{-x+r+b}\right\} - \mathbf{P}^{r,b}\left\{\beta_x \le \rho_x\right\} \\
&= \sum_{[x]\in\mathbf{Z}^d/n\mathbf{Z}^d} \mathbf{P}^{0,0}\left\{\rho_{b-x} \le \beta_{r-x}\right\} - \mathbf{P}^{0,0}\left\{\beta_{x-b} \le \rho_{x-r}\right\} \\
&= 0.
\end{aligned}$$

Clearly, $\mathbf{E}^{r,b}\Delta^2 \le n^{2d}$. The main results are nontrivial bounds on the variance $\mathbf{E}^{r,s}\Delta^2$.

4.1 An upper bound

The goal of this section is to prove the following result.

Theorem 4.1. *Suppose $d \ge 3$. Then*

$$\mathbf{E}^{r,b}\Delta^2 = O(n^{d+2}).$$

Using the Markov property, proving Theorem 4.1 is reduced to determining which walk, $R(t)$ or $B(t)$, reaches a point $[x] \in \mathbf{Z}^d/n\mathbf{Z}^d$ first, given the initial positions $R(t) = r$ and $B(t) = b$. Indeed, expanding,

$$\begin{aligned}
\mathbf{E}^{r,b}\Delta^2 &= \sum_{[x],[y]\in\mathbf{Z}^d/n\mathbf{Z}^d} \mathbf{P}^{r,b}\{[x],[y] \text{ are the same color}\} \\
&\qquad - \mathbf{P}^{r,b}\{[x],[y] \text{ are different colors}\} \qquad (4.1)
\end{aligned}$$

where

$$\{[x],[y] \text{ are the same color}\} = \{[x],[y] \in \mathscr{R}\} \cup \{[x],[y] \in \mathscr{B}\}$$

and

$$\{[x],[y] \text{ are different colors}\} = \{[x] \in \mathscr{R}, [y] \in \mathscr{B}\} \cup \{[y] \in \mathscr{R}, [x] \in \mathscr{B}\}.$$

For $x, y \in \mathbf{Z}^d$ define $\eta_{x,y} = \rho_x \wedge \beta_x \wedge \rho_y \wedge \beta_y$ to be the first time x or y is colored. Then, by the Markov property,

$$\begin{aligned} &\mathbf{P}^{r,b}\{[x],[y] \text{ are the same color}|\eta_{x,y} = \rho_x\} \\ &\qquad = \mathbf{E}^{r,b}\left[\mathbf{P}^{x,B(\eta_{x,y})}\{\rho_y \le \beta_y\}\middle|\eta_{x,y} = \rho_x\right] \end{aligned} \tag{4.2}$$

and

$$\begin{aligned} &\mathbf{P}^{r,b}\{[x],[y] \text{ are different colors}|\eta_{x,y} = \rho_x\} \\ &\qquad = \mathbf{E}^{r,b}\left[\mathbf{P}^{x,B(\eta_{x,y})}\{\beta_y \le \rho_y\}\middle|\eta_{x,y} = \rho_x\right]. \end{aligned} \tag{4.3}$$

Similar equations hold when we condition on $\eta_{x,y} = \beta_x$, ρ_y, or β_y. By bounding terms like

$$\mathbf{P}^{r,b}\{\rho_x \le \beta_x\} - \mathbf{P}^{r,b}\{\beta_x \le \rho_x\} = \mathbf{P}^{r,b}\{[x] \text{ is red}\} - \mathbf{P}^{r,b}\{[x] \text{ is blue}\} \tag{4.4}$$

and using equations (4.2) and (4.3) we will obtain the desired bound on $\mathbf{E}^{r,b}\Delta^2$.

Define $\sigma^j = \inf\{t \ge 0;\ S^j(t) \in [0]\}$, $j = 1, ..., d$ and define $\sigma = \sigma^1 \vee \cdots \vee \sigma^2$. The following proposition makes precise the idea that if neither the red walk nor the blue walk hits the point x quickly, then the two SRWs are equally likely to hit x first. It is useful in bounding (4.4).

Proposition 4.2. *If* $r, b, x \in \mathbf{Z}^d$*, then*

$$\mathbf{P}^{r,b}\{\sigma \le \rho_x \le \beta_x\} = \mathbf{P}^{r,b}\{\sigma \le \beta_x \le \rho_x\}.$$

Proof. Define the processes $\check{R}(t) = (\check{R}^1(t), ..., \check{R}^d(t)), \check{B}(t) = (\check{B}^1(t), ..., \check{B}^d(t)) \in \mathbf{Z}^d$ by

$$\begin{aligned}\check{R}^j(t) &= R^j(t \wedge \sigma^j) + \left(B^j(t \vee \sigma^j) - B^j(\sigma^j)\right), \\ \check{B}^j(t) &= B^j(t \wedge \sigma^i) + \left(R^j(t \vee \sigma^j) - R^j(\sigma^j)\right),\end{aligned}$$

$j = 1, ..., d$. By comparing the inter-arrival times of the processes $R(t), B(t), \check{R}(t), \check{B}(t)$ one finds that $[\check{R}(t)]$ and $[\check{B}(t)]$ are independent compound Poisson processes with $([\check{R}(t)], [\check{B}(t)]) \stackrel{\mathscr{L}}{=} ([R(t)], [B(t)])$. Let $\check{\rho}_x, \check{\beta}_x, \check{\sigma}$ be stopping times for the processes $\check{R}(t), \check{B}(t)$ corresponding to ρ_x, β_x, σ. Then $\check{\sigma} = \sigma$ and

$$\{\check{\sigma} \leq \check{\rho}_x \leq \check{\beta}_x\} = \{\sigma \leq \beta_x \leq \rho_x\}.$$

It follows that

$$\mathbf{P}^{r,b}\{\sigma \leq \rho_x \leq \beta_x\} = \mathbf{P}^{r,b}\{\check{\sigma} \leq \check{\rho}_x \leq \check{b}_x\} = \mathbf{P}^{r,b}\{\sigma \leq \beta_x \leq \rho_x\}.$$

□

Applying Proposition 4.2 to (4.4) gives

$$\begin{aligned}\mathbf{P}^{r,b}\{\rho_x \leq \beta_x\} - \mathbf{P}^{r,b}\{\beta_x \leq \rho_x\} &= \mathbf{P}^{r,b}\{\rho_x \leq \beta_x,\ \rho_x < \sigma\} \\ &\quad -\mathbf{P}^{r,b}\{\beta_x \leq \rho_x,\ \beta_x < \sigma\} \\ &\leq \mathbf{P}^{r,b}\{\rho_x < \sigma\}. \qquad (4.5)\end{aligned}$$

We work towards bounding $\mathbf{P}^{r,b}\{\rho_x < \sigma\}$.

Lemma 4.3. *There is a constant $c > 0$ such that*

$$\mathbf{P}^{r,b}\{\sigma > kn^2\} = O(e^{-ck}).$$

Proof. Observe that

$$\mathbf{P}^{r,b}\{\sigma > kn^2\} \leq \sum_{j=1}^{d} \mathbf{P}^{r,b}\{\sigma^j > kn^2\}.$$

Thus, to prove the lemma it suffices to show that $\mathbf{P}^{x,y}\{\sigma^1 > kn^2\} = O(e^{-kj})$. We proved an analogous result for discrete time SRW in Proposition 3.4 and, after discretizing $S^1(t)$, we make use of it presently.

Define $\xi_0 = 0$ and, for $j = 1, 2...$ define $\xi_j = \inf\{t \geq \xi_{j-1};\ S^1(t) \neq S^1(\xi_{j-1})\}$. Let $S_j = S^1(\xi_j)$ and let $\tau_n = \inf\{j \geq 0;\ S_j \notin (-n, n)\}$. Then $S^1(t)$ is a rate λ/d compound Poisson process and S_j is a discrete time SRW. Furthermore,

$$\sigma^1 \leq \sum_{j=0}^{\tau_n} \xi_j.$$

By Propositions 3.4 and 3.12,

$$\begin{aligned} \mathbf{P}^{r,b}\{\sigma^1 > kn^2\} &\leq \mathbf{P}^{r,b}\left\{\sum_{j=0}^{\tau} \xi_j > kn^2\right\} \\ &\leq \mathbf{P}^{r,b}\left\{\sum_{j=0}^{\lfloor \lambda kn^2/(2d)\rfloor} \xi_j > kn^2\right\} + \mathbf{P}^{r,b}\{\tau > \lfloor \lambda kn^2/(2d)\rfloor\} \\ &= O(e^{-ck}) \end{aligned}$$

for some constant $c > 0$. □

Proposition 3.6 concerns discrete time SRW and $||\cdot||_2$-balls. We adapt this result to fit a slightly different situation: continuous time SRW and $||\cdot||_\infty$-balls. Define $B_m = \{z \in \mathbf{Z}^d;\ ||z||_\infty < m\}$ and $\nu_k = \inf\{t \geq 0;\ R(t) \in \partial B_{kn+n/2}\}$.

Lemma 4.4. *If $|r| \leq n/2$, then there is a constant $c > 0$ such that*

$$\mathbf{P}^{r,b}\{\nu_k \leq kn^2\} = O(\exp(-ck^{1/2})).$$

Proof. The stopping time ν_k is analogous to the discrete stopping time τ_m considered above and the present proof is similar to that of the previous lemma: we discretize $R(t)$ and apply Propositions 3.6 and 3.12. The only new ingredient is our reliance on the equivalence of the $||\cdot||_\infty$ and $||\cdot||_2$ norms in $\mathbf{R}^d$.

Take $c_0 > 0$, a constant such that $c_0||x||_\infty \leq ||x||_2$ for all $x \in \mathbf{Z}^d$. Let $\xi_0 = 0$ and, for $j = 1, 2, ...$ let $\xi_j = \inf\{t \geq \xi_{j-1};\ R(t) \neq R(\xi_{j-1})\}$. Let $R_j = R(\xi_j)$ and let $\tau_k = \inf\{j \geq 0;\ R_j \in \partial C_{c_0(kn+n/2)}\}$. Then, using Propositions 3.6 and 3.12,

$$\begin{aligned}
\mathbf{P}^{r,b}\{\nu_k \leq kn^2\} &\leq \mathbf{P}^{r,b}\left\{\sum_{j=0}^{\tau_k} \xi_j \leq kn^2\right\} \\
&\leq \mathbf{P}^{r,b}\left\{\sum_{j=0}^{\lfloor 2\lambda kn^2/d \rfloor} \xi_j \leq kn^2\right\} + \mathbf{P}^{r,b}\{\tau_k \leq 2\lambda kn^2/d\} \\
&= O(\exp(-ck^{1/2}))
\end{aligned}$$

for some constant $c > 0$. □

With the two previous lemmas in hand, we can bound $\mathbf{P}^{r,b}\{\rho_x < \sigma\}$.

Proposition 4.5. *Suppose $d \geq 3$. Let $r, b \in \mathbf{Z}^d$, $||r||_\infty \leq n/2$. Then*

$$\mathbf{P}^{r,b}\{\rho_0 < \sigma\} = \frac{G(r)}{G(0)} + O(n^{2-d}) = O(|r|^{2-d}). \tag{4.6}$$

Proof. We have

$$\mathbf{P}^{r,b}\{\rho_0 < \sigma\} = \mathbf{P}^{r,b}\{\rho_0 < \sigma,\ \rho_0 < \nu_0\} + \sum_{k=0}^{\infty} \mathbf{P}^{r,b}\{\rho_0 < \sigma,\ \nu_k \leq \rho_0 < \nu_{k+1}\}. \tag{4.7}$$

We estimate the terms on the right side above. Since $n\mathbf{Z}^d \cap B_{n/2} = \{0\}$, Proposition 3.9 gives

$$\mathbf{P}^{r,b}\{\rho_0 < \sigma,\ \rho_0 < \nu_0\} \leq \mathbf{P}^{r,b}\{\rho_0 < \nu_0\} \leq \frac{G(r)}{G(0)} + O(n^{2-d}). \tag{4.8}$$

Let $\rho_0^{(k)} = \inf\{t \geq 0;\ R(t) \in n\mathbf{Z}^d \cap B_{kn+n/2}\}$. To estimate the terms $\mathbf{P}^{r,b}\{\rho_0 < \sigma,\ \nu_k \leq \rho_0 < \nu_{k+1}\}$, first note that

$$\begin{aligned} \mathbf{P}^{r,b}\{\rho_0 < \sigma,\ \nu_k \leq \rho_0 < \nu_{k+1}\} &\leq \mathbf{P}^{r,b}\{\nu_k < \sigma,\ \nu_k \leq \rho_0 < \nu_{k+1}\} \\ &\leq \mathbf{P}^{r,b}\{\nu_k < \sigma,\ \nu_k \leq \rho_0^{(k+1)} < \infty\} \end{aligned} \tag{4.9}$$

and

$$\#(n\mathbf{Z}^d \cap B_{kn+n/2}) = O(k^d).$$

From the definition of ν_k,

$$|R(\nu_k) - nx| \geq n/4.$$

for all $x \in \mathbf{Z}^d$. By the Markov property and Theorem 3.8

$$\begin{aligned} \mathbf{P}^{r,b}\{\nu_k < \sigma,\ \nu_k \leq \rho_0^{(k+1)} < \infty\} &\leq \mathbf{P}^{r,b}\{\nu_k < \sigma\} \sum_{z \in n\mathbf{Z}^d \cap B_{(k+1)n+n/2}} G(R(\nu_k), z) \\ &= n^{2-d}\mathbf{P}^{r,b}\{\nu_k < \sigma\}O(k^d). \end{aligned} \tag{4.10}$$

Applying Lemmas 4.3 and 4.4,

$$\mathbf{P}^{r,b}\{\nu_k < \sigma\} \leq \mathbf{P}^{r,b}\{\nu_k \leq jn^2\} + \mathbf{P}^{r,b}\{kn^2 < \sigma^2\} = O(\exp(-ck^{1/2})).$$

Combining this with (4.9) and (4.10) gives

$$\begin{aligned} \mathbf{P}^{r,b}\{\rho_0 < \sigma,\ \nu_k \leq \rho_0 < \nu_{k+1}\} &\leq \mathbf{P}^{r,b}\{\nu_k < \sigma,\ \nu_k \leq \rho_0^{(k+1)} < \infty\} \\ &= n^{2-d}O(k^d \exp(-ck^{1/2})). \end{aligned}$$

Note we can sum over k above; (4.7) and (4.8) give the first equality in (4.6), the second equality follows from Theorem 3.8. $\square$

The previous proposition contains the bulk of Theorem 4.1. An elementary calculation is all that is left.

Proof of Theorem 4.1. Suppose that $x \neq y$ and $||x||_\infty, ||y||_\infty \leq n/2$. By (4.2), (4.3), (4.5), and Proposition 4.5,

$$\mathbf{P}^{r,b}\{[x],[y] \text{ are the same color}\} - \mathbf{P}^{r,b}\{[x],[y] \text{ are different colors}\} = O(|x-y|^{2-d}).$$

By equivalence of norms and (4.1),

$$\mathbf{E}^{r,b}\Delta^2 = O\left(\sum_{x,y\in B_{n/2},\ x\neq y} ||x-y||_\infty^{2-d}\right) + O(n^d).$$

To approximate the sum above, note that for fixed $y \in B_{n/2}$,

$$\sum_{x\in B_{n/2}\setminus\{y\}} ||x-y||_\infty^{2-d} \leq \sum_{x\in B_n\setminus\{0\}} ||x||_\infty^{2-d}.$$

Thus,

$$\begin{aligned}\sum_{x,y\in B_{n/2},\ x\neq y} ||x-y||_\infty^{2-d} &= O\left(n^d \sum_{x\in B_n\setminus\{0\}} ||x||_\infty^{2-d}\right)\\ &= O\left(n^d \sum_{k=1}^{n} j\right)\\ &= O(n^{d+2}).\end{aligned}$$

The proposition follows. $\square$

4.2 A lower bound

Suppose that rather than coloring points of $\mathbf{Z}^d/n\mathbf{Z}^d$ by random walks, we flip independent fair coins to determine the color of each point of $\mathbf{Z}^d/n\mathbf{Z}^d$. That is, suppose that a site $x \in \mathbf{Z}^d/n\mathbf{Z}^d$ is colored red with probability 1/2 and blue with probability 1/2, independent of all other sites. Then it is easily seen that the variance of the number of red sites minus the number of blue sites is n^d. Using Proposition 4.5 we will prove that the variance $\mathbf{E}^{r,b}\Delta^2$ is, up to a constant multiple, at least the variance of the number of red sites minus the number of blue sites when colors are determined by coin flips.

Theorem 4.6. *Suppose $d \geq 3$. There is a constant $c > 0$ such that*

$$\mathbf{E}^{rb}\Delta^2 \geq cn^d.$$

To prove Theorem 4.6 we consider instances where one of the two random walks neighbors an uncolored site. Let

$$\mathscr{U}(t) = (\mathbf{Z}^d/n\mathbf{Z}^d) \setminus (\mathscr{R}(t) \cup \mathscr{B}(t)),$$

be the collection of uncolored points at time t and define the following stopping times:

$$
\begin{aligned}
\eta_0 &= 0, \\
\xi_1^{(0)} &= \inf\{t \geq 0;\ S(t) \neq S(0)\}, \\
\xi_2^{(0)} &= \inf\{t \geq \xi_1^{(0)};\ S(t) \neq S(\xi_1^{(0)})\}, \\
&\vdots \\
\eta_j &= \inf\{t \geq \xi_2^{(j-1)};\ [R(t)] \in \partial\mathscr{U}(t) \text{ or } [B(t)] \in \partial\mathscr{U}(t)\}, \\
\xi_1^{(j)} &= \inf\{t \geq \eta_j;\ S(t) \neq S(\eta_j)\}, \\
\xi_2^{(j)} &= \inf\{t \geq \xi_1^{(j)};\ S(t) \neq S(\xi_1^{(j)})\} \\
&\vdots
\end{aligned}
$$

Define the event

$$A_j = \left\{R(\eta_j) = R(\xi_2^{(j)}) \text{ and } B(\eta_j) = B(\xi_2^{(j)})\right\} \cap \left\{\mathscr{U}(\eta_j) \neq \mathscr{U}(\xi_2^{(j)})\right\} \cap \{\eta_j < \infty\}$$

for $j = 0, 1, \ldots$ In words, η_j is a time when one of the walks $R(t)$ or $B(t)$ neighbor an uncolored site and A_j is the event that in the two steps immediately following η_j, one of the walks jumps to an uncolored site (thus, coloring the site) and then jumps back to its previous position. Also define the martingale

$$M(t) = \mathbf{E}^{r,b}[\Delta|\mathcal{F}_t]$$

where $\mathcal{F}_t = \sigma(R(t), B(t); 0 \leq s \leq t)$.

Observe that if $j \leq n^d/2$, then $\eta_j < \infty$ and

$$\mathbf{P}^{r,b}\{A_j\} \geq \frac{1}{16d^2}. \tag{4.11}$$

By (4.5) and Proposition 4.5, when $d \geq 3$ and $j \leq n^d/2$,

$$\begin{aligned}\left[M(\xi_2^{(j)}) - M(\eta_j)\right]^2 &\geq \left[M(\xi_2^{(j)}) - M(\eta_j)\right]^2 I\{A_j\} \\ &= \left(1 - \frac{G(e)}{G(0)} + O(n^{2-d})\right)^2 I\{A_j\} \qquad (4.12)\end{aligned}$$

where $e \in \mathbf{Z}^d$, $|e| = 1$. (In fact, both (4.11) and (4.12) hold whenever $\eta_j < \infty$.) The inequalities (4.11) and (4.12) are key to proving Theorem 4.6. The other main ingredient is the following lemma.

Lemma 4.7.

$$\mathbf{E}^{r,b}\Delta^2 \geq \mathbf{E}^{r,b}\left[\sum_{j=0}^{\infty}\left(M(\xi_2^{(j)}) - M(\eta_j)\right)^2\right].$$

Proof. The Lemma is a consequence of the optional sampling theorem ([4], Theorem 77.5), however, some care is required to justify the interchange of limits along the way.

By Jensen's inequality,

$$\mathbf{E}^{r,b}M(t)^2 \leq \mathbf{E}^{r,b}\Delta^2 \leq n^{d+2}.$$

Hence, standard martingale convergence theorems imply that

$$\lim_{t\to\infty} M(t) = \Delta \text{ a.s. and in } L^2$$

and, consequently,

$$\mathbf{E}^{r,b}\Delta^2 = \lim_{t\to\infty}\mathbf{E}^{r,b}M(t)^2. \qquad (4.13)$$

The next step is to look more closely at $M(t)$ and $\mathbf{E}^{r,b}M(t)^2$. For $j \geq 0$, define

$$D_j(t) = M(t \wedge \eta_{j+1}) - M(t \wedge \eta_j).$$

Then

$$M(t) = \sum_{j=0}^{\infty} D_j(t)$$

and

$$\mathbf{E}^{r,b} M(t)^2 = \mathbf{E}^{r,b} \left[\sum_{j=0}^{\infty} D_j(t) \right]^2 = \mathbf{E}^{r,b} \left[\sum_{j,k=0}^{\infty} D_j(t) D_k(t) \right]. \tag{4.14}$$

By Lemma 3.11, there is $\rho \in (0, 1)$ such that

$$\mathbf{P}^{r,b}\{D_j(t) \neq 0\} \leq \mathbf{P}^{r,b}\{\eta_{j+1} \leq t\} \leq e^t O(\rho^j)$$

Since $|D_j(t)| \leq 2n^d$ and $\sum_{j,k=0}^{\infty} \rho^{j+k} < \infty$, we can apply the dominated convergence theorem on the right side of (4.14) and get

$$\mathbf{E}^{r,b} M(t)^2 = \sum_{j,k=0}^{\infty} \mathbf{E}^{r,b} \left[D_j(t) D_k(t) \right]. \tag{4.15}$$

By the optional sampling theorem, if $j < k$, then

$$\mathbf{E}^{r,b} \left[D_j(t) D_k(t) \right] = \mathbf{E}^{r,b} \left\{ D_j(t) \mathbf{E}^{r,b} [D_k(t) | \mathcal{F}_{\eta_k}] \right\} = 0.$$

Again by the optional sampling theorem,

$$\mathbf{E}^{r,b} \left[\left(M(t \wedge \eta_{j+1}) - M(t \wedge \xi_2^{(j)}) \right) \left(M(t \wedge \xi_2^{(j)}) - M(t \wedge \eta_j) \right) \right] = 0,$$

hence,

$$\mathbf{E}^{r,b} D_j(t)^2 = \mathbf{E}^{r,b} \left[M(t \wedge \eta_{j+1}) - M(t \wedge \xi_2^{(j)}) \right]^2 + \mathbf{E}^{r,b} \left[M(t \wedge \xi_2^{(j)}) - M(t \wedge \eta_j) \right]^2.$$

Now using (4.13), (4.15) and Fatou's lemma,

$$\begin{aligned} \mathbf{E}^{r,b}\Delta^2 &= \lim_{t\to\infty}\sum_{j=0}^{\infty}\mathbf{E}^{r,b}\left[M(t\wedge\eta_{j+1})-M(t\wedge\xi_2^{(j)})\right]^2 \\ &\qquad\qquad +\mathbf{E}^{r,b}\left[M(t\wedge\xi_2^{(j)})-M(t\wedge\eta_j)\right]^2 \\ &\geq \sum_{j=0}^{\infty}\mathbf{E}^{r,b}\left[M(\eta_{j+1})-M(\xi_2^{(j)})\right]^2+\mathbf{E}^{r,b}\left[M(\xi_2^{(j)})-M(\eta_j)\right]^2 \\ &\geq \sum_{j=0}^{\infty}\mathbf{E}^{r,b}\left[M(\xi_2^{(j)})-M(\eta_j)\right]^2 \\ &= \mathbf{E}^{r,b}\left[\sum_{j=0}^{\infty}\left(M(\xi_2^{(j)})-M(\eta_j)\right)^2\right]. \end{aligned}$$

□

Proof of Theorem 4.6. We bound

$$\mathbf{E}^{r,b}\left[\sum_{j=0}^{\infty}\left(M(\xi_2^{(j)})-M(\eta_j)\right)^2\right]$$

from below. By (4.11) and (4.12),

$$\begin{aligned} \mathbf{E}^{r,b}\left[\sum_{j=0}^{\infty}\left(M(\xi_2^{(j)})-M(\eta_j)\right)^2\right] &\geq \sum_{0\leq j\leq n^d/2}\mathbf{E}^{r,b}\left[M(\xi_2^{(j)})-M(\eta_j)\right]^2 \\ &\geq \sum_{0\leq j\leq n^d/2}\left(1-\frac{G(e)}{G(0)}+O(n^{2-d})\right)^2\mathbf{P}^{r,b}\{A_j\} \\ &\geq \left(1-\frac{G(e)}{G(0)}+O(n^{2-d})\right)^2\frac{n^d}{32d^2} \\ &\geq cn^d \end{aligned}$$

for some constant $c>0$. The Theorem follows from Lemma 4.7. □

5 Discussion

5.1 $d = 2$

When $d = 2$, simple random walk on $\mathbf{Z}^d$ is recurrent, hence the Green's function is infinite, and, at a glance, the proofs leading to an upper bound on the variance of the number of red points minus the number of blue points on $\mathbf{Z}^d/n\mathbf{Z}^d$ do not work. However, examining the proof of Theorem 4.1, it's clear that we need not make reference to the Green's function and the proof goes through provided that we understand asymptotics of $\mathbf{P}^x\{S_{\tau_m^0} = 0\}$ (Proposition 3.9). Lawler, [2] Proposition 1.6.7 is a relevant result:

Proposition. *Suppose $d = 2$ and $x \in C_m$. Then*

$$\mathbf{P}^x\{S_{\tau_m^0} = 0\} = \frac{\log n - \log|x|}{\log n} + O((\log n)^{-1}).$$

Using the above proposition in place of Proposition 3.9, we can essentially mimic the proof of Theorem 4.1 to get, for $d = 2$,

$$\mathbf{E}^{r,b}\Delta^2 = O(n^4/(\log n)).$$

5.2 The conjectured rate

When $d \geq 3$, Theorems 4.1 and 4.6 imply that there are positive constants c and C such that

$$cn^d \leq \mathbf{E}^{r,b}\Delta^2 \leq Cn^{d+2}.$$

It has been conjectured in personal communications between Pemantle, Peres, and Revelle that

$$\mathbf{E}^{r,b}\Delta^2 \asymp \begin{cases} n^4/(\log n) & \text{if } d = 2, \\ n^4 & \text{if } d = 3, \\ n^4 \log n & \text{if } d = 4, \\ n^d & \text{if } d \geq 5. \end{cases}$$

If we consider the process of coloring the torus in blocks of n^2 time steps, then sites colored in different blocks are roughly independent. It is thought that considering the variance arising from sites colored within each of these time blocks will lead to the conjectured rates.

References

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