SHARP THRESHOLDS FOR THE FROG MODEL AND RELATED SYSTEMS

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ABSTRACT

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The frog model refers to a system of interacting random walks on a rooted graph. It begins with a single active particle (i.e. frog) at the root, and some distribution of inactive particles among the non-root vertices. Active particles perform discrete-time-nearest neighbor random walks on the graph and activate passive particles upon landing on them. Once activated, the trajectories of distinct particles are independent. In this thesis, we examine the frog model in several different environments, and in each case, work towards identifying conditions under which the model is recurrent, transient, or neither, in terms of the number of distinct frogs that return to the root. We begin by looking at a continuous analog of the model on $\mathbb{R}$ in chapter 2, following which I analyze several different models on $\mathbb{Z}$ in chapters 2 and 3. I then conclude by examining the frog model on trees in chapter 4. The strategy used for analyzing the model on $\mathbb{R}$ primarily revolves around looking at a closely related birth-death process. Somewhat similar techniques are then used for the model on $\mathbb{Z}$. For the frog model on trees I exploit some of the self-similarity properties of the model in order to construct an operator which is used to analyze its long term behaviour, as it relates to questions of recurrence vs. transience.

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Chapter 1

Introduction

1.1 Definitions and background on the frog model

The topic of this thesis is a derivative of activated random walk known as the frog model. Since its inception, the term frog model has referred to a system of interacting random walks on a rooted graph. Specifically, it begins with one active "frog" (i.e. particle) at the root and sleeping frogs distributed among the non-root vertices, where the number of sleeping frogs at each non-root vertex are independent random variables (not necessarily identically distributed). The active frog performs a discrete-time nearest neighbor random walk on the graph. Any time an active frog lands on a vertex containing a sleeping frog, the sleeping frog wakes up and begins performing its own discrete-time nearest neighbor random walk independent of those of the other active frogs.

Much of the existing research that has been done involving the frog model on various infinite rooted graphs has focused on distinguishing conditions under which the model is recurrent (meaning that infinitely many particles return to the root with probability 1) from conditions under which it is transient. In the first published result to address this topic [14], Telcs and Wormald showed that for every $d \geq 1$ the frog model on $\mathbb{Z}^d$ beginning with one sleeping frog per non-root vertex (and where active frogs perform simple random walks) is recurrent. More recent variations of the frog model that have been studied have included the frog model with drift on $\mathbb{Z}$ [5, 2], the frog model on trees [6, 7], and continuous analogs of the frog model on $\mathbb{R}^d$ [1]. In each of these cases as well, primary emphasis has been placed on analyzing the long term behavior of the model as it pertains to questions of recurrence vs. transience, and in some cases, on identifying the critical threshold values of certain parameters where the model transitions between these two states.

One way the frog model can be interpreted is as an activated random walk model in which the particle deactivation rate is 0. Interest in activated random walk models has appeared to grow in recent years in reaction to progress on a number of different models, several of which are featured in recent work by Leonardo T. Rolla [9]. Applications for these models can be found in both the biological and social sciences, where they have been used to model the spread of an infectious disease, and the propagation of information. They've also been used as conservative lattice gas models, and in a variety of contexts in physics literature (see [3]).

The fact that the frog model differs from all other activated random walk models in that active particles never deactivate, endows it with its own qualitatively distinct dynamics. In particular, the frog model stands out within this larger family of models in that time functions as a dummy variable. By this I mean that when it comes to questions of recurrence and transience for the frog model, all of the relevant information relates to the paths traversed by activated particles, rather than the relative times at which the movement of different particles occurs. While this distinctive feature of the frog model can present its own set of challenges, it also allows for a certain amount of freedom and flexibility compared to other activated random walk models. In fact it is this very freedom to permute, or simply diregard the order of events, that is behind many of the techniques that are used in the analysis of the frog model in both this work, as well as the wider literature.

1.2 Intro: Part II

1.2.1 The Gantert and Schmidt model

The results in this thesis come from three papers written by the author [10, 11, 12]. Each of these papers examines the frog model in a distinct context, and each one forms the basis for one of the chapters 2-4. In chapter 2, which is based on the results in [10], two related models are analyzed. The first is a continuous analog of the frog model situated on the real line, and the second is a discretized counterpart to this model situated on $\mathbb{Z}$. The inspiration for looking at each of these models came from [5], which addresses a particular version of the frog model in which activated frogs perform random walks with some positive leftward drift on the integers, and the numbers of sleeping frogs at each vertex (aside from the origin which begins with a single activated frog) are i.i.d. random variables. In [5] Nina Gantert and Philipp Schmidt establish tight conditions on the distribution function for the number of sleeping frogs per vertex, that determine whether the system is recurrent or transient. Specifically, they prove that if η refers to a nonnegative integer valued random variable that has the same distribution as the number of sleeping frogs at any nonzero vertex, then

$$\mathbb{P}_\eta(\text{the origin is visited i.o.}) = \begin{cases} 0 & \text{if } \mathbb{E}[\log^+ \eta] < \infty \\ 1 & \text{if } \mathbb{E}[\log^+ \eta] = \infty \end{cases} \tag{1.2.1}$$

(Note that this result does not depend on the particular value of the leftward drift. It is only required that the leftward drift be positive).

1.2.2 The continuous model on $\mathbb{R}$

The continuous frog model analog featured in chapter 2 starts with a single active frog at the origin on the real line that begins performing Brownian motion with leftward drift $\lambda > 0$. The sleeping frogs all reside to the right of the origin according to a Poisson process with intensity $f : [0, \infty) \to [0, \infty)$. Any time an active frog hits a sleeping frog, the sleeping frog wakes up and also begins performing Brownian motion with leftward drift λ, independent of that of the other active frogs. The main result presented concerning this model establishes sharp conditions involving the Poisson intensity function f and drift λ, distinguishing between transience (meaning the probability that the origin is hit by infinitely many different frogs is 0), and non-transience (meaning this probability is greater than 0). Specifically, it is shown that the model is transient if and only if

$$\int_0^\infty e^{-\frac{f(t)}{2\lambda}} f(t)dt = \infty \tag{1.2.2}$$

(where it's assumed $\lambda > 0$ and f is monotonically increasing). The proof of this result involves using the model to construct a continuous-time-inhomogeneous birth-death process, and then showing that transience of the model on $\mathbb{R}$ corresponds to this process having survival probability 0. Once the problem has been translated into one about birth-death processes, we use a coupling argument and some analysis to achieve the desired result.

1.2.3 The non-uniform model on $\mathbb{Z}$

Following a short discussion about some of the implications of this last result, in which several examples of near-critical cases are provided, the focus shifts to a discretized version of the model on $\mathbb{R}$, which I call the *non-uniform frog model with drift on* $\mathbb{Z}$. In this model, sleeping frogs are distributed among the positive integer vertices, where for each $j \geq 1$ the number of sleeping frogs at $x = j$ at time $t = 0$ is denoted as η_j. The η_j's are to be independent Poisson random variables with $\mathbb{E}[\eta_j] = f(j)$ for some function $f : \mathbb{Z}^+ \to [0, \infty)$. The process begins with a single active frog at the origin and, once activated, frogs perform random walks (independently of each other) which at each step move one unit to the left with probability p (where $\frac{1}{2} < p < 1$) and one unit to the right with probability $1 - p$. The main result presented concerning this model states that it is transient if and only if

$$\sum_{j=1}^{\infty} e^{-\frac{1-p}{2p-1} f(j)} = \infty \tag{1.2.3}$$

(where f is assumed to be monotonically increasing, and the terms "transience" and "non-transience" are to have the same meaning with respect to this model that they had for the model on $\mathbb{R}$). We prove this result by first using the model to construct a discrete-time-inhomogeneous Markov process, after which it is shown that transience of the model on $\mathbb{Z}$ is equivalent to this Markov process eventually hitting the absorbing state 0 with probability 1. Showing that this last outcome occurs if and only if (1.2.3) holds is then achieved via a coupling argument similar to the one employed for the continuous-time birth-death process referenced above.

After completing the proof for the non-uniform frog model (along with an accompanying lemma), I conclude the chapter with a short subsection in which I present a pair of counterexamples where f is not monotone increasing, and the tight conditions for both the model on $\mathbb{R}$ and the non-uniform model on $\mathbb{Z}$ cease to apply.

1.3 Intro: Part III

1.3.1 The nonhomogeneous model on $\mathbb{Z}$

Chapter 3 of this thesis, which is based on the results in [12], begins by addressing a more general frog model on the integers which I refer to as the nonhomogeneous frog model on $\mathbb{Z}$. This model encompasses both the non-uniform frog model on $\mathbb{Z}$ and the model looked at by Gantert and Schmidt in [5], as well as a third model on $\mathbb{Z}$ which appears in [2] that features a single sleeping frog at every positive integer point $x = n$ who, upon activation, perform random walks that go left with probability $p_n > \frac{1}{2}$ (note this value depends on n) and right with probability $1 - p_n$. In the nonhomogeneous frog model on $\mathbb{Z}$ points to the left of the origin contain no sleeping frogs and, for $j \geq 1$, the number of sleeping frogs at $x = j$ is a random variable X_j, where the X_j's are independent, non-zero with positive probability, and where $X_{j+1} \succeq X_j$ (here " $\succeq$ " represents stochastic dominance). In addition, for each $j \geq 1$ frogs originating at $x = j$ (if activated) go left with probability p_j (where $\frac{1}{2} < p_j < 1$) and right with probability $1 - p_j$, where the p_j's are decreasing and the random walks are independent. The frog beginning at the origin goes left with probability p_0 and right with probability $1 - p_0$ (where p_0 also satisfies $\frac{1}{2} < p_0 < 1$).

The main result that I establish for the nonhomogeneous frog model on $\mathbb{Z}$ extends both the result by Gantert and Schmidt (see (1.2.1)) and my own result discussed above involving the non-uniform frog model with drift on $\mathbb{Z}$. It also builds on a result by Bertacchi, Machado, and Zucca which established that the model from [2] described in the previous paragraph is non-transient if there exists some increasing sequence of positive integers $\{n_k\}_{k\in\mathbb{N}}$ such that

$$\sum_{k=0}^{\infty}\prod_{i=0}^{n_k}\Big(1-\Big(\frac{1-p_i}{p_i}\Big)^{n_{k+1}-i}\Big)<\infty \tag{1.3.1}$$

(note this condition is proven to be sufficient, rather than sufficient *and* necessary). The result that I prove, which is a sharp condition distinguishing between transience and non-transience for the nonhomogeneous frog model on $\mathbb{Z}$, states that the model is transient if and only if

$$\sum_{n=2}^{\infty}\prod_{j=1}^{n-1}f_j\Big(1-\Big(\frac{1-p_j}{p_j}\Big)^{n-j}\Big)=\infty \tag{1.3.2}$$

(where f_j represents the probability generating function of X_j, the number of sleeping frogs at $x=j$). The proof of this result incorporates many of the same concepts that are used in the proof of the sharp condition for the non-uniform model on $\mathbb{Z}$. However, significant adjustments have to be made, particularly on account of the non-constant leftward drift involved in the nonhomogeneous case. Further discussion of these details will be left for chapter 3.

1.3.2 Applications of the nonhomogeneous model

After establishing the above result for the nonhomogeneous frog model on $\mathbb{Z}$, the focus will shift towards showing how it can be applied in a number of more specific cases. The first application of the theorem will involve the Gantert and Schmidt model from [5], and will entail showing how (1.2.1) can be achieved quite easily using the formula (3.1.1). After this (3.1.1) is used to obtain a formula that provides a sharp condition distinguishing between transience and non-transience in the case where the X_j's are i.i.d. and which, for the particular case where $X_j = 1$, builds on the result from [2] by giving a sharp result that supersedes the soft condition in (1.3.1) and, for the case where $p_j = \frac{1}{2} + \frac{C}{\log j}$ (for all but finitely many j), implies the existence of a phase transition at $C = \frac{\pi^2}{24}$. Finally, (3.1.1) will also be employed to obtain a formula that builds on the result involving the non-uniform frog model by generalizing to cases where the p_j's are not constant. For these last two results, the proofs will require some light assumptions relating to the concavity of the sequences $\left\{p_j^{-1}\right\}$ and $\{\lambda_j\}$ (where λ_j represents the Poisson mean of the distribution of X_j in the non-uniform model).

1.4 Intro: Part IV

In the final chapter of this thesis, which is based on the work in [11], I move past looking at the frog model in 1-dimensional environments, focusing instead on the frog model on trees. Specifically, I establish recurrence of the frog model on an infinite tree for which vertices on even levels have three children and vertices on odd levels have two (this is referred to as the 3,2-alternating tree), and where non-root vertices each begin with a single sleeping frog. The inspiration for looking at this model came from [7], in which Hoffman, Johnson, and Junge proved that the same model, when taken on the regular n-ary tree, is recurrent for $n = 2$ and transient for $n \geq 5$. The cases of the 3-ary and 4-ary trees, which remain open, were conjectured in their paper to be recurrent and transient respectively.

The proof of recurrence on $\mathbb{T}_{3,2}$ (the 3,2-alternating tree) largely involves adapting the methods that were employed by Hoffman, Johnson, and Junge in [7] to prove recurrence on $\mathbb{T}_2$, to the demands of this more complex model. The approach they pioneered entails embedding another model with nice self-similarity properties inside the standard model.

This self-similar model is then shown to be recurrent via a bootstrapping argument which is applied to it's generating function (for the number of distinct frogs that hit the root). Since this model is dominated by the original, this in turn implies recurrence of the original model on $\mathbb{T}_2$ (or $\mathbb{T}_{3,2}$ in the case of my result).

Chapter 2

A continuous frog model analog

2.1 The frog model with drift on $\mathbb{R}$

As noted in the introduction, the frog model with drift on $\mathbb{R}$ features sleeping frogs situated to the right of the origin according to a Poisson process with intensity $f : [0, \infty) \to [0, \infty)$. We begin with a single active frog at the origin that performs Brownian motion with leftward drift $\lambda > 0$. Whenever a sleeping frog is hit by an active frog, it becomes active and also begins performing Brownian motion with leftward drift λ, independent of that of the other active frogs (see Figure 2.1 for an illustration). A more formal construction of this process is not needed, as most of the analysis involves a related and easily constructed birth-death process. In this section we establish sharp conditions distinguishing between transience (meaning the probability that the origin is hit by infinitely many distinct frogs is 0), and non-transience (meaning this probability is greater than 0).

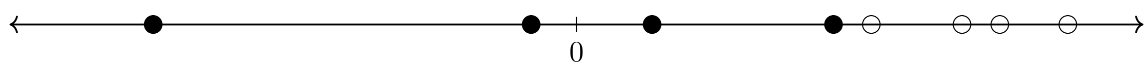

Figure 2.1: The model on $\mathbb{R}$, with black circles as active frogs and white ones sleeping frogs.

The main result for the frog model with drift on $\mathbb{R}$ will be the following theorem. It is assumed here, as well as in the discrete case, that f is not the zero function.

Theorem 2.1.1. *For any $\lambda > 0$ and f monotonically increasing, the frog model with drift on $\mathbb{R}$ is transient if and only if*

$$\int_0^\infty e^{-\frac{f(t)}{2\lambda}} f(t)dt = \infty. \tag{2.1.1}$$

Remark 1. Note that the decision to restrict our focus to the case where no sleeping frogs reside to the left of the origin was not made in order to simplify the problem. In fact, if the domain of f is expanded to $(-\infty, \infty)$ and it is allowed to take positive values to the left of the origin, then provided

$$\int_{-\infty}^0 e^{2\lambda t} f(t)dt < \infty \ ,$$

the transience/non-transience of the model depends on the same integral condition from Theorem 2.1.1. This follows from the theorem, along with the fact that $\mathbb{E}[L_{(-\infty,0)}]$ (where $L_{(a,b)}$ denotes the number of distinct frogs originating in (a, b) that hit the origin) is equal to the above integral. Alternatively, because $L_{(a,b)}$ (for $b \leq 0$) has a Poisson distribution with mean $\int_a^b e^{2\lambda t} f(t)dt$, divergence of the improper integral above will imply that $L_{(-\infty,0)}$ dominates a Poisson r.v. of any finite mean, thus implying recurrence of the model.

The proof of Theorem 2.1.1 proceeds as follows. It is first noted that by virtue of a simple rescaling, it suffices to prove the theorem for the specific case $\lambda = \frac{1}{2}$. A continuous-time-inhomogeneous birth-death process $\{X_t\}$ is then defined with birth rate $f(t)1_{(X_t>0)}$, and death rate X_t. Transience for the frog model with drift $\frac{1}{2}$ and Poisson intensity of sleeping frogs f, is shown to coincide with X_t eventually arriving at the absorbing state 0 with probability 1. A related process $\{Y_t\}$ is then defined, which is identical to $\{X_t\}$ except that 0 is not an absorbing state (i.e. $\{Y_t\}$ has birth rate $f(t)$ and death rate Y_t). The primary task in proving that (2.1.1) corresponds to transience of the model consists of proving that $\{Y_t\}$ will jump from 0 to 1 infinitely often with probability 1, if and only if (2.1.1) holds. To achieve this, it is first shown that as $t \to \infty$ the distribution of Y_t behaves increasingly like that of a Poisson r.v. with mean $\int_0^t e^{-(t-u)} f(u)du$. This is then used to show that the expected number of jumps from 0 to 1 (made by $\{Y_t\}$) is infinite if and only if the integral expression in (2.1.1) diverges. Together with a proof that this quantity is infinite with probability 1 as long as it has infinite expectation, this is sufficient for establishing that the model is transient if and only if (2.1.1) holds.

2.1.1 The process $\{X_t\}$

We start by setting $\lambda = \frac{1}{2}$ (note by virtue of a simple rescaling, it suffices to prove the theorem for a single value of $\lambda > 0$). For any $t \geq 0$ let X_t denote the number of frogs originating in $[0, t]$ that ever pass the point $x = t$. Now note that $\lim_{t\to\infty} X_t > 0$ if and only if (i) the set of points (initially) containing sleeping frogs is unbounded (f being nonnegative and monotonically increasing implies it is bounded on compact sets, which guarantees that no bounded region will contain infinitely many sleeping frogs) and (ii) all of these frogs are eventually awakened. Since a Brownian motion with leftward drift $\frac{1}{2}$ is continuous and goes to $-\infty$ with probability 1, it follows that

$$\lim_{t\to\infty} X_t > 0 \Longleftrightarrow \{\text{infinitely many frogs return to the origin}\} \tag{2.1.2}$$

Having established this equivalence, we now present the following proposition.

Proposition 2.1.2. $\{X_t\}$ *is a continuous-time-inhomogeneous birth-death process with birth rate* $f(t)1_{(X_t>0)}$ *and death rate* X_t*.*

Proof. By a straight forward argument involving an exponential martingale it is known that the right most point reached by a Brownian motion (beginning at the origin) with leftward drift $\frac{1}{2}$, has an exponential distribution with mean 1. By the strong Markov property it follows that if $0 < a < b$, then the probability that a frog originating in $[0, a]$ ever passes $x = b$ (conditioned on its passing $x = a$) equals $e^{-(b-a)}$. Therefore, if we let $R_{(t,dt)}$ represent the number of frogs originating in $[0, t)$ that reach $x = t$, but not $x = t+dt$, then $\left(R_{(t,dt)}|X_t\right)$ has distribution $\text{Bin}(X_t, 1-e^{-dt})$. Since $1-e^{-dt} = dt+o(dt)$ as $dt \to 0$, it then follows that $\{X_t\}$ has "death rate" X_t. Furthermore, since the number of sleeping frogs in $(t, t+dt)$ has distribution $\text{Poiss}(\lambda_{(t,dt)})$ (where $\lambda_{(t,dt)} = \int_t^{t+dt} f(u)du$), and the probability all such frogs are awoken and reach $x = t + dt$, approaches 1 as $dt \to 0$ (provided $X_t > 0$), this means $\{X_t\}$ has "birth rate" $f(t)1_{(X_t>0)}$. Hence, the proof is complete. □

Remark 2. While the subscript t denoted a spatial parameter in the original definition of $\{X_t\}$, it will be referred to as time from here on out in order to maintain consistency with the expression "continuous-time birth-death process". In addition, the elements in the process $\{X_t\}$ will be called particles, rather than frogs. A particle will be said to "die" at time t_0 if the point furthest to the right reached by that particle is $x = t_0$.

2.1.2 The process $\{Y_t\}$

The statement (2.1.2), along with the scale invariance of the original model with respect to λ discussed in the introduction, together imply that Theorem 2.1.1 can be proven by showing that the process $\{X_t\}$ goes extinct with probability 1 if and only if formula (2.1.1) holds (for the case $\lambda = \frac{1}{2}$). So let $\{Y_t\}$ be another continuous-time-inhomogeneous birth-death process with birth rate $f(t)$ and death rate Y_t (hence, it differs from $\{X_t\}$ only in the sense that 0 is not an absorbing state). $\{Y_t\}$ is identified with a triple $(\Omega, \mathcal{F}, \mathbf{P})$ defined as follows: Ω will represent the set of all functions $\omega : [0, \infty) \to \mathbb{N}$ such that $\omega(0) = 1$ and where ω is constant everywhere except at a countable collection of points $p_1 < p_2 < \ldots$, where for each $i \geq 1$, $\omega(p_i) = \lim_{t \to p_i^-} \omega(t) \pm 1$ (note that Ω can be thought of as the collection of all possible paths $\{Y_t\}$ can take). Let $\mathcal{F}$ denote the σ-field on Ω generated by the finite dimensional sets $\{\omega : \omega(s_i) = C_i \text{ for } 1 \leq i \leq n\}$ (where $0 < s_1 < \cdots < s_n$ and $C_i \in \mathbb{N}$ for each i). Finally, $\mathbf{P}$ will refer to the probability measure on $(\Omega, \mathcal{F})$ associated with $\{Y_t\}$. The primary task involved in moving towards a proof of Theorem 2.1.1 will consist of proving a statement about $\{Y_t\}$. This comes in the form of the following theorem.

Theorem 2.1.3. *Assume $f : [0, \infty) \to [0, \infty)$ is monotonically increasing and define the value $V(\omega) = \#\{\text{points where } \omega \text{ jumps from } 0 \text{ to } 1\}$. Then $\mathbf{P}(V = \infty) = 1$ if and only if*

$$\int_0^\infty e^{-f(t)} f(t) dt = \infty \tag{2.1.3}$$

If (2.1.3) *does not hold, then $\mathbf{P}(V = \infty) = 0$.*

The proof of Theorem 2.1.3 has three main steps. The first one entails proving the following proposition. Note that in the statement of this proposition, and those following it, $\mathbf{E}$ will denote expectation with respect to the probability measure $\mathbf{P}$ and f is assumed to be monotonically increasing.

Proposition 2.1.4.

$$\int_0^\infty e^{-f(t)} f(t)dt = \infty \implies \mathbf{E}[V] = \infty \tag{2.1.4}$$

After Proposition 2.1.4 is established, it is then shown how the result can be used to prove one direction of Theorem 2.1.3. This entails establishing the following implication.

Proposition 2.1.5. $\mathbf{E}[V] = \infty \implies \mathbf{P}(V = \infty) = 1.$

After establishing Proposition 2.1.5, we address the issue of proving the other direction of Theorem 2.1.3. This comes in the form of the proceeding proposition.

Proposition 2.1.6.

$$\int_0^\infty e^{-f(t)} f(t)dt < \infty \implies \mathbf{P}(V < \infty) = 1 \tag{2.1.5}$$

Proof of Proposition 2.1.4. First note that for any $t > 0$, Y_t is a random variable of the form $\text{Bern}(e^{-t}) + \text{Poiss}(\lambda_t)$ (with the two parts of the sum independent) where

$$\lambda_t = \int_0^t f(u)e^{-(t-u)}du = f(t) - f(t)e^{-t} - \int_0^t (f(t) - f(u))e^{-(t-u)}du \tag{2.1.6}$$

This follows from the fact that the single particle we began with at time zero remains "alive" at time t with probability e^{-t} (hence the term $\text{Bern}(e^{-t})$), along with the fact that, if f is continuous at u (note that f being increasing implies it is continuous a.e.), then the event of a particle being "born" inside the time interval $[u, u + du)$ and surviving until at least time t, has probability $(f(u)e^{-(t-u)} + o(1))du$ as $du \to 0$ (where disjoint intervals are independent). It is then implied by (2.1.6) that $\lambda_t \leq f(t) \; \forall \, t \in [0, \infty)$, from which it follows that

$$\mathbf{P}(\omega(t) = 0) \geq e^{-f(t)}(1 - e^{-t}) \tag{2.1.7}$$

Since the probability $\{Y_t\}$ jumps from 0 to 1 on an interval $[t, t + dt)$ is $(1 + o(1))\mathbf{P}(\omega(t) = 0)f(t)dt$ as $dt \to 0$, it follows that

$$\mathbf{E}[V] = \int_0^\infty \mathbf{P}(\omega(t) = 0) f(t) dt \tag{2.1.8}$$

Combining this with (2.1.7) then establishes the implication

$$\int_0^\infty e^{-f(t)} f(t) dt = \infty \implies \mathbf{E}[V] = \infty$$

Hence, the proof is complete. □

Proof of Proposition 2.1.5. Let $U_x = \{\omega \in \Omega : \omega$ never jumps from 0 to 1 in $(x, \infty)\}$. If it's assumed that $\mathbf{P}(V = \infty) < 1$, then this implies that there exists $x, L > 0$ (with $L \in \mathbb{Z}$) such that $\mathbf{P}(U_{x+1} | \; \omega(x + 1) = L) > 0$. Since

$$\mathbf{P}(\{\omega(t) > 0 \text{ on } (x, x + 1)\} \cap \{\omega(x + 1) = L\} \mid \omega(x) = 1) > 0 \tag{2.1.9}$$

we get $\mathbf{P}(U_x | \; \omega(x) = 1) > 0$. Since f is monotonically we know $\big(\{Y_{t_1+t}\} \, | (Y_{t_1} = 1)\big)$ can be coupled with $\big(\{Y_{t_2+t}\} \, | (Y_{t_2} = 1)\big)$ (for $t_2 > t_1$) so that the former is dominated by the latter. From this it follows that $\mathbf{P}(U_t | \; \omega(t) = 1)$ is increasing w.r.t. t. Now if we let $V_x(\omega) = \# \{$points in (x, ∞) where ω jumps from 0 to 1$\}$, the fact that $\mathbf{P}(U_t | \omega(t) = 1)$ is positive (for $t \geq x$) and increasing implies that $\mathbf{P}(V_x \geq T + 1 | \; V_x \geq T) \leq 1 - \mathbf{P}(U_x | \; \omega(x) = 1) \; \forall \, T \in \mathbb{N}$. Hence,

$$\mathbf{E}[V_x] \leq \sum_{j=0}^\infty (1 - \mathbf{P}(U_x | \; \omega(x) = 1))^j = \frac{1}{\mathbf{P}(U_x | \; \omega(x) = 1)} < \infty \implies \mathbf{E}[V] < \infty$$

Therefore, we've established the contrapositive of Proposition 2.1.5, which establishes the proposition. □

Proof of Proposition 2.1.6. It follows from (2.1.7) and (2.1.8) that

$$\mathbf{E}[V] = \int_0^\infty e^{-\lambda_t}(1 - e^{-t})f(t)dt \tag{2.1.10}$$

Since $1 - e^{-t} \to 1$ as $t \to \infty$, in order to show that $\mathbf{E}[V] < \infty$ it suffices to establish the following implication.

$$\int_0^\infty e^{-f(t)} f(t)dt < \infty \implies \int_0^\infty e^{-\lambda_t} f(t)dt < \infty \tag{2.1.11}$$

Using the integral formula for λ_t in (2.1.6), we see that if f is continuous at t, then λ_t is differentiable at t with

$$\frac{d\lambda_t}{dt} = \frac{d\Big(e^{-t}\int_0^t e^u f(u)du\Big)}{dt} = f(t) - e^{-t}\int_0^t e^u f(u)du = f(t) - \lambda_t$$

Hence, at all continuity points of f, we have $f(t) = \lambda_t + \lambda'_t$. Since f is monotonically increasing, this means it has only countably many discontinuity points, which means it is continuous a.e (as was mentioned in the proof of 2.1.4). It then follows that $f(t) = \lambda_t + \lambda'_t$ a.e. Hence, we can write

$$\int_0^\infty e^{-f(t)} f(t)dt = \int_0^\infty e^{-(\lambda_t + \lambda'_t)} f(t)dt = \int_0^\infty e^{-\lambda_t} f(t) e^{-\lambda'_t} dt \tag{2.1.12}$$

Now let $0 \le t_1 < t_2$ and note that

$$\lambda_{t_2} - \lambda_{t_1} = \int_0^{t_2 - t_1} e^{-(t_2-u)} f(u)du + \int_0^{t_1} e^{-(t_1-u)}\Big(f(t_2 - t_1 + u) - f(u)\Big)du > 0 \tag{2.1.13}$$

Hence, λ_t is monotonically increasing. Also note that if $0 \le t_1 < t_2 \le N$ (for $N < \infty$) then

$$\begin{aligned} \lambda_{t_2} &= \int_0^{t_2} e^{-(t_2-u)} f(u)du \\ &= e^{(t_1-t_2)} \int_0^{t_1} e^{-(t_1-u)} f(u)du + \int_{t_1}^{t_2} e^{-(t_2-u)} f(u)du \\ &\le \lambda_{t_1} + (t_2 - t_1) f(N) \end{aligned}$$

Along with (2.1.13) this implies $|\lambda_{t_2} - \lambda_{t_1}| \le (t_2 - t_1) f(N)$, which means that λ_t is absolutely continuous on $[0, N]$. Coupled with λ_t being monotonically increasing and satisfying $\lambda_0 = 0$, this implies that if $g : [0, N] \to [0, \infty)$ is any Lebesgue measurable function, then

$$\int_0^{\lambda_N} g(x)dx = \int_0^N g(\lambda_t)\lambda_t' dt \tag{2.1.14}$$

(see [13], para. 2, pg. 156). Now since $\lim_{t\to\infty} f(t) = \infty$ (otherwise it could not hold that $\int_0^\infty e^{-f(t)} f(t)dt < \infty$) this means $\lambda_t = \int_0^t e^{-(t-u)} f(u)du \to \infty$ as $t \to \infty$. Therefore, if $g \ge 0$ is Lebesgue measurable with $\int_0^\infty g(x)dx < \infty$, then letting $N \to \infty$ in (2.1.14) gives

$$\int_0^\infty g(x)dx = \int_0^\infty g(\lambda_t)\lambda_t' dt$$

Specifically looking at the cases $g_1(x) = e^{-x}$ and $g_2(x) = xe^{-x}$, gives the two formulas

$$\int_0^\infty e^{-\lambda_t}\lambda_t' dt = \int_0^\infty e^{-x} dx = 1$$
$$\int_0^\infty e^{-\lambda_t}\lambda_t\lambda_t' dt = \int_0^\infty xe^{-x} dx = 1$$

Combining these formulas with (2.1.12), and using the fact that $f(t) = \lambda_t + \lambda_t'$ a.e., then gives

$$\begin{aligned} \int_0^\infty e^{-\lambda_t} f(t)dt &= \int_0^\infty e^{-\lambda_t} f(t) 1_{(\lambda_t' \le 1)} dt + \int_0^\infty e^{-\lambda_t}(\lambda_t + \lambda_t') 1_{(\lambda_t' > 1)} dt \\ &\le e \int_0^\infty e^{-\lambda_t} f(t) e^{-\lambda_t'} dt + \int_0^\infty e^{-\lambda_t}\lambda_t\lambda_t' dt \ + 1 \\ &= e \int_0^\infty e^{-f(t)} f(t)dt \ + 2 < \infty \end{aligned}$$

Hence, this establishes (2.1.11) which, as was shown, implies $\mathbf{E}[V] < \infty$, from which it follows that $\mathbf{P}(V < \infty) = 1$. Hence, the proof is complete. □

Proof of Theorem 2.1.3. The theorem follows immediately from Propositions 2.1.4, 2.1.5, and 2.1.6. □

2.1.3 Proving Theorem 2.1.1

With Theorem 2.1.3 established, we can now complete the proof of Theorem 2.1.1. Due to the relationship discussed earlier between the process $\{X_t\}$ and the frog model with drift on $\mathbb{R}$, as well as scale invariance of the original model, it suffices to prove the following claim.

Claim: For $f : [0, \infty) \to [0, \infty)$ monotonically increasing, the process $\{X_t\}$ dies out with probability 1 if and only if

$$\int_0^\infty e^{-f(t)} f(t) dt = \infty$$

Proof. First couple the process $\{X_t\}$ with the familiar process $\{Y_t\}$ so that the two processes are identical until $\{X_t\}$ dies out. Since Theorem 2.1.3 established the implication

$$\int_0^\infty e^{-f(t)} f(t) dt = \infty \implies \mathbf{P}(\{Y_t\} \text{ jumps from 0 to 1 i.o.}) = 1 \qquad (2.1.15)$$

this means that if the left side of (2.1.15) holds, then with probability 1, $\{X_t\}$ will eventually die out (i.e. the coupled processes $\{Y_t\}$ and $\{X_t\}$ will eventually hit 0). Conversely, since Theorem 2.1.3 also states that if the integral in (2.1.15) is finite then $V < \infty$ with probability 1, this means that if we let $T_0(\omega) = \{t \in [0, \infty) : \omega(t) = 0\}$, then $\mathbf{P}(\sup T_0 < \infty) = 1$. Letting $\mathbb{P}$ represent the law of $\{X_t\}$ on $(\Omega, \mathcal{F})$, it now follows that there must be some $t > 0$ and some positive integer M s.t.

$$\mathbb{P}(X_s > 0 \ \forall \ s \geq t | X_t = M) = \mathbf{P}(Y_s > 0 \ \forall \ s \geq t | Y_t = M) > 0 \qquad (2.1.16)$$

and $\mathbf{P}(Y_t = M) > 0$. This then implies that $\mathbf{P}(X_t = M) > 0$, which along with (2.1.16), gives

$$\int_0^\infty e^{-f(t)} f(t)dt < \infty \implies \mathbb{P}(X_s > 0 \,\forall\, s \geq 0) \geq \mathbb{P}(X_t = M)\mathbb{P}(X_s > 0 \,\forall\, s \geq t | X_t = M) > 0$$

Alongside the first part of the proof, this last result establishes that $\{X_t\}$ dies out with probability 1 if and only if the integral on the left side of (2.1.15) diverges. Thus we have established the above claim, which completes the proof of Theorem 2.1.1. □

Remark 3. Note that the result of Theorem 2.1.1 can easily be extended to all measurable functions $f : [0, \infty) \to [0, \infty)$ for which $\exists\, r \in (0, \infty)$ such that f is bounded on $[0, r)$ and increasing on $[r, \infty)$. This is established by first noting that it follows from (2.1.2) that the process is non-transient if and only if

$$\mathbb{P}(\lim_{t\to\infty} X_t > 0) > 0 \iff \mathbb{P}(X_r > 0)\mathbb{P}(\lim_{t\to\infty} X_t > 0 | X_r > 0) > 0 \tag{2.1.17}$$

Because $\mathbb{P}(X_r > 0) \geq e^{-r}$ (since the particle beginning at time 0 remains "alive" at time r with probability e^{-r}), and because $\mathbb{P}(\lim_{t\to\infty} X_t > 0 | X_r = L) > 0$ (for some $L > 0$) if and only if $\mathbb{P}(\lim_{t\to\infty} X_t > 0 | X_r = 1) > 0$, it follows from (2.1.17) and Theorem 2.1.1 that the process is non-transient if and only if

$$\mathbb{P}(\lim_{t\to\infty} X_t > 0 | X_r = 1) > 0 \iff \int_r^\infty e^{-f(t)} f(t)dt < \infty \iff \int_0^\infty e^{-f(t)} f(t)dt < \infty$$

2.2 The non-uniform frog model with drift on $\mathbb{Z}$

Recall that the non-uniform frog model with drift on $\mathbb{Z}$ contains η_j sleeping frogs at $x = j$ for every positive integer j, where each η_j has distribution Poiss$(f(j))$ for some function $f : \mathbb{Z}^+ \to [0, \infty)$, and the η_j's are independent. Upon becoming activated (via being landed on by an active frog), a frog performs a random walk independent of all other active frogs, that at each step goes left with probability p (for some $p > \frac{1}{2}$) and right with probability $1-p$. We start with a single active frog at the origin which also goes left (or right) with probability p (or $1-p$ respectively). We again establish a sharp condition distinguishing between transience and non-transience in the form of the following theorem.

Theorem 2.2.1. *For $\frac{1}{2} < p < 1$ and f monotonically increasing, the non-uniform frog model with drift on $\mathbb{Z}$ is transient if and only if*

$$\sum_{j=1}^{\infty} e^{-\frac{1-p}{2p-1} f(j)} = \infty \tag{2.2.1}$$

Remark 4. As with the model on $\mathbb{R}$, allowing sleeping frogs to also reside to the left of the origin does not complicate matters significantly in the case of the model on $\mathbb{Z}$. If the domain of f is expanded to all of $\mathbb{Z}\setminus\{0\}$, then the condition given in Theorem 2.2.1 continues to apply as long as

$$\sum_{j=1}^{\infty} \left(\frac{1-p}{p}\right)^j f(-j) < \infty$$

since the above sum is equal to $\mathbb{E}[L^*_{(-\infty,0)}]$ (where $L^*_{(a,b)}$ denotes the number of distinct frogs originating in $(a,b)\cap\mathbb{Z}$ that ever hit the origin). Conversely, since $L^*_{(-N-1,0)}$ has a Poisson distribution with mean equal to the sum of the first N terms in the expression above, the divergence of this sum will, as with the continuous case (see *Remark* 1), imply recurrence of the model.

The proof that transience of the non-uniform frog model corresponds to (2.2.1) nearly mirrors that of the continuous case. A discrete-time-inhomogeneous Markov process $\{M_j\}$

is defined, where

$$(M_{j+1}|M_j) = \begin{cases} \text{Bin}\big(M_j, \frac{1-p}{p}\big) + \text{Poiss}\big(\frac{1-p}{p} f(j)\big) & \text{if } M_j \geq 1 \\ 0 & \text{if } M_j = 0 \end{cases}$$

Transience of the non-uniform frog model is shown to hold if and only if $\{M_j\}$ eventually arrives at the absorbing state 0 with probability 1. $\{N_j\}$ will then represent a process just like $\{M_j\}$, except $(N_{j+1}|N_j = 0) = \text{Poiss}(\frac{1-p}{p} f(j))$ (i.e. 0 is not an absorbing state). Most of the focus is devoted to showing that $\{N_j\}$ will attain the value 0 infinitely often with probability 1 if and only if the sum in (2.2.1) diverges. The proof involves establishing a series of three propositions which are essentially the discrete analogs of a series of propositions that were used when dealing with the continuous model.

2.2.1 The processes $\{M_j\}$ and $\{N_j\}$

We begin by using the non-uniform frog model to define the process $\{M_j\}$ as follows. Let $M_0 = 1$ and, for $j \geq 1$, let M_j equal the number of frogs originating in $\{0, 1, \ldots, j-1\}$ that ever hit $x = j$. Much like with the process $\{X_t\}$, we find that

$$\lim_{j \to \infty} M_j > 0 \Longleftrightarrow \{\text{infinitely many frogs return to the origin}\} \tag{2.2.2}$$

Examining the process $\{M_j\}$, we also obtain this next proposition.

Proposition 2.2.2. *$\{M_j\}$ is a discrete-time-inhomogeneous Markov process with $M_0 = 1$, $M_1 = Bern(\frac{1-p}{p})$, and for $j \geq 1$*

$$(M_{j+1}|M_j) = \begin{cases} Bin\big(M_j, \frac{1-p}{p}\big) + Poiss\big(\frac{1-p}{p} f(j)\big) & \textit{if } M_j \geq 1 \\ 0 & \textit{if } M_j = 0 \end{cases}$$

(where the two parts of the above sum are independent).

Proof. By a simple martingale argument the probability an active frog residing at $x = j$ ever makes it to $x = j + 1$ is $\frac{1-p}{p}$. Therefore, the expression for M_1 follows. This also implies that if we condition on M_j, then for $j \geq 1$ the distribution of the number of frogs that make it to $x = j+1$ which originate in $\{0, 1, \ldots, j-1\}$, is $\text{Bin}\big(M_j, \frac{1-p}{p}\big)$. Adding this to the number of frogs originating at $x = j$ that ever make it to $x = j + 1$, while again using the first line of this proof along with the fact that η_j (the number of sleeping frogs starting at $x = j$) has distribution $\text{Poiss}\big(f(j)\big)$, gives us the above piecewise expression for $\big(M_{j+1}|M_j\big)$. □

We now introduce the process $\{N_j\}$ which, as stated in the introduction, will represent a process identical to $\{M_j\}$ except that $(N_{j+1}|N_j = 0)$ has distribution $\text{Poiss}(\frac{1-p}{p} f(j))$ (meaning 0 is not an absorbing state). $\{N_j\}$ is identified with a triple $(\Omega^*, \mathcal{F}^*, \mathbf{P}^*)$ defined as follows. Ω^* will represent the set of all functions $\omega : \mathbb{N} \to \mathbb{N}$, $\mathcal{F}^*$ will represent the σ-field on Ω^* generated by the finite dimensional sets, and $\mathbf{P}^*$ will refer to the probability measure on $(\Omega^*, \mathcal{F}^*)$ associated with $\{N_j\}$ (note that $\mathbf{P}^*$ is supported on $\{\omega \in \Omega^* : \omega(0) = 1, \omega(1) \leq 1\}$). Using the formalism defined above, we now present a result which constitutes the main step in proving Theorem 2.2.1.

Theorem 2.2.3. *If* $\frac{1}{2} < p < 1$, f *is monotonically increasing, and we let* $K(\omega) = \#\{j \in \mathbb{Z}^+ : \omega(j) = 0\}$, *then* $\mathbf{P}^*(K = \infty) = 1$ *if and only if*

$$\sum_{j=1}^{\infty} e^{-\frac{1-p}{2p-1} f(j)} = \infty \tag{2.2.3}$$

If (2.2.3) *does not hold then* $\mathbf{P}^*(K = \infty) = 0$.

2.2.2 Proving Theorem 2.2.1

We begin this section by presenting Propositions 2.2.4, 2.2.5, and 2.2.6. In places where the proofs bear an especially strong resemblance to the proofs of the corresponding propositions for the model on $\mathbb{R}$ (2.1.4, 2.1.5, and 2.1.6 respectively) some details are omitted. In what follows, f is always assumed to be monotonically increasing, and $\mathbf{E}^*$ will represent expectation with respect to $\mathbf{P}^*$.

Proposition 2.2.4.

$$\sum_{j=1}^{\infty} e^{-\frac{1-p}{2p-1}f(j)} = \infty \implies \mathbf{E}^*[K] = \infty$$

Proof. As a random variable (for $j \geq 1$)

$$N_j = \text{Bern}\left(\left(\frac{1-p}{p}\right)^j\right) + \text{Poiss}(\tau_j) \tag{2.2.4}$$

where

$$\tau_j = \sum_{i=1}^{j-1} \left(\frac{1-p}{p}\right)^{j-i} f(i)$$

By an argument similar to the one employed in the proof of Proposition 2.1.4, we find that it follows from the fact f is increasing that $\tau_j \leq \frac{1-p}{2p-1} f(j) \ \forall \ j$. Combining this with (2.2.4) establishes that

$$\mathbf{P}^*(\omega(j) = 0) \geq \left(1 - \left(\frac{1-p}{p}\right)^j\right) e^{-\frac{1-p}{2p-1}f(j)} \tag{2.2.5}$$

Since $\left(\frac{1-p}{p}\right)^j \to 0$ as $j \to \infty$ and

$$\mathbf{E}^*[K] = \sum_{j=1}^{\infty} \mathbf{P}^*(\omega(j) = 0)$$

the proposition follows. □

Proposition 2.2.5.

$$\mathbf{E}^*[K] = \infty \implies \mathbf{P}^*(K = \infty) = 1$$

Proof. Proceed by proving the contrapositive. Let $U_j = \{\omega \in \Omega^* : \omega(i) > 0 \;\forall\; i > j\}$. Assume $\mathbf{P}^*(K = \infty) < 1$. It will follow that $\exists\; L \geq 1$ such that $\mathbf{P}^*(U_L|\omega(L) = 0) > 0$. Along with the fact that $\mathbf{P}^*(U_L|\omega(L) = 0)$ is monotonically increasing (which follows from the fact that f is increasing), this implies that $\mathbf{E}^*[K] - L$ can be bounded above by the sum of a geometric series with base $1 - \mathbf{P}^*(U_L|\omega(L) = 0) < 1$. The contrapositive of the proposition then follows, which establishes the proposition. □

Proposition 2.2.6.

$$\sum_{j=1}^{\infty} e^{-\frac{1-p}{2p-1}f(j)} < \infty \implies \mathbf{P}^*(K < \infty) = 1$$

Proof. Since (2.2.4) implies that $\mathbf{P}^*(\omega(j) = 0) = e^{-\tau_j}\big(1 - \big(\frac{1-p}{p}\big)^j\big)$, it follows that

$$\mathbf{E}^*[K] = \sum_{j=1}^{\infty} e^{-\tau_j}\left(1 - \left(\frac{1-p}{p}\right)^j\right) < \sum_{j=1}^{\infty} e^{-\tau_j}$$

Hence, to show that

$$\sum_{j=1}^{\infty} e^{-\frac{1-p}{2p-1}f(j)} < \infty \implies \mathbf{E}^*[K] < \infty$$

it suffices to show that

$$\sum_{j=1}^{\infty} e^{-\frac{1-p}{2p-1}f(j)} < \infty \implies \sum_{j=1}^{\infty} e^{-\tau_j} < \infty \tag{2.2.6}$$

To establish (2.2.6), first note that

$$\begin{aligned}
\tau_{j+1} - \tau_j &= \left(\frac{1-p}{p}\right)^{j+1} \sum_{i=1}^{j} \left(\frac{1-p}{p}\right)^{-i} f(i) - \left(\frac{1-p}{p}\right)^{j} \sum_{i=1}^{j-1} \left(\frac{1-p}{p}\right)^{-i} f(i) \\
&= \left[\left(\frac{1-p}{p}\right)^{j+1} - \left(\frac{1-p}{p}\right)^{j}\right] \sum_{i=1}^{j-1} \left(\frac{1-p}{p}\right)^{-i} f(i) + \frac{1-p}{p} f(j) \\
&= \frac{1-2p}{p}\tau_j + \frac{1-p}{p} f(j) \implies \frac{1-p}{2p-1} f(j) = \frac{p}{2p-1}\Delta\tau_j + \tau_j
\end{aligned}$$

(where $\Delta\tau_j$ denotes $\tau_{j+1} - \tau_j$). Hence,

$$\sum_{j=1}^{\infty} e^{-\frac{1-p}{2p-1}f(j)} = \sum_{j=1}^{\infty} e^{-\tau_j} \cdot e^{-\frac{p}{2p-1}\Delta\tau_j} \tag{2.2.7}$$

Now if the sum on the right in (2.2.6) is written as

$$\sum_{j=1}^{\infty} e^{-\tau_j} = \sum_{j=1}^{\infty} e^{-\tau_j} 1_{(\Delta\tau_j \le 1)} + \sum_{j=1}^{\infty} e^{-\tau_j} 1_{(\Delta\tau_j > 1)} \tag{2.2.8}$$

then (2.2.7) and the left side of (2.2.6) imply that

$$\sum_{j=1}^{\infty} e^{-\tau_j} \le e^{\frac{p}{2p-1}} \sum_{j=1}^{\infty} e^{-\frac{1-p}{2p-1} f(j)} + \frac{e}{e-1} < \infty$$

(where the $\frac{e}{e-1}$ term follows from the fact that the last sum on the right in (2.2.8) can be bounded above by the sum of the geometric series with base e^{-1}). Therefore, this establishes (2.2.6), which implies $\mathbf{E}^*[K] < \infty$, from which it follows that $\mathbf{P}^*(V < \infty) = 1$. Hence, the proof is complete. □

Proof of Theorem 2.2.3. The theorem is an immediate consequence of Propositions 2.2.4-2.2.6. □

Theorem 2.2.3 is now used to establish Theorem 2.2.1. Note that on account of (2.2.2), establishing Theorem 2.2.1 reduces to proving the following claim.

Claim: For $f : \mathbb{Z}^+ \to [0, \infty)$ monotonically increasing, the process $\{M_j\}$ dies out with probability 1 if and only if

$$\sum_{j=1}^{\infty} e^{-\frac{1-p}{2p-1} f(j)} = \infty$$

Proof. By a coupling of $\{M_j\}$ with $\{N_j\}$, it follows from Theorem 2.2.3 that if the above sum diverges, then $\{M_j\}$ dies out with probability 1. For the other direction, we can apply an argument exactly like the one we used in the continuous case, but where we replace the integral with the sum and replace $\{X_t\}$ and $\{Y_t\}$ with $\{M_j\}$ and $\{N_j\}$ respectively. Alongside the first part of the proof, this establishes Theorem 2.2.1. □

Remark 5. Much like with the continuous case, the result of Theorem 2.2.1 extends to all functions $f : \mathbb{Z}^+ \to [0, \infty)$ for which $\exists\, q \in \mathbb{Z}^+$ such that f is increasing on $\{q, q+1, q+2, \dots\}$. Due to its strong similarity to the argument given in *Remark* 3, the explanation for this is omitted.

2.3 Counterexamples and additional comments

In this final section I'll discuss a scenario in which $f : [0, \infty) \to [0, \infty)$ is not monotonically increasing, and the tight condition of Theorem 2.1.1 ceases to hold. A similar case for the discrete model is also mentioned.

Example 1. Define $f : [0, \infty) \to [0, \infty)$ as

$$f(t) = \begin{cases} 1 & \text{if } t \in [2^n, 2^n + 1) \text{ for } n \in \mathbb{Z}^+ \\ t & \text{otherwise} \end{cases}$$

Since $\mathbf{E}[V] = \int_0^\infty e^{-\lambda_t} f(t)(1 - e^{-t})dt$ (see (2.1.10)), to show that $\mathbf{E}[V] < \infty$ it suffices to show that $\int_0^\infty e^{-\lambda_t} f(t)dt < \infty$. Recalling from (2.1.6) that $\lambda_t = \int_0^t e^{-(t-u)} f(u)du$, we'll seek to achieve a lower bound for λ_t. Note first that if $n \in \mathbb{Z}^+$ then

$$\int_{2^n+1}^{2^{n+1}} e^u f(u)du = \int_{2^n+1}^{2^{n+1}} ue^u du = \big(2^{n+1} - 1\big)e^{2^{n+1}} - 2^n \cdot e^{2^n+1}$$

Hence, for $t = 2^{n+1}$ (for $n \in \mathbb{Z}^+$)

$$\begin{aligned} \lambda_t &= e^{-t}\Big(\int_0^2 ue^u du + \sum_{j=1}^n \int_{2^j}^{2^j+1} e^u du + \sum_{j=1}^n \big(2^{j+1} - 1\big)e^{2^{j+1}} - 2^j \cdot e^{2^j+1}\Big) \quad (2.3.1) \\ &\geq e^{-t}\Big(\big(t-1\big)e^t - \frac{t}{2}e^{\frac{t}{2}+1}\Big) \geq t - 2 \end{aligned}$$

(where the last inequality follows from the fact that $\frac{t}{2}e^{\frac{t}{2}+1} < e^t$ for $t \geq 4$). Since $\lambda_{t+r} \geq e^{-r}\lambda_t$, it follows that for $0 \leq r \leq 1$ (with $t = 2^{n+1}$ as above) we have

$$\lambda_{t+r} \geq e^{-1}(t-2) \quad (2.3.2)$$

Furthermore, note that if $t_o \in (t+1, 2t)$ then λ'_{t_o} exists (since f is continuous in $(2^{n+1} + 1, 2^{n+2})$) with $\lambda'_{t_o} = f(t_o) - \lambda_{t_o}$. Since $\lambda_{t_o} \le e^{-t_o} \int_0^{t_o} ue^u du = t_o - 1 + e^{-t_o}$, this means

$$\lambda_{t_o} \le t_o - e^{-1} \implies \lambda'_{t_o} = t_o - \lambda_{t_o} \ge e^{-1}$$

which along with (2.3.2), implies $\lambda_{t_o} \ge e^{-1}(t_o - 2)$. Combining this with (2.3.1) and (2.3.2) then tells us that $\lambda_s \ge e^{-1}(s-2) \; \forall \; s \in [2^{n+1}, 2^{n+2})$ (for $n \in \mathbb{Z}^+$), and therefore $\lambda_s \ge e^{-1}(s-2) \; \forall \; s \in [4, \infty)$. Using this inequality, along with the fact that $f(s) \le s \; \forall \; s \in [0, \infty)$, we find that

$$\int_0^\infty e^{-\lambda_t} f(t) dt \le \int_0^4 t dt + \int_4^\infty e^{-e^{-1}(t-2)} t dt < \infty \implies \mathbf{E}[V] < \infty \implies \mathbf{P}(V < \infty) = 1$$

By the argument that was employed in subsection 2.1.3 to establish the implication $\mathbf{P}(V < \infty) = 1 \implies$ {non-transience}, it follows that for the given Poisson intensity function f (with drift $\frac{1}{2}$) the model is non-transient. Noting now that

$$\int_0^\infty e^{-f(t)} f(t) dt \ge \sum_{j=1}^\infty \int_{2^j}^{2^j+1} e^{-1} dt = \sum_{j=1}^\infty e^{-1} = \infty$$

we find that the tight condition from Theorem 2.1.1 does indeed fail to apply in this case.

Remark 6. Notice that the tight condition of Theorem 2.1.1 also fails to hold when $f : [0, \infty) \to [0, \infty)$ is a bounded (nonzero) function such that $\int_0^\infty f(x) dx < \infty$ (since then $\int_0^\infty e^{-f(x)} f(x) dx < \infty$, but the model is transient). However, if the integral in (2.1.1) is changed to $\int_0^\infty e^{-f(x)}(1 + f(x)) dx$, then 2.1.1 remains valid, yet functions in $L^1([0, \infty))$ that are bounded, nonzero, and nonnegative, no longer violate the new condition. Hence, such functions offer far less insight into the limits to which the result of Theorem 2.1.1 can be stretched, than does the case examined in Example 1.

Example 2. Define $f : \mathbb{Z}^+ \to \mathbb{N}$ as

$$f(j) = \begin{cases} 1 & \text{if } j = 2^n \text{ for } n \in \mathbb{Z}^+ \\ j & \text{otherwise} \end{cases}$$

It follows from (2.2.4) that in order to show that $\mathbf{E}^*[K] < \infty$ it suffices to show that $\sum_{j=1}^{\infty} e^{-\tau_j} < \infty$ (with τ_j defined as in the proof of Proposition 2.2.4). From the formulas for τ_j and f we see that $\tau_j \geq \left(\frac{1-p}{p}\right)^2 (j-2)\ \forall\ j \geq 1$ (recall $\frac{1}{2} < p < 1$). Hence,

$$\sum_{j=1}^{\infty} e^{-\tau_j} < \infty \implies \mathbf{E}^*[K] < \infty \implies \mathbf{P}^*(K < \infty) = 1$$

As we saw in the proofs of Theorems 2.1.1 and 2.2.1, this implies non-transience of the model. Combining this with the fact that

$$\sum_{j=1}^{\infty} e^{-\frac{1-p}{2p-1} f(j)} \geq \sum_{j=1}^{\infty} e^{-\frac{1-p}{2p-1} f(2^j)} = \sum_{j=1}^{\infty} e^{-\frac{1-p}{2p-1}} = \infty$$

we see that the tight condition of Theorem 2.2.1 does not apply in this case.

Chapter 3

Several frog model variants on $\mathbb{Z}$

3.1 The nonhomogeneous frog model

Refreshing the definition of the nonhomogeneous frog model on $\mathbb{Z}$, it begins with no frogs to the left of the origin, and one active frog at the origin which performs a random walk that at each step goes left with probability p_0 (where $\frac{1}{2} < p_0 < 1$) and right with probability $1-p_0$. For $j \geq 1$, the number of sleeping frogs at $x = j$ is a random variable X_j, where the X_j's are independent, non-zero with positive probability, and where $X_{j+1} \succeq X_j$ (here " $\succeq$ " represents stochastic dominance). In addition, for each $j \geq 1$ frogs originating at $x = j$ (if activated) go left with probability p_j (where $\frac{1}{2} < p_j < 1$) and right with probability $1 - p_j$, where the p_j's are decreasing and the random walks are independent. My main result for the nonhomogeneous frog model on $\mathbb{Z}$ is the following sharp condition that distinguishes between transience and non-transience of the model.

Theorem 3.1.1. *Let f_j be the probability generating function of X_j for the nonhomogeneous* *frog model on $\mathbb{Z}$. The model is transient if and only if*

$$\sum_{n=2}^{\infty}\prod_{j=1}^{n-1} f_j\Big(1-\Big(\frac{1-p_j}{p_j}\Big)^{n-j}\Big)=\infty \tag{3.1.1}$$

3.1.1 The processes $\{M_j\}$ and $\{N_j\}$

In order to move towards a proof of Theorem 3.1.1, we begin by defining the process $\{M_j\}$ where, for each $j \geq 1$, M_j represents the number of frogs originating in $\{0, 1, \ldots, j-1\}$ that ever hit the point $x = j$. $\{M_j\}$ is now identified with a triple $(\Omega, \mathcal{F}, \mathbf{P})$ defined as follows: Ω will represent the set of all functions $\omega : \mathbb{Z}^+ \to \mathbb{N}$ (i.e. the set of all possible trajectories of $\{M_j\}$), $\mathcal{F}$ will represent the σ-field on Ω generated by the finite dimensional sets, and $\mathbf{P}$ will refer to the probability measure induced on $(\Omega, \mathcal{F})$ by the process $\{M_j\}$. Since $\mathbb{P}(X_n \geq 1) \geq \mathbb{P}(X_1 \geq 1) > 0 \ \forall\ n \geq 1$ (recall $X_{j+1} \succeq X_j \ \forall\ j \geq 1$) and the X_j's are independent, it follows from Borel-Cantelli II that $\{X_j \geq 1 \text{ i.o.}\}$ a.s. Additionally, since each activated frog performs a random walk with nonzero leftward drift, this means that each activated frog will eventually hit the origin with probability 1. Coupling this with the fact that $\{X_j \geq 1 \text{ i.o.}\}$ a.s. $\implies \sum_{j=1}^{\infty} X_j = \infty$ a.s., along with the implication $M_l = 0 \implies M_j = 0 \ \forall\ j > l$, we find that

$$\{\text{infinitely many frogs hit the origin}\} \iff \min M_j > 0 \tag{3.1.2}$$

Now on account of (3.1.2), it follows that in order to establish Theorem 3.1.1, it suffices to show that

$$\min M_j = 0 \ \ \mathbf{P}-\text{a.s.} \iff \sum_{n=2}^{\infty}\prod_{j=1}^{n-1} f_j\Big(1-\Big(\frac{1-p_j}{p_j}\Big)^{n-j}\Big)=\infty \tag{3.1.3}$$

With this in mind, we define a new model which we'll call the F^+ model. This model will resemble the non-homogeneous frog model on $\mathbb{Z}$ in that the distribution of the number of frogs beginning at every vertex will be the same in the two cases, as will the drifts of the active frogs. The only difference will be that in the F^+ model all frogs will begin as active frogs (i.e. they do not need to be landed on to be activated). The next step is to now use the F^+ model to define the process $\{N_j\}$ where, for each $j \geq 1$, N_j equals the number of frogs originating in $\{0, 1, \ldots, j-1\}$ that ever hit the point $x = j$ in the F^+ model (i.e. $\{N_j\}$ is identical to $\{M_j\}$ except that the F^+ model replaces the non-homogeneous frog model on $\mathbb{Z}$ in the definition). $\{N_j\}$ can now be identified with the triple $(\Omega, \mathcal{F}, \mathbf{Q})$, where $\mathbf{Q}$ will refer to the probability measure induced on $(\Omega, \mathcal{F})$ by the process $\{N_j\}$. Having defined this construction, we'll now establish the following proposition, which will serve as the key step in proving Theorem 3.1.1.

Proposition 3.1.2. *Define the random variable* $K(\omega) = \#\{j \in \mathbb{Z}^+ : \omega(j) = 0\}$. *Then* $\mathbf{Q}(K = \infty) = 1$ *if and only if*

$$\sum_{n=2}^{\infty} \prod_{j=1}^{n-1} f_j \Big(1 - \Big(\frac{1-p_j}{p_j}\Big)^{n-j}\Big) = \infty \tag{3.1.4}$$

If (3.1.4) *does not hold then* $\mathbf{Q}(K = \infty) = 0$.

Remark. It is worth noting that it cannot be assumed that $\{M_j\}$ and $\{N_j\}$ are Markov processes, since M_j (N_j resp.) only gives the number of frogs originating to the left of the point $x = j$ that ever hit $x = j$, rather than also providing the information about where each such frog originated (a significant detail, since frog origin determines drift). Nevertheless, because the only conditioning we will do with respect to these two processes will involve conditioning on M_j (N_j resp.) equalling 0, they prove to be sufficient for our purposes.

Proof of Proposition 3.1.2. By a simple martingale argument the probability a frog starting at $x = j$ ever hits $x = n$ (for $n > j$) is $\left(\frac{1-p_j}{p_j}\right)^{n-j}$. Hence, the probability that no frogs beginning at $x = j$ ever hit $x = n$ is

$$\sum_{i=0}^{\infty} \mathbb{P}(X_j = i)\Big(1 - \Big(\frac{1-p_j}{p_j}\Big)^{n-j}\Big)^i = f_j\Big(1 - \Big(\frac{1-p_j}{p_j}\Big)^{n-j}\Big)$$

It then follows that for every $n \geq 1$ we have

$$\mathbf{Q}(\omega(n) = 0) = \Big(1 - \Big(\frac{1-p_0}{p_0}\Big)^n\Big) \prod_{j=1}^{n-1} f_j\Big(1 - \Big(\frac{1-p_j}{p_j}\Big)^{n-j}\Big)$$

$$\implies \mathbf{E}[K] = \frac{2p_0 - 1}{p_0} + \sum_{n=2}^{\infty}\Big(1 - \Big(\frac{1-p_0}{p_0}\Big)^n\Big) \prod_{j=1}^{n-1} f_j\Big(1 - \Big(\frac{1-p_j}{p_j}\Big)^{n-j}\Big)$$

(where $\mathbf{E}$ refers to expectation with respect to the probability measure $\mathbf{Q}$). Since $\Big(1 - \left(\frac{1-p_0}{p_0}\right)^n\Big) \to 1$ as $n \to \infty$, this means

$$\mathbf{E}[K] < \infty \iff \sum_{n=2}^{\infty} \prod_{j=1}^{n-1} f_j\Big(1 - \Big(\frac{1-p_j}{p_j}\Big)^{n-j}\Big) < \infty \tag{3.1.5}$$

It now immediately follows that if the right side of (3.1.5) holds, then $\mathbf{Q}(K = \infty) = 0$. Hence, to prove the proposition it suffices to establish the implication $\mathbf{Q}(K = \infty) < 1 \implies \mathbf{E}[K] < \infty$ (note this is just the contrapositive of $\mathbf{E}[K] = \infty \implies \mathbf{Q}(K = \infty) = 1$).

Now since the event $\{K = \infty\}$ cannot depend on the behavior of the frogs from any finite collection of vertices (for the process $\{N_j\}$), it follows that $\mathbf{Q}(K = \infty | \omega(1) = 0) = \mathbf{Q}(K = \infty | \omega(1) = 1)$, which in turn establishes the implication

$$\mathbf{Q}(K = \infty) < 1 \implies \mathbf{Q}(1 \leq K < \infty) > 0 \tag{3.1.6}$$

Next define $V_n = \{\omega \in \Omega : \omega(j) > 0 \;\forall\; j > n\}$ and assume $\mathbf{Q}(K = \infty) < 1$. Letting $\mathbf{Q}^{(n)}$ denote the probability measure obtained by conditioning on the event $\omega(n) = 0$, (3.1.6) then implies that there must exist $L \geq 1$ such that $\mathbf{Q}^{(L)}(V_L) > 0$. Additionally, because

$X_{i_1+i_2} \succeq X_{i_1}\ \forall\ i_1, i_2 \geq 1$ (since $X_{i+1} \succeq X_i\ \forall\ i \geq 1$ and $\succeq$ is transitive) and because the sequence of drifts $\{p_j\}$ is decreasing with respect to j, this implies that for any $L' > L$ there exists a coupling of the models $\left(\mathrm{F}^+|N_L = 0\right)$ and $\left(\mathrm{F}^+|N_{L'} = 0\right)$ (i.e. the F^+ model with all frogs to the left of the point $x = L'$ removed) with the following properties: (i) Every frog originating at $x = L + j$ in $\left(\mathrm{F}^+|N_L = 0\right)$ has a particular frog that corresponds to it originating at $x = L' + j$ in the coupled model $\left(\mathrm{F}^+|N_{L'} = 0\right)$ (note that unless X_{L+j} and $X_{L'+j}$ are identically distributed, there can be frogs originating at $x = L' + j$ in $\left(F^+|N_{L'} = 0\right)$ that do not correspond to frogs originating at $x = L + j$ in $\left(F^+|N_L = 0\right)$), and (ii) whenever a frog in $\left(\mathrm{F}^+|N_L = 0\right)$ takes a step to the right, the corresponding frog in $\left(\mathrm{F}^+|N_{L'} = 0\right)$ does as well (and where if a frog with drift p_{L+j} in $\left(F^+|N_L = 0\right)$ takes a step to the left, then the corresponding frog in $\left(F^+|N_{L'} = 0\right)$ must have it's step go to the right with probability $\frac{p_{L'+j}-p_{L+j}}{1-p_{L+j}}$). Letting $K_n(\omega) = \#\{j > n : \omega(j) = 0\}$, the above coupling then implies that

$$\left(K_L|\omega(L) = 0\right) \succeq \left(K_{L'}|\omega(L') = 0\right) \implies \mathbf{Q}^{(L')}(V_{L'}) \geq \mathbf{Q}^{(L)}(V_L) \tag{3.1.7}$$

Now if we define the stopping times T_n where $T_1(\omega) = \min\{j \geq 1 : \omega(L + j) = 0\}$ and, for $n \geq 2$, $T_n(\omega) = \min\{j > T_{n-1}(\omega) : \omega(L + j) = 0\}$, we find that for every $n \geq 2$

$$\mathbf{Q}^{(L)}(K_L \geq n) = \sum_{j=1}^{\infty} \mathbf{Q}^{(L)}(T_{n-1} = j)\mathbf{Q}^{(L+j)}(V^c_{L+j}) \leq \mathbf{Q}^{(L)}(V^c_L)\mathbf{Q}^{(L)}(K_L \geq n-1) \tag{3.1.8}$$

(where the inequality follows from (3.1.7)). From this it then follows that for $n \geq 1$

$$\begin{aligned} \mathbf{Q}^{(L)}(K_L \geq n) &\leq \left(1 - \mathbf{Q}^{(L)}(V_L)\right)^n \\ \implies \mathbf{E}[K_L|\omega(L) = 0] &\leq \sum_{n=1}^{\infty} \left(1 - \mathbf{Q}^{(L)}(V_L)\right)^n \\ &= \frac{1 - \mathbf{Q}^{(L)}(V_L)}{\mathbf{Q}^{(L)}(V_L)} < \infty \end{aligned}$$

Since $\mathbf{E}[K_L] \le \mathbf{E}[K_L|\omega(L)=0]$ and $\mathbf{E}[K] \le L + \mathbf{E}[K_L]$, we find that

$$\mathbf{E}[K] \le L + \frac{1-\mathbf{Q}^{(L)}(V_L)}{\mathbf{Q}^{(L)}(V_L)} < \infty$$

Hence, we've established the implication $\mathbf{Q}(K=\infty) < 1 \implies \mathbf{E}[K] < \infty$, which then gives the implication $\mathbf{E}[K] = \infty \implies \mathbf{Q}(K=\infty) = 1$, thus completing the proof of the proposition. □

3.1.2 Proving Theorem 3.1.1

Proof of Theorem 3.1.1. Coupling the fact that Theorem 3.1.1 is equivalent to (3.1.3) with the fact that $\mathbf{P}(\min \omega(j) = 0) = 1 \iff \mathbf{Q}(K \ge 1) = 1$, we find the task of proving Theorem 3.1.1 is reduced to establishing that

$$\mathbf{Q}(K \ge 1) = 1 \iff \sum_{n=2}^{\infty} \prod_{j=1}^{n-1} f_j\Big(1 - \Big(\frac{1-p_j}{p_j}\Big)^{n-j}\Big) = \infty \tag{3.1.9}$$

Noting that the implication

$$\sum_{n=2}^{\infty} \prod_{j=1}^{n-1} f_j\Big(1 - \Big(\frac{1-p_j}{p_j}\Big)^{n-j}\Big) = \infty \implies \mathbf{Q}(K \ge 1) = 1 \tag{3.1.10}$$

follows immediately from Proposition 3.1.2, as does the fact that

$$\sum_{n=2}^{\infty} \prod_{j=1}^{n-1} f_j\Big(1 - \Big(\frac{1-p_j}{p_j}\Big)^{n-j}\Big) < \infty \implies \mathbf{Q}(K < \infty) = 1$$

our task is reduced to establishing the implication $\mathbf{Q}(K<\infty) = 1 \implies \mathbf{Q}(K=0) > 0$. Now recalling that (3.1.6) implies that if $\mathbf{Q}(K<\infty) = 1$ then $\exists\, L$ such that $\mathbf{Q}(V_L|\omega(L) = 0) > 0$, we find that $\mathbf{Q}(K=0) \ge \left(\frac{1-p_0}{p_0}\right)^L \mathbf{Q}(V_L|\omega(L)=0) > 0$ (where $\left(\frac{1-p_0}{p_0}\right)^L$ is the probability that the frog starting at $x=0$ in the F^+ model ever hits the point $x=L$), thus completing the final step of the proof. □

3.2 A simple proof of Gantert and Schmidt's result

In order to demonstrate the utility of Theorem 3.1.1, in this section we show how it can be used to obtain a simple two step proof of Gantert and Schmidt's result from [5] described in subsection 1.2.1. Recall that in the Gantert and Schmidt model the number of sleeping frogs at every non-zero vertex are i.i.d. copies of some random variable η. It begins with a single active frog at the origin and, once activated, a frog performs a random walk, independent of the other active frogs, that at each step goes left with probability p (where $\frac{1}{2} < p < 1$) and right with probability $1-p$. Their result stated that the model is recurrent if and only if $\mathbb{E}[\log^{+}\eta] = \infty$ (and otherwise it is transient).

Part 1 of my proof uses a method similar to Gantert and Schmidt's, while part 2 employs a more novel approach which simplifies matters considerably.

Part 1: WTS: $\mathbb{E}[\log^{+}\eta] = \infty \implies$ recurrence

Begin by defining the process $\{A_j\}$ where for every $j \in \mathbb{Z}/\{0\}$ A_j represents the number of distinct frogs originating at $x = j$ that ever hit the origin in the Gantert-Schmidt model. Next we define the triple $(\Omega^*, \mathcal{F}^*, \mathbf{P}^*)$ where Ω^* represents the set of functions $\omega : \mathbb{Z}/\{0\} \to \mathbb{N}$ (i.e. the possible trajectories of $\{A_j\}$), $\mathcal{F}^*$ represents the σ-field on Ω^* generated by the finite dimensional sets, and $\mathbf{P}^*$ represents the probability measure induced on $(\Omega^*, \mathcal{F}^*)$ by the process $\{A_j\}$. Additionally, denoting the two sided sequence $\{..., \eta_{-2}, \eta_{-1}, \eta_1, \eta_2, \dots\}$ that gives the number of sleeping frogs beginning at every nonzero vertex as H, we define the process $\left\{A_j^{(H)}\right\}$ in the same way as $\{A_j\}$, but where the number of sleeping frogs starting at each vertex is given by the terms of H. As with $\{A_j\}$, each such process can be identified with a triple $(\Omega^*, \mathcal{F}^*, \mathbf{P}_H^*)$, where $\mathbf{P}_H^*$ represents the probability measure that $\left\{A_j^{(H)}\right\}$ induces on $(\Omega^*, \mathcal{F}^*)$. Now since the activated frogs in this model all have nonzero leftward drift, this means all frogs that begin to the left of the origin are activated with probability 1. Hence, for $j \geq 1$ and $H = \{..., \eta_{-2}, \eta_{-1}, \eta_1, \eta_2, \dots\}$, we find that $\mathbf{P}_H^*(\omega(-j) > 0) = 1 - \left(1 - \left(\frac{1-p}{p}\right)^j\right)^{\eta_{-j}}$. Now defining $U(\omega) = \#\{j \in \mathbb{Z}^+ : \omega(-j) > 0\}$, noting that the random variables $\omega(-j)$ are independent with respect to $\mathbf{P}_H^*$, and noting that if $\eta_{-j} \geq \left(\frac{p}{1-p}\right)^j$ then $\mathbf{P}_H^*(\omega(-j) > 0) = 1 - \left(1 - \left(\frac{1-p}{p}\right)^j\right)^{\eta_{-j}} \geq 1 - e^{-1} > 0$, we see that the implication

$$\left\{\eta_{-j} \geq \left(\frac{p}{1-p}\right)^j \text{ i.o.}\right\} \implies \mathbf{P}_H^*(U = \infty) = 1 \tag{3.2.1}$$

follows from B.C. II. Furthermore, if we define $\Gamma = \left\{H \in (\eta_j)_{j \in \mathbb{Z}^*} : \eta_{-j} \geq \left(\frac{p}{1-p}\right)^j \text{ i.o.}\right\}$ and let μ represent the probability measure associated with $(\eta_j)_{j \in \mathbb{Z}^*}$, then since

$$\begin{aligned}\sum_{j=1}^{\infty} \mathbb{P}\left(\eta \geq \left(\frac{p}{1-p}\right)^j\right) &= \sum_{j=1}^{\infty} \mathbb{P}\left(\log^+\eta \geq j\log\left(\frac{p}{1-p}\right)\right) \geq \sum_{j=1}^{\infty} \mathbb{P}\left(\log^+\eta \geq j\left\lceil\log\left(\frac{p}{1-p}\right)\right\rceil\right) \\ &\geq \frac{1}{\left\lceil\log\left(\frac{p}{1-p}\right)\right\rceil}\left(\mathbb{E}[\log^+\eta] - \left\lceil\log\left(\frac{p}{1-p}\right)\right\rceil\right)\end{aligned}$$

we find that another application of B.C. II gives the implication $\mathbb{E}[\log^+\eta] = \infty \implies \mu(\Gamma) = 1$. Alongside (3.2.1), this establishes part 1.

Part 2: WTS: $\mathbb{E}[\log^+\eta] < \infty \implies$ transience

Choose a constant C such that $0 < C < 1$ and $C \cdot \frac{p}{1-p} > 1$. Noting that

$$\sum_{j=1}^{\infty} \mu\left(\eta_{-j} \geq C^j\left(\frac{p}{1-p}\right)^j\right) = \sum_{j=1}^{\infty} \mathbb{P}\left(\log^+\eta \geq j\log\left(\frac{Cp}{1-p}\right)\right) \leq \frac{1}{\log\left(\frac{Cp}{1-p}\right)}\mathbb{E}[\log^+\eta]$$

it follows from B.C. I that

$$\mathbb{E}[\log^+\eta] < \infty \implies \mu\left(\eta_{-j} \geq C^j\left(\frac{p}{1-p}\right)^j \text{ i.o.}\right) = 0 \tag{3.2.2}$$

In addition, since for $j \geq 1$ we have $\mathbf{P}^*_H(\omega(-j) > 0) = 1 - \left(1 - \left(\frac{1-p}{p}\right)^j\right)^{\eta_{-j}}$ (see line preceding (3.2.1)) and

$$1 - \left(1 - \left(\frac{1-p}{p}\right)^j\right)^{C^j\left(\frac{p}{1-p}\right)^j} = (1+o(1))C^j \text{ as } j \to \infty$$

we find that if $\eta_{-j} \geq C^j\left(\frac{p}{1-p}\right)^j$ at only finitely many points, then $\sum_{j=1}^{\infty} \mathbf{P}^*_H(\omega(-j) > 0) < \infty$. Now coupling this with (3.2.2) and employing B.C. I, we get (for $j \geq 1$)

$$\mathbb{E}[\log^+ \eta] < \infty \implies \mathbf{P}^*(\omega(-j) > 0 \text{ i.o.}) = 0 \tag{3.2.3}$$

Letting $\mathcal{A} = \sum_{j=1}^{\infty} \omega(-j)$, it follows from (3.2.3) that $\mathbb{E}[\log^+ \eta] < \infty \implies \mathbf{P}^*(\mathcal{A} < \infty) = 1$. If we now let $\mathcal{B} = \sum_{j=1}^{\infty} \omega(j)$, we find that in order to prove that $\mathbb{E}[\log^+ \eta] < \infty$ implies transience, it suffices to establish that for each k with $0 \leq k < \infty$ the following implication holds.

$$\mathbb{E}[\log^+ \eta] < \infty \implies \mathbf{P}^*(\mathcal{B} < \infty | \mathcal{A} = k) = 1 \tag{3.2.4}$$

Now note that in terms of whether or not $\mathcal{B} = \infty$, the only relevant detail regarding the frogs beginning to the left of the origin is how far the one(s) that travels the furthest to the right of the origin gets. Denoting this value as $\mathcal{C}$, if we assume $\mathbf{P}^*(\mathcal{B} = \infty) > 0$, then there would have to exist $r \geq 0$ such that $\mathbf{P}^*(\mathcal{B} = \infty | \mathcal{C} = r) > 0$. Since the frog beginning at the origin reaches the point $x = r$ with positive probability, it would follow that $\mathbf{P}^*(\mathcal{B} = \infty | \mathcal{A} = 0) > 0$. Hence, in order to establish (3.2.4), it suffices to establish the implication $\mathbb{E}[\log^+ \eta] < \infty \implies \mathbf{P}^*(\mathcal{B} < \infty | \mathcal{A} = 0) = 1$.

The next step is to observe that $\big(\mathcal{B} | \mathcal{A} = 0\big)$ has the same distribution as the number of distinct (initially sleeping) frogs that hit the origin in the non-homogeneous model on $\mathbb{Z}$ (in the case where $p_j = p$ for each $j \geq 0$ and the X_j's are i.i.d. copies of η). Using Theorem

3.1.1, it then follows that in order to establish that $\mathbb{E}[\log^+\eta] < \infty$ implies transience, it is sufficient to establish the implication

$$\mathbb{E}[\log^+\eta] < \infty \implies \sum_{n=2}^{\infty}\prod_{j=1}^{n-1} f\Big(1-\Big(\frac{1-p}{p}\Big)^j\Big) = \infty \tag{3.2.5}$$

(where f represents the probability generating function of η). Now noting that

$$\sum_{n=2}^{\infty}\prod_{j=1}^{n-1} f\Big(1-\Big(\frac{1-p}{p}\Big)^j\Big) = \mathbb{E}\big[\sum_{n=2}^{\infty} e^{\sum_{j=1}^{n-1}\log(1-(\frac{1-p}{p})^j)X_j}\big] \tag{3.2.6}$$

we observe that because $\log\big(1-\big(\frac{1-p}{p}\big)^j\big) = -(1+o(1))\big(\frac{1-p}{p}\big)^j$ as $j\to\infty$, it follows that if we have $0 < C < 1$ such that $\frac{Cp}{1-p} > 1$ and $X_j \le \Big(\frac{Cp}{1-p}\Big)^j$ for all but finitely many j, then

$$\sum_{n=2}^{\infty} e^{\sum_{j=1}^{n-1}\log(1-(\frac{1-p}{p})^j)X_j} = \infty$$

When coupled with (3.2.2) (where we replace η_{-j} with η_j on the right) and (3.2.6), this establishes (3.2.5) which, as we saw, indicates that the left side of (3.2.5) implies transience, thus completing the proof.

3.3 Applications of Theorem 3.1.1

3.3.1 Sharp conditions for the i.i.d. case

Having shown in the previous subsection how Theorem 3.1.1 can be used to obtain a concise proof of Gantert and Schmidt's result from [5], this subsection is devoted to establishing a new result that involves a model similar to the one from [5], but where the drifts of the individual frogs are dependent on where they originated (it will be assumed that no sleeping frogs reside to the left of the origin). This result comes in the form of the following theorem.

Theorem 3.3.1. *For any version of the nonhomogeneous frog model on $\mathbb{Z}$ for which the X_j's are i.i.d. with $\mathbb{E}[X_1] < \infty$, $p_j = \frac{1}{2} + a_j$ with $g(j) = \frac{1}{a_j}$ being concave, and $d = min\{j : \mathbb{P}(X_1 = j) > 0\}$, the model is transient if and only if $\sum_{n=1}^{\infty} \frac{e^{-\frac{\mathcal{K}}{4a_n}}}{(a_n)^{d/2}} = \infty$ (where f represents the generating function of X_j and $\mathcal{K} = -\int_0^{\infty} log[f(1-e^{-x})]dx$).*

Remark 1. Note that X_1 having finite first moment (as stated in the theorem) gives us

$$\mathbb{E}[X_1] < \infty \implies f'(1) = \mathbb{E}[X_1] < \infty \implies \log[f(1-e^{-x})] = -q \cdot e^{-x} + o(e^{-x}) \implies \mathcal{K} < \infty$$

(where $q = f'(1)$).

Remark 2. One noteworthy (and immediate) consequence of Theorem 3.3.1 is that for fixed f, $a_n = \frac{\mathcal{K}/4}{\text{log}n}$ (for all but finitely many n) represents a natural critical case in the sense that for $a_n = \frac{C}{\text{log}n}$ the model is transient if and only if $C \geq \mathcal{K}/4$. An instance of particular significance is the case where $X_j = 1 \; \forall \; j$ (i.e. each positive integer point begins with exactly one sleeping frog). Since in this scenario $f(x) = x$, we find that

$$\mathcal{K} = \int_0^{\infty} |\log(1-e^{-x})| dx = \int_0^{\infty} \sum_{n=1}^{\infty} \frac{e^{-nx}}{n} dx = \sum_{n=1}^{\infty} \int_0^{\infty} \frac{e^{-nx}}{n} dx = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

Hence, it follows that if $a_n = \frac{C}{\text{log}n}$, then the model is transient if and only if $C \geq \frac{\pi^2}{24}$, thus providing a new phase transition for the model from [2] that was mentioned in subsection 1.3.1 of the introduction.

Proof of Theorem 3.3.1. Given our result in Theorem 3.1.1, it follows that in order to establish this new result, it will suffice to show that

$$\sum_{n=1}^{\infty} \frac{e^{-\frac{\mathcal{K}}{4a_n}}}{(a_n)^{d/2}} = \infty \iff \sum_{n=2}^{\infty} \prod_{j=1}^{n-1} f\Big(1 - \Big(1 - \frac{4a_{n-j}}{1+2a_{n-j}}\Big)^j\Big) = \infty \tag{3.3.1}$$

(where the expression on the right in (3.3.1) is obtained by substituting $\frac{1}{2}+a_j$ for p_j and switching j and $n-j$ in (3.1.1)). Furthermore, if we define $w_n = \frac{4a_n}{1+2a_n}$ and note that

$$\frac{e^{-\frac{\mathcal{K}}{w_n}}}{(w_n)^{d/2}} \Bigg/ \frac{e^{-\frac{\mathcal{K}}{4a_n}}}{(a_n)^{d/2}} \to Ae^{-\frac{\mathcal{K}}{2}} \text{ as } n\to\infty \tag{3.3.2}$$

(where $A = \lim_{n\to\infty} \left(\frac{1+2a_n}{4}\right)^{d/2}$) we find that (3.3.1) is equivalent to the following:

$$\sum_{n=1}^{\infty} \frac{e^{-\frac{\mathcal{K}}{w_n}}}{(w_n)^{d/2}} = \infty \iff \sum_{n=2}^{\infty}\prod_{j=1}^{n-1} f(1-(1-w_{n-j})^j) = \infty \tag{3.3.3}$$

We'll first establish (3.3.1) (via (3.3.3)) under the condition that a_n^{-1} is $O\big(\sqrt{n}\big)$ (see steps (i)-(iv)), following which we'll address the general case.

(i) $\sum_{n=2}^{\infty}\prod_{j=1}^{n-1} f(1-(1-w_n)^j) = \infty \iff \sum_{n=2}^{\infty}\prod_{j=1}^{n-1} f(1-(1-w_{n-j})^j) = \infty$

Since a_n is decreasing this means w_n is as well, from which it follows that

$$\prod_{j=1}^{n-1} f(1-(1-w_{n-j})^j) \geq \prod_{j=1}^{n-1} f(1-(1-w_n)^j)$$

for all n. Hence, in order to establish (i) it suffices to show that

$$\operatorname{limsup} \sum_{j=1}^{n-1} \log[f(1-(1-w_{n-j})^j)] - \log[f(1-(1-w_n)^j)] < \infty \tag{3.3.4}$$

Rewriting the expression in (3.3.4) (see below) we now get the inequality

$$\operatorname{limsup} \sum_{j=1}^{n-1} \log[f(1-(1-w_n)^j + \big((1-w_n)^j - (1-w_{n-j})^j\big))] - \log[f(1-(1-w_n)^j)] \tag{3.3.5}$$

$$\leq \operatorname{limsup} \sum_{j=1}^{n-1} \Big(\log[f(1-(1-w_n)^j + \big((j\cdot(w_{n-j}-w_n)) \wedge (1-w_n)\big)\cdot(1-w_n)^{j-1})] \\ -\log[f(1-(1-w_n)^j)]\Big)$$

Since a_n^{-1} being $O\big(\sqrt{n}\big)$ implies w_n^{-1} is as well, this means the larger expression in (3.3.5) is equal to

$$\operatorname*{limsup}_{n-1>\frac{1}{w_n}} \sum_{j=1}^{n-1} \Big(\log[f(1-(1-w_n)^j + \big((j\cdot(w_{n-j}-w_n)) \wedge (1-w_n)\big)\cdot(1-w_n)^{j-1})] \tag{3.3.6}$$

$$-\log[f(1-(1-w_n)^j)]\Big)$$

$$\leq \limsup_{n-1>\frac{1}{w_n}} \sum_{j\leq\frac{1}{w_n}} \Big(\log[f(1-(1-w_n)^j + \big((j\cdot(w_{n-j}-w_n))\wedge(1-w_n)\big)\cdot(1-w_n)^{j-1})]$$

$$-\log[f(1-(1-w_n)^j)]\Big) + \limsup_{n-1>\frac{1}{w_n}} \sum_{\frac{1}{w_n}<j\leq n-1} \frac{q\cdot(w_{n-j}-w_n)\cdot j\cdot(1-w_n)^{j-1}}{f(1-(1-w_n)^{\frac{1}{w_n}})}$$

(recall that $q = f'(1)$). The second term to the right of the inequality in (3.3.6) can now be bounded above by

$$\frac{q}{f(1-e^{-1})} \limsup_{n-1>\frac{1}{w_n}} \sum_{\frac{1}{w_n}<j\leq n-1} (w_{n-j}-w_n)\cdot j\cdot(1-w_n)^{j-1} \tag{3.3.7}$$

$$\leq \frac{q}{f(1-e^{-1})} \limsup_{n-1>\frac{1}{w_n}} \sum_{\frac{1}{w_n}<j\leq n-1} \left(1-\frac{w_n}{w_{n-j}}\right)\cdot\frac{w_{n-j}}{w_n}\cdot w_n\cdot j\cdot e^{-w_n(j-1)}$$

$$\leq \frac{q\cdot e}{f(1-e^{-1})} \limsup_{n-1>\frac{1}{w_n}} \sum_{\frac{1}{w_n}<j\leq n-1} \frac{1}{w_n\cdot(n-j)}\cdot(w_n\cdot j)^2\cdot e^{-w_n j}$$

(where the final inequality in (3.3.7) follows from the fact that $\frac{w_{n-j}}{w_n} \leq \frac{n}{n-j}$, which follows from the concavity of $\frac{1}{w_n}$, which in turn follows from the concavity of $\frac{1}{a_n}$). Next we bound the final term in (3.3.7) by

$$\frac{q\cdot e}{f(1-e^{-1})\cdot\liminf nw_n^2} \limsup_{n-1>\frac{1}{w_n}} \sum_{\frac{1}{w_n}<j\leq n-1} \frac{1}{1-\frac{j}{n}}\cdot w_n\cdot(w_n j)^2\cdot e^{-w_n j} \tag{3.3.8}$$

$$\leq \frac{q\cdot e}{f(1-e^{-1})\cdot\liminf nw_n^2} \limsup_{n-1>\frac{1}{w_n}} \sum_{\frac{1}{w_n}<j\leq n-1} w_n\cdot(w_n j)^2\cdot e^{\frac{-3w_n j}{4}}$$

(with the last inequality following from the fact, implied by w_n^{-1} being $O\big(\sqrt{n}\big)$, that for sufficiently large n we have $\frac{1}{1-\frac{j}{n}} \leq e^{n^{-\frac{2}{3}}j} \leq e^{\frac{w_n j}{4}}$ for $1 \leq j \leq n-1$). Finally, comparing the last sum to the integral of $x^2 e^{-\frac{3}{4}x}$, we see there must exist $K < \infty$ (independent of n) such that the sum is bounded above by $\int_0^\infty x^2 e^{\frac{-3}{4}x}dx + K$. Combining this with w_n^{-1} being $O(\sqrt{n})$ then implies that the bottom expression in (3.3.8) is finite which, coupled with (3.3.7) and (3.3.8), establishes that the second term on the right of the inequality in (3.3.6) is finite as well.

To complete the proof of (i), it now just needs to be shown that the first term on the right of the inequality in (3.3.6) is finite as well. Now because for any probability generating function of a non negative integer valued random variable with finite mean $\frac{f'(x)}{f(x)}$ is $O\big(\frac{1}{x}\big)$, this means there must exist a constant $C < \infty$ such that $\frac{f'(x)}{f(x)} \leq \frac{C}{x}$ $\forall\, x \in\ (0,1]$, from which it follows that the term in question is bounded above by

$$\limsup_{n-1>\frac{1}{w_n}} \sum_{j\leq\frac{1}{w_n}} \frac{C\cdot(w_{n-j}-w_n)\cdot j\cdot(1-w_n)^{j-1}}{1-(1-w_n)^j} \tag{3.3.9}$$

Next noting that for $x \in (0,1]$ and $m \in \mathbb{Z}^+$ we have $\frac{1-(1-x)^m}{mx} = \frac{1}{m}\big(1+(1-x)+\cdots+(1-x)^{m-1}\big) \geq (1-x)^{m-1}$, it follows that (3.3.9) can be bounded above by

$$C\cdot\limsup_{n-1>\frac{1}{w_n}} \sum_{j\leq\frac{1}{w_n}} \frac{(w_{n-j}-w_n)}{w_n} = C\cdot\limsup_{n-1>\frac{1}{w_n}} \sum_{j\leq\frac{1}{w_n}} \left(\frac{1}{w_n}-\frac{1}{w_{n-j}}\right)\cdot w_{n-j}$$

On account of the concavity of $\frac{1}{w_n}$, this last expression can itself be bounded above by

$$C\cdot\limsup_{n-1>\frac{1}{w_n}} \sum_{j\leq\frac{1}{w_n}} \frac{j}{n}\cdot\frac{1}{w_n}\cdot\frac{n}{n-j}\cdot w_n = C\cdot\limsup_{n-1>\frac{1}{w_n}} \sum_{j\leq\frac{1}{w_n}} \frac{j}{n-j} = \frac{C}{2}\cdot\limsup_{n-1>\frac{1}{w_n}} \frac{1}{n\cdot w_n^2} < \infty$$

(where the second equality along with the finiteness of the last term both follow from the fact that w_n^{-1} is $O\big(\sqrt{n}\big)$). Hence, this establishes that (3.3.9), as well as the first term to the right of the inequality in (3.3.6), is finite. Now if this is combined with the finiteness of the second expression to the right of the inequality in (3.3.6), along with the inequality in (3.3.5), we see that (3.3.4) follows, thus completing the proof of (i).

(ii) $\sum_{n=2}^{\infty}\prod_{j=1}^{n-1} f(1-e^{-w_n\cdot j}) = \infty \iff \sum_{n=2}^{\infty}\prod_{j=1}^{n-1} f(1-(1-w_n)^j) = \infty$

Because we know that

$$1-w_n \leq e^{-w_n} \implies \sum_{n=2}^{\infty}\prod_{j=1}^{n-1} f(1-e^{-w_n\cdot j}) \leq \sum_{n=2}^{\infty}\prod_{j=1}^{n-1} f(1-(1-w_n)^j)$$

it follows that in order to establish (ii), it suffices to show (much like in the case of (i)) that

$$\limsup \sum_{j=1}^{n-1} \log[f(1-(1-w_n)^j)] - \log[f(1-e^{-w_n \cdot j})] < \infty \tag{3.3.10}$$

Defining $C_n = \frac{e^{-w_n}-(1-w_n)}{w_n^2}$, we have the following string of inequalities (where the expression on the first line equals the expression in (3.3.10), and with $S(n,j)$ representing the summand on the second line).

$$\limsup \sum_{j=1}^{n-1} \log[f(1-e^{-w_n \cdot j} + \big((1-w_n+C_n w_n^2)^j - (1-w_n)^j\big)] - \log[f(1-e^{-w_n \cdot j})] \tag{3.3.11}$$

$$\leq \limsup \sum_{j=1}^{n-1} \log[f(1-e^{-w_n \cdot j} + \big((j \cdot C_n \cdot w_n^2) \wedge (1-w_n)\big) \cdot e^{-w_n(j-1)})] - \log[f(1-e^{-w_n \cdot j})]$$

$$\leq \limsup_{n-1>\frac{1}{w_n}} \sum_{j \leq \frac{1}{w_n}} S(n,j) + \limsup_{n-1>\frac{1}{w_n}} \sum_{\frac{1}{w_n}<j\leq n-1} S(n,j)$$

If we can show that both of the expressions on the last line of (3.3.11) are finite, then (3.3.10) will immediately follow. Beginning with the first expression, observe that if we use the fact (referenced in the proof of (i)) that there must exist $C < \infty$ such that $\frac{f'(x)}{f(x)} \leq \frac{C}{x}$ $\forall\, x \in (0,1]$, then we can obtain the string of inequalities

$$\begin{aligned} \limsup_{n-1>\frac{1}{w_n}} \sum_{j \leq \frac{1}{w_n}} S(n,j) &\leq \limsup_{n-1>\frac{1}{w_n}} \sum_{j \leq \frac{1}{w_n}} \frac{C \cdot C_n \cdot j \cdot w_n^2 \cdot e^{-w_n(j-1)}}{1-e^{-w_n \cdot j}} \\ &\leq \frac{C}{2} \limsup_{n-1>\frac{1}{w_n}} \sum_{j \leq \frac{1}{w_n}} \frac{j \cdot w_n^2}{1-e^{-w_n \cdot j}} \end{aligned} \tag{3.3.12}$$

(where the second inequality follows from the fact that $C_n \leq \frac{1}{2}$ $\forall\, n$). Now using the fact that

$$1-e^{-w_n \cdot j} = \left(1-e^{-w_n}\right) \cdot \left(1+e^{-w_n} + \cdots + \left(e^{-w_n}\right)^{j-1}\right) \geq j \cdot \left(1-e^{-w_n}\right) \cdot \left(e^{-w_n}\right)^{j-1}$$

and that $\frac{1-e^{-w_n}}{w_n} \geq 1-e^{-1}$ (since $0 < w_n < 1$ $\forall\, n$), we find that the expression on the second line in (3.3.12) is bounded above by

$$\frac{C}{2(1-e^{-1})} \limsup_{n-1>\frac{1}{w_n}} \sum_{j\leq\frac{1}{w_n}} \frac{j\cdot w_n^2}{j\cdot w_n\cdot e^{-1}} = \frac{C\cdot e}{2(1-e^{-1})} \limsup_{n-1>\frac{1}{w_n}} \sum_{j\leq\frac{1}{w_n}} w_n \leq \frac{C\cdot e}{2(1-e^{-1})} < \infty$$

thus establishing that the first sum on the last line of (3.3.11) is finite.

In order to establish (3.3.10), and thus complete the proof of (ii), it only remains to show that the second sum on the last line of (3.3.11) is finite as well. We accomplish this via the following string of inequalities:

$$\begin{aligned}\limsup_{n-1>\frac{1}{w_n}} \sum_{\frac{1}{w_n}<j\leq n-1} S(n,j) &\leq \frac{C}{2(1-e^{-1})} \limsup_{n-1>\frac{1}{w_n}} \sum_{\frac{1}{w_n}<j<\infty} w_n\cdot(w_n j)\cdot e^{-w_n(j-1)} \\ &\leq \frac{C\cdot e}{2(1-e^{-1})}\int_0^\infty x\cdot e^{-x}dx + K\end{aligned}$$

(where the first inequality follows from the same argument used in (3.3.12)). Hence, the proof of (ii) is complete.

(iii) $\sum_{n=2}^{\infty}\prod_{j=1}^{\infty} f(1-e^{-w_n\cdot j}) = \infty \iff \sum_{n=2}^{\infty}\prod_{j=1}^{n-1} f(1-e^{-w_n\cdot j}) = \infty$

Since one direction is immediate, we're left with just having to show that

$$\limsup \sum_{j=n}^{\infty} -\log[f(1-e^{-w_n\cdot j})] < \infty \tag{3.3.13}$$

Observing that

$$\begin{aligned}\limsup \sum_{j=n}^{\infty} -\log[f(1-e^{-w_n\cdot j})] &= \limsup \frac{1}{w_n}\sum_{j=n}^{\infty} -w_n\log[f(1-e^{-w_n\cdot j})] \\ &\leq \limsup \frac{1}{w_n}\int_{(n-1)\cdot w_n}^{\infty} -\log[f(1-e^{-x})]dx\end{aligned}$$

we find that, as a consequence of the fact that $f'(1) = q < \infty$ and w_n^{-1} is $O(\sqrt{n})$, we have

$$\limsup \frac{1}{w_n}\int_{(n-1)\cdot w_n}^{\infty} -\log[f(1-e^{-x})]dx = \limsup \frac{1}{w_n}\int_{(n-1)\cdot w_n}^{\infty} q\cdot e^{-x}dx$$

$$= \limsup \frac{1}{w_n} \cdot q \cdot e^{-(n-1)\cdot w_n} \leq \limsup \frac{\sqrt{n}}{\sqrt{l}} \cdot q \cdot e^{-\frac{n-1}{n} \cdot \sqrt{n} \cdot \sqrt{l}} = 0$$

(where l denotes the value of $\liminf n \cdot w_n^2$). Hence, this establishes (3.3.13), thus completing the proof of (iii).

(iv) $\sum_{n=1}^{\infty} \frac{e^{\frac{-\mathcal{K}}{w_n}}}{(w_n)^{d/2}} = \infty \iff \sum_{n=2}^{\infty} \prod_{j=1}^{\infty} f(1 - e^{-w_n \cdot j}) = \infty$

Denoting $c_d = \mathbb{P}(X_1 = d)$ (recall $d = \min\{j : \mathbb{P}(X_1 = j) > 0\}$), observe that

$$\begin{aligned} \frac{d(\log[f(x)])}{dx} &= \frac{f'(x)}{f(x)} = \frac{dc_d + (d+1)c_{d+1}x + \dots}{c_d x + c_{d+1}x^2 + \dots} \\ &= \frac{d}{x} \cdot \frac{1 + \frac{d+1}{d}\frac{c_{d+1}}{c_d}x + \dots}{1 + \frac{c_{d+1}}{c_d}x + \dots} = \frac{d}{x} + O(1) \end{aligned} \tag{3.3.14}$$

Now we want to approximate

$$\frac{-\mathcal{K}}{w_n} - \log\Big[\prod_{j=1}^{\infty} f(1 - e^{-w_n \cdot j})\Big] = \frac{1}{w_n}\int_0^{\lfloor \frac{1}{w_n} \rfloor w_n} \log[f(1 - e^{-x})]dx \tag{3.3.15}$$

$$-\frac{1}{w_n}\sum_{j=1}^{\lfloor \frac{1}{w_n} \rfloor} w_n \log[f(1-e^{-w_n \cdot j})] + \frac{1}{w_n}\int_{\lfloor \frac{1}{w_n} \rfloor w_n}^{\infty} \log[f(1-e^{-x})]dx - \frac{1}{w_n}\sum_{\lceil \frac{1}{w_n} \rceil}^{\infty} w_n \log[f(1-e^{-w_n \cdot j})]$$

within an order of $O(1)$. First noting that the expression on the second line of (3.3.15) is $O(1)$ as $n \to \infty$ (this follows from the fact that it is bounded above by 0 and below by $\log\Big[f\Big(1 - e^{-w_n \cdot \lfloor \frac{1}{w_n} \rfloor}\Big)\Big]$), we see that our task is reduced to approximating

$$\frac{1}{w_n}\int_0^{\lfloor \frac{1}{w_n} \rfloor w_n} \log[f(1 - e^{-x})]dx - \frac{1}{w_n}\sum_{j=1}^{\lfloor \frac{1}{w_n} \rfloor} w_n \cdot \log[f(1 - e^{-w_n \cdot j})] \tag{3.3.16}$$

$$= \frac{1}{w_n}\sum_{j=2}^{\lfloor \frac{1}{w_n} \rfloor} \int_0^{w_n} \log[f(1 - e^{-(w_n \cdot j - t)})] - \log[f(1 - e^{-w_n \cdot j})]dt + O(1)$$

(where the $O(1)$ term represents $\frac{1}{w_n}\int_0^{w_n} \log[f(1-e^{-x})]dx - \log[f(1-e^{-w_n})]$). Using (3.3.14), we then find that the integrand in the bottom expression equals

$$-\int_{1-e^{-(w_n \cdot j - t)}}^{1-e^{-w_n \cdot j}} \frac{d}{x} + O(1)dx = d\log\Big[\frac{1 - e^{-(w_n \cdot j - t)}}{1 - e^{-w_n \cdot j}}\Big] + O(e^{-(w_n \cdot j - t)} - e^{-w_n \cdot j})$$

$$= d\log\Big[\frac{1-e^{-(w_n\cdot j-t)}}{1-e^{-w_n\cdot j}}\Big] + O(t) = d\log\left[1+\frac{e^{-w_n\cdot j}(1-e^t)}{1-e^{-w_n\cdot j}}\right] + O(t)$$

$$= d\log\Big[1-\frac{t}{w_n\cdot j}+O(t)\Big] + O(t) = d\log\Big[1-\frac{t}{w_n\cdot j}\Big] + O(t)$$

(with the final equality following from the fact that $n \geq 2 \implies 1 - \frac{t}{w_n\cdot j} \geq \frac{1}{2} > 0\ \forall\ t$).

Plugging this back into the expression on the second line of (3.3.16) now gives

$$\frac{1}{w_n}\sum_{j=2}^{\lfloor\frac{1}{w_n}\rfloor}\int_0^{w_n} d\log\Big[1-\frac{t}{w_n\cdot j}\Big] + O(t)dt = \frac{d}{w_n}\sum_{j=2}^{\lfloor\frac{1}{w_n}\rfloor} -(j-1)\cdot w_n\cdot\log\Big[1-\frac{1}{j}\Big] - w_n + O(w_n^2)$$

$$= d\sum_{j=2}^{\lfloor\frac{1}{w_n}\rfloor} -(j-1)\cdot\log\Big[1-\frac{1}{j}\Big]-1+O(w_n) = d\sum_{j=2}^{\lfloor\frac{1}{w_n}\rfloor}\frac{-1}{2j}+O\Big(\frac{1}{j^2}\Big)+O(w_n) = \frac{-d}{2}\log\Big[\frac{1}{w_n}\Big]+O(1)$$

(where the $O(t)$ expressions indicate that the absolute value of the term in question is bounded above by ct for some $c < \infty$ that is independent of both n and t). Looking back now at the first line of (3.3.15), we find that

$$\frac{-\mathcal{K}}{w_n} - \log\Big[\prod_{j=1}^{\infty} f(1-e^{-w_n\cdot j})\Big] = \frac{-d}{2}\log\Big[\frac{1}{w_n}\Big] + O(1)$$

$$\implies C_1\cdot\frac{e^{\frac{-\mathcal{K}}{w_n}}}{(w_n)^{d/2}} \leq \prod_{j=1}^{\infty} f(1-e^{-w_n\cdot j}) \leq C_2\cdot\frac{e^{\frac{-\mathcal{K}}{w_n}}}{(w_n)^{d/2}}$$

(for some $0 < C_1 < C_2 < \infty$ independent of n), thus completing the proof of (iv).

Having now established (3.3.1) via (i)-(iv) when a_n^{-1} is $O(\sqrt{n})$, our final task is to address the general case. To do this, we first note that because the proof of (iv) does not use that a_n^{-1} is $O(\sqrt{n})$, it follows that it continues to hold without this assumption. Coupling this with (3.3.2), along with the fact that

$$\sum_{n=2}^{\infty}\prod_{j=1}^{\infty} f(1-e^{-w_n\cdot j}) \leq \sum_{n=2}^{\infty}\prod_{j=1}^{n-1} f(1-e^{-w_n\cdot j})$$

$$\leq \sum_{n=2}^{\infty}\prod_{j=1}^{n-1} f(1-(1-w_n)^j) \leq \sum_{n=2}^{\infty}\prod_{j=1}^{n-1} f(1-(1-w_{n-j})^j)$$

we find that the implication going from left to right in (3.3.1) holds regardless of whether or not a_n^{-1} is $O\big(\sqrt{n}\big)$. Hence, to complete the proof of the theorem we simply need to show that when a_n^{-1} is not $O\big(\sqrt{n}\big)$, finiteness of the expression on the left side of (3.3.1), still implies finiteness of the expression on the right.

If we define the sequence $\tilde{a}_n$ so that

$$\frac{1}{\tilde{a}_n} = \begin{cases} \frac{1}{a_n} & \text{if } \frac{1}{a_n} < 3\sqrt{n} \\ 3\sqrt{n} & \text{otherwise} \end{cases}$$

it then follows that $\frac{1}{\tilde{a}_n}$ is concave and $O\big(\sqrt{n}\big)$ (also note $\frac{1}{2} < \frac{1}{2} + \tilde{a}_n < 1$ still holds). In addition, since we're assuming that the expression on the left side of (3.3.1) is finite, this means

$$\sum_{n=1}^{\infty} \frac{e^{\frac{-\mathcal{K}}{4\tilde{a}_n}}}{(\tilde{a}_n)^{d/2}} \leq \sum_{n=1}^{\infty} \frac{e^{\frac{-\mathcal{K}}{4a_n}}}{(a_n)^{d/2}} + \sum_{n=1}^{\infty} e^{\frac{-\mathcal{K}\cdot 3\sqrt{n}}{4}} \cdot 3^{d/2} \cdot n^{d/4} < \infty$$

Hence, the proof of (3.3.1), for the case where a_n^{-1} is $O\big(\sqrt{n}\big)$, implies that

$$\sum_{n=2}^{\infty} \prod_{j=1}^{n-1} f\Big(1 - \Big(1 - \frac{4\tilde{a}_{n-j}}{1 + 2\tilde{a}_{n-j}}\Big)^j\Big) < \infty$$

Coupling this with the fact that $a_n \leq \tilde{a}_n$, we can now conclude that

$$\sum_{n=2}^{\infty} \prod_{j=1}^{n-1} f\Big(1 - \Big(1 - \frac{4a_{n-j}}{1 + 2a_{n-j}}\Big)^j\Big) < \infty$$

which, along with the argument in the previous paragraph, establishes that (3.3.1) continues to hold when a_n^{-1} is not $O\big(\sqrt{n}\big)$. Hence, the proof of the theorem is complete. □

3.3.2 Sharp conditions for the Poiss(λ_j) scenario

In this section we'll address the non-uniform frog model with drift on $\mathbb{Z}$ (see Section 2.2), establishing sharp conditions for the case where the drift values of individual frogs are dependent on where they originate. The result is as follows.

Theorem 3.3.2. *For $X_j = Poiss(\lambda_j)$ and $p_j = \frac{1}{2} + a_j$ (with the sequences $\frac{1}{a_j}$ and λ_j both being concave), the nonhomogeneous frog model on $\mathbb{Z}$ is transient if and only if*

$$\sum_{n=1}^{\infty} e^{-\lambda_n\left(\frac{1}{4a_n}-\frac{1}{2}\right)} = \infty \tag{3.3.17}$$

Proof. Since Poiss(λ_j) has generating function $e^{\lambda_j(x-1)}$, applying Theorem 3.1.1 reduces our task to showing that

$$\sum_{n=1}^{\infty} e^{-\lambda_n\left(\frac{1}{4a_n}-\frac{1}{2}\right)} = \infty \iff \sum_{n=2}^{\infty} e^{-\sum_{j=1}^{n-1}\lambda_{n-j}\left(1-\frac{4a_{n-j}}{1+2a_{n-j}}\right)^j} = \infty$$

Noting also that

$$\sum_{n=2}^{\infty} e^{-\lambda_n\left(\frac{1}{4a_n}-\frac{1}{2}\right)} = \sum_{n=2}^{\infty} e^{-\sum_{j=1}^{\infty}\lambda_n\left(1-\frac{4a_n}{1+2a_n}\right)^j}$$
$$\leq \sum_{n=2}^{\infty} e^{-\sum_{j=1}^{n-1}\lambda_n\left(1-\frac{4a_n}{1+2a_n}\right)^j} \leq \sum_{n=2}^{\infty} e^{-\sum_{j=1}^{n-1}\lambda_{n-j}\left(1-\frac{4a_{n-j}}{1+2a_{n-j}}\right)^j}$$

we see that it will in fact suffice to establish the implication

$$\sum_{n=1}^{\infty} e^{-\lambda_n\left(\frac{1}{4a_n}-\frac{1}{2}\right)} < \infty \implies \sum_{n=2}^{\infty} e^{-\sum_{j=1}^{n-1}\lambda_{n-j}\left(1-\frac{4a_{n-j}}{1+2a_{n-j}}\right)^j} < \infty \tag{3.3.18}$$

To do this we'll begin by proving (3.3.18) for the case where λ_n and a_n^{-1} are both $O\big(n^{1/3}\big)$. Much like with the proof of Theorem 3.3.1, this will be accomplished by showing that

$$\limsup\ \lambda_n\Big(\frac{1}{4a_n}-\frac{1}{2}\Big) - \sum_{j=1}^{n-1}\lambda_{n-j}\Big(1-\frac{4a_{n-j}}{1+2a_{n-j}}\Big)^j < \infty \tag{3.3.19}$$

As a first step towards establishing (3.3.19), we observe the following string of inequalities (with ϵ_j denoting $\frac{a_j}{1+2a_j}$).

$$\limsup \sum_{j=1}^{n-1}\lambda_n(1-4\epsilon_{n-j})^j - \lambda_{n-j}(1-4\epsilon_{n-j})^j = \limsup \sum_{j=1}^{n-1}(\lambda_n-\lambda_{n-j})\cdot(1-4\epsilon_{n-j})^j \tag{3.3.20}$$

$$\leq \limsup \sum_{j=1}^{n-1}(\lambda_n-\lambda_{n-j})\cdot(1-4\epsilon_n)^j \leq \limsup \sum_{j=1}^{n-1}\frac{j}{n}\cdot\lambda_n\cdot e^{-4j\epsilon_n}$$

$$\leq \text{limsup} \frac{\lambda_n/\epsilon_n^2}{n} \sum_{j=1}^{\infty} \epsilon_n \cdot (\epsilon_n j) \cdot e^{-4j\epsilon_n} < \infty$$

(where the inequality between the two sums on the second line of (3.3.20) follows from the fact that λ_j is concave and $(1 - 4\epsilon_n)^j \leq e^{-4j\epsilon_n}$, and the finiteness of the last expression derives from the fact that λ_n and ϵ_n^{-1} are both $O\big(n^{1/3}\big)$, along with the fact that the sum is bounded above by $\int_0^\infty xe^{-4x}dx + K$ for some $K < \infty$). Next, we present another string of inequalities as shown.

$$\text{limsup} \sum_{j=1}^{n-1} \lambda_n(1-4\epsilon_n)^j - \sum_{j=1}^{n-1} \lambda_n(1-4\epsilon_{n-j})^j = \text{limsup}\lambda_n \sum_{j=1}^{n-1}(1-4\epsilon_n)^j - (1-4\epsilon_{n-j})^j \quad (3.3.21)$$

$$\leq \text{limsup } 4\lambda_n \sum_{j=1}^{n-1}(\epsilon_{n-j}-\epsilon_n)\cdot j\cdot(1-4\epsilon_n)^{j-1} = \text{limsup } 4\lambda_n \sum_{j=1}^{n-1}(\epsilon_n^{-1}-\epsilon_{n-j}^{-1})\cdot\epsilon_n\epsilon_{n-j}\cdot j\cdot(1-4\epsilon_n)^{j-1}$$

Because ϵ_n^{-1} is concave (since it equals $a_n^{-1}+2$), it follows that the expression on the second line of (3.3.21) is less than or equal to

$$\text{limsup } 4\lambda_n \sum_{j=1}^{n-1} \frac{j}{n} \cdot \epsilon_{n-j} \cdot j \cdot (1 - 4\epsilon_n)^{j-1} \leq \text{limsup } 4\lambda_n \sum_{j=1}^{n-1} \epsilon_{n-j} \cdot \frac{j^2}{n} \cdot e^{-4(j-1)\epsilon_n}$$

$$\leq \text{limsup } 4e\lambda_n \sum_{j=1}^{n-1} \frac{\epsilon_n}{1 - \frac{j}{n}} \cdot \frac{j^2}{n} \cdot e^{-4j\epsilon_n} = \text{limsup } \frac{4e\lambda_n/\epsilon_n^2}{n} \sum_{j=1}^{n-1} \frac{1}{1 - \frac{j}{n}} \cdot (j\epsilon_n)^2 \cdot e^{-4j\epsilon_n}\epsilon_n$$

$$\leq \text{limsup } \frac{4e\lambda_n/\epsilon_n^2}{n} \sum_{j=1}^{\infty} \epsilon_n \cdot (\epsilon_n j)^2 \cdot e^{-3j\epsilon_n} < \infty$$

(where the inequality on the second line follows from the fact that for sufficiently large n we have $\frac{1}{1-\frac{j}{n}} < e^{j\epsilon_n}$ for all j with $1 \leq j < n$, and where the finiteness of the last term follows from λ_n and ϵ_n^{-1} both being $O\big(n^{1/3}\big)$, along with the fact that the sum is once again bounded above by $\int_0^\infty x^2e^{-3x}dx + K$ for some $K < \infty$). Combining this last string of inequalities with (3.3.21), we see that

$$\text{limsup} \sum_{j=1}^{n-1} \lambda_n(1 - 4\epsilon_n)^j - \sum_{j=1}^{n-1} \lambda_n(1 - 4\epsilon_{n-j})^j < \infty \quad (3.3.22)$$

Finally, we observe that

$$\text{limsup}\ \lambda_n\Big(\frac{1}{4a_n}-\frac{1}{2}\Big)-\sum_{j=1}^{n-1}\lambda_n(1-4\epsilon_n)^j=\text{limsup}\sum_{j=1}^{\infty}\lambda_n(1-4\epsilon_n)^j-\sum_{j=1}^{n-1}\lambda_n(1-4\epsilon_n)^j \quad (3.3.23)$$

$$=\text{limsup}\sum_{j=n}^{\infty}\lambda_n(1-4\epsilon_n)^j=\text{limsup}\lambda_n\cdot\frac{(1-4\epsilon_n)^n}{4\epsilon_n}\leq\text{limsup}\frac{\lambda_n}{4\epsilon_n}\cdot e^{-4n\epsilon_n}=0$$

(where the last equality again follows from λ_n and ϵ_n^{-1} both being $O\big(n^{1/3}\big)$). Now putting (3.3.20), (3.3.22), and (3.3.23) together, we see that (3.3.19) (and therefore (3.3.18)) does indeed hold if λ_n and a_n^{-1} are $O\big(n^{1/3}\big)$.

To complete the proof of the theorem, (3.3.18) now just needs to be proven for the general case (i.e. without the condition that λ_n and a_n^{-1} are $O\big(n^{1/3}\big)$). To do this we begin by defining $\tilde{\lambda}_n$ and $\tilde{a}_n$ as

$$\tilde{\lambda}_n=\begin{cases}\lambda_n & \text{if } \lambda_n<n^{1/3}\\ n^{1/3} & \text{otherwise}\end{cases}$$

and

$$\frac{1}{\tilde{a}_n}=\begin{cases}\frac{1}{a_n} & \text{if } \frac{1}{a_n}<3n^{1/3}\\ 3n^{1/3} & \text{otherwise}\end{cases}$$

(again the coefficient 3 has been chosen so that $\frac{1}{2}<\frac{1}{2}+\tilde{a}_n<1\ \forall\ n$). Now noting that

$$\sum_{n=1}^{\infty}e^{-\tilde{\lambda}_n\left(\frac{1}{4\tilde{a}_n}-\frac{1}{2}\right)}\leq\left(\sum_{n=1}^{\infty}e^{-n^{1/3}\left(\frac{3n^{1/3}}{4}-\frac{1}{2}\right)}+\sum_{n=1}^{\infty}e^{-n^{1/3}\left(\frac{1}{4a_n}-\frac{1}{2}\right)}\right.$$
$$\left.+\sum_{n=1}^{\infty}e^{-\lambda_n\left(\frac{3n^{1/3}}{4}-\frac{1}{2}\right)}+\sum_{n=1}^{\infty}e^{-\lambda_n\left(\frac{1}{4a_n}-\frac{1}{2}\right)}\right)<\infty$$

(where the finiteness of the middle two sums on the right of the inequality follows from the fact that $a_n<\frac{1}{2}$ and $\lambda_n>0\ \forall\ n\geq 1$), it follows from the proof of (3.3.18) for the case where λ_n and a_n^{-1} are $O\big(n^{1/3}\big)$, that

$$\sum_{n=2}^{\infty}e^{-\sum_{j=1}^{n-1}\lambda_{n-j}\left(1-\frac{4a_{n-j}}{1+2a_{n-j}}\right)^j}\leq\sum_{n=2}^{\infty}e^{-\sum_{j=1}^{n-1}\tilde{\lambda}_{n-j}\left(1-\frac{4\tilde{a}_{n-j}}{1+2\tilde{a}_{n-j}}\right)^j}<\infty$$

(where the first inequality follows from the fact that $\tilde{\lambda}_j \leq \lambda_j$ and $\tilde{a}_j^{-1} \leq a_j^{-1}$). Hence, this establishes (3.3.18) for the general case, and thus completes the proof of the theorem. $\square$

Chapter 4

The frog model on trees

4.1 Recurrence on $\mathbb{T}_{3,2}$

As noted in the introduction, the frog model on $\mathbb{T}_{3,2}$ features one sleeping frog at every non-root vertex, and a single active frog starting at the root. Once activated, a frog performs an unbiased random walk on the tree, independent of those performed by the other active frogs. The main result that will be presented for this model is the following theorem.

Theorem 4.1.1. *The frog model on* $\mathbb{T}_{3,2}$ *is recurrent.*

Since the proof of this result centers around modifying the approach used by Hoffman, Johnson, and Junge in [7] to prove recurrence on $\mathbb{T}_2$, I will begin by providing a brief outline of their proof. They begin by constructing a new self-similar model on $\mathbb{T}_2$ (the 2-ary tree), by having active particles perform non-backtracking random walks that can terminate under certain carefully chosen conditions. After the introduction of the self-similar model it is shown how it can be coupled with the original model so that the number of returns to the root in the original case always dominates that of the self-similar, thus reducing the problem to establishing recurrence in the self-similar case. From here, the self-similarity properties of the new model are exploited in order to show that the generating function f of the number of particles that return to the root is a fixed point of an operator $\mathcal{A}$ given by

$$\mathcal{A}g(x) = \frac{x+2}{3} g\Big(\frac{x+1}{2}\Big)^2 + \frac{x+1}{3} g\Big(\frac{x}{2}\Big)\bigg(1 - g\Big(\frac{x+1}{2}\Big)\bigg).$$

A bootstrapping argument called Poisson thinning is then employed, wherebye $\mathcal{A}$ is repeatedly applied to the generating functions of a series of Poisson random variables of gradually increasing mean, in order to show that the only fixed point of $\mathcal{A}$ is the zero function, thus establishing recurrence for both the self-similar and original models.

The difficulty in extending the above result to the 3-ary tree seems to derive from the 3-ary case (it appears) being very close to criticality. Specifically, various constructions of alternate models that are dominated by the original and possess some reasonable degree of self similarity, when analyzed both through simulations and iterative applications of their corresponding operators on generating functions, appear to produce cases that cease to be recurrent. Hence the decision to address the intermediate case of the 3,2-alternating tree. The strategy that I use in order to establish recurrence of the frog model on $\mathbb{T}_{3,2}$ (the 3,2-alternating tree) is similar to the one just described. However, many of the techniques have to be modified substantially in order to accommodate the additional complexity of the new model. First, a set of constraints similar to those used to define the self-similar model on the 2-ary tree need to be chosen very carefully in the 3,2 case. This is because we face the competing necessities of both preserving recurrence and also obtaining enough self-similarity so that the corresponding operator on generating functions is simple enough to work with. In attempting to perform this delicate balancing act, we obtain a quasi-self-similar model which appears to be recurrent. In order to confirm recurrence for the new model via a bootstrapping argument similar to the one used for $\mathbb{T}_2$, it is necessary to establish that the corresponding operator is monotone (i.e. that $h \geq g \implies \mathcal{A}h \geq \mathcal{A}g$). This turns out to be one of the more difficult parts of the proof, and is accomplished by expressing $\mathcal{A}$ as a composition of several other operators, each of which is then shown to be monotone increasing. Once this has been done, a combination of techniques, including the use of a Python program in which an interval arithmetic package is imported in order to escape rounding errors, are used to complete the proof. A print out of the Python program can be found in Appendix A at the end of this dissertation.

4.1.1 The non-backtracking frog model

In order to construct the self-similar frog model referenced above, we first introduce an intermediate model called the non-backtracking frog model, in which individual frogs perform uniformly random non-backtracking walks (i.e. at each step a frog chooses randomly from the set of all adjacent vertices *except* the one from which it just came) that are stopped at the root. As with the original model, we start with one sleeping frog at each non-root vertex and a single active frog at the root (note that due to the non-backtracking property, the frog starting at the root will only take steps away from the root). Ultimately, we will show that the non-backtracking frog model can be embedded inside of the original model, but we will first need to address some technical details related to non-backtracking random walks on $\mathbb{T}_{3,2}$.

We start by defining the Markov process $\Upsilon : \mathbb{N} \to \mathbb{T}_{3,2}$ as follows: If $\Upsilon(0) = \varnothing$ (where $\varnothing$ represents the root) then Υ simply proceeds as an unbiased random walk on $\mathbb{T}_{3,2}$. If $\Upsilon(0) \neq \varnothing$ then Υ proceeds as an unbiased random walk *except* that if $\Upsilon(n) = \varnothing$ (for some

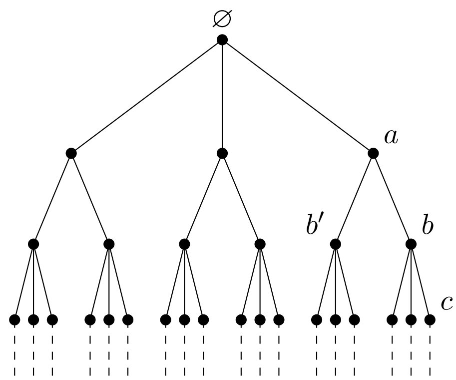

Figure 4.1: The first four levels of $\mathbb{T}_{3,2}$ with relevant nodes labeled.

n) and $\Upsilon(n+1)$ is one of the two nodes that does not belong to the sub-tree containing $\Upsilon(0)$, then with probability $\frac{5}{8}$ the process terminates at $\Upsilon(n+1)$ (i.e. $\Upsilon(j) = \Upsilon(n+1)\ \forall\, j \geq n+1$). Next Υ is used to define the sequence $\{t_n\}$ in the following way: Let $t_0 = 0$ and, for $k \geq 0$, let $s_k = \sup\{s \geq t_k : \Upsilon(s) = \Upsilon(t_k)\}$. If $s_k < \infty$ let $t_{k+1} = s_k + 1$. Otherwise, let $t_{k+1} = t_k$ (note that, modulo a set of measure 0, s_k only equals infinity in the case where Υ is stopped at one of the children of the root as described above).

We now use Υ and $\{t_k\}$ to define the new process $\Phi : \mathbb{N} \to \mathbb{T}_{3,2}$ as follows: First, if $\Upsilon(0) = \varnothing$ then we just let $\Phi(k) = \Upsilon(t_k)\ \forall\ k \geq 0$. Otherwise, let $\Phi(0) = \Upsilon(0)$ and for each $k \geq 0$ let

$$\Phi(k+1) = \begin{cases} \varnothing & \text{if } \Phi(k) = \varnothing \\ \Upsilon(t_{k+1}) & \text{otherwise} \end{cases}$$

Next an important result regarding the process Φ will be established.

Proposition 4.1.2. *The process Φ is identical (in terms of its transition probabilities) to an unbiased non-backtracking random walk on $\mathbb{T}_{3,2}$ that terminates upon hitting the root.*

Proof. In the case where $\Phi(0) = \varnothing$ the process Φ moves one step away from the root each time, so the conclusion follows by symmetry. When $\Phi(0) \neq \varnothing$ a more complicated argument will be required. We start by making some preliminary computations. Let p_1 represent the probability that an unbiased random walk on $\mathbb{T}_{3,2}$ that starts at a (see Figure 4.1 above) ever hits the root. Likewise, let p_2 represent the probability that an unbiased random walk on $\mathbb{T}_{3,2}$ starting at b ever hits a. More generally we see by symmetry that the probability an unbiased random walk on $\mathbb{T}_{3,2}$ starting at a node on an odd numbered level (even resp.) ever hits the parent of this node is p_1 (p_2 resp.). Calculating these values we get the expressions

$$p_1 = \frac{1}{3} + \frac{2}{3}p_2p_1,\ p_2 = \frac{1}{4} + \frac{3}{4}p_1p_2 \implies p_1 = \frac{1}{3-2p_2},\ p_2 = \frac{1}{4-3p_1}$$

Solving for p_1 then gives

$$p_1 = \frac{1}{3 - \frac{2}{4-3p_1}} = \frac{4-3p_1}{10-9p_1} \implies 9p_1^2 - 13p_1 + 4 = 0 \implies p_1 = \frac{4}{9} \text{ or } 1$$

Since the value 1 can clearly be disregarded this then gives $p_1 = \frac{4}{9}$. Plugging this into the formula for p_2 above we get $p_2 = \frac{3}{8}$.

Returning now to the task of establishing that the transition probabilities of Φ match those of the non-backtracking random walk, we begin by addressing the task of showing that $\mathbb{P}(\Phi(1) = \varnothing | \Phi(0) = a) = \frac{1}{3}$. Denoting $\mathbb{P}(\Phi(1) = \varnothing | \Phi(0) = a)$ as p and $\mathbb{P}(\Upsilon(n+j) = a \text{ for some } j > 0 | \Upsilon(n) = \varnothing)$ as q (where we're assuming here that Υ originates in the sub-tree rooted at a), we find (based on the definition of Φ) that

$$p = \frac{2}{3}p_2p + \frac{1}{3}(1-q) + \frac{1}{3}qp = \frac{1}{4}p + \frac{1}{3}(1-q) + \frac{1}{3}qp \implies p = \frac{4-4q}{9-4q}$$

Noting that

$$q = \frac{1}{3} + \frac{1}{4}p_1q = \frac{1}{3} + \frac{1}{9}q \implies q = \frac{3}{8}$$

it then follows from the above formula for p in terms of q, that indeed $p = \frac{1}{3}$. Using this, symmetry implies that $\mathbb{P}(\Phi(1) = b | \Phi(0) = a) = \mathbb{P}(\Phi(1) = b' | \Phi(0) = a) = \frac{1}{3}$. Hence, we find that in the case where $t = 0$ and $\Phi(0) = a$, the transition probabilities of Φ do agree with those of the non-backtracking random walk that is stopped at the root.

Moving on, we now want to show that $\mathbb{P}(\Phi(1) = a|\Phi(0) = b) = \frac{1}{4}$. Denoting this last probability as p and the value $\mathbb{P}(\Upsilon(n+j) = b$ for some $j > 0|\Upsilon(n) = a)$ as q (again assuming Υ originates in the sub-tree rooted at a), it follows from the definition of Φ that

$$p = \frac{1}{4}(1-q) + \frac{1}{4}qp + \frac{3}{4}p_1 p = \frac{1}{4}(1-q) + \frac{1}{4}qp + \frac{1}{3}p \implies p = \frac{3-3q}{8-3q}$$

Using the fact that

$$q = \frac{1}{3} + \frac{1}{3}p_2 q + \frac{1}{8}q = \frac{1}{3} + \frac{1}{4}q \implies q = \frac{4}{9}$$

our formula for p in terms of q then tells us that $p = \frac{1}{4}$. Again using symmetry, we find that if Φ starts at b at time $t = 0$, it then goes to each of the four adjacent nodes with equal probability. Generalizing these results, if we now let p'_n represent the probability that the first step made by Φ is towards the root (given that $\Phi(0)$ resides at level n) and let q'_n represent the probability that Υ (starting at level n-1) ever hits a particular child node of its starting node (e.g. the rightmost node), we find that it follows from induction, along with the computations for the base cases $p'_1, p'_2, q'_1,$ and q'_2 given above, that

$$p'_n = \begin{cases} \frac{1}{3} & \text{for n odd} \\ \frac{1}{4} & \text{for n even} \end{cases}$$

Once again exploiting symmetry, we find that the above result implies that when beginning at a non-root vertex, Φ moves to each of the adjacent vertices with equal probability.

Now note that by the same symmetry considerations which ensure that the transition probabilities for Φ, when begun at the root, match those in the non-backtracking case, it also follows that, following a down step, Φ's transition probabilities again match those of the non-backtracking random walk (stopped at $\varnothing$). Coupling this with the results from the previous paragraph, the only remaining task involved in establishing the proposition is addressing the case of Φ's transition probabilities after it has just taken a step *towards* the root. Since Φ always stops upon hitting the root, the case where its previous step brought it to $\varnothing$ is immediate. Now if we let r_n (for $n \geq 1$) represent the probability of Φ taking a step towards the root, conditioned on its previous step having brought it from level $n+1$ of $\mathbb{T}_{3,2}$ to level n, we find that

$$r_n = \begin{cases} \frac{p_2 p'_n}{p'_{n+1}} & \text{if n is odd} \\ \frac{p_1 p'_n}{p'_{n+1}} & \text{if n is even} \end{cases}$$

Plugging in the values for p_1, p_2, and p'_n, then gives $r_n = \frac{1}{2}$ for n odd and $r_n = \frac{1}{3}$ for n even. From this it then follows that, conditioned on having just moved from a node to its parent (not the root), Φ then moves to each of the available adjacent nodes (other than the one it just came from) with equal probability. Hence, we've completed the task of showing that the transition probabilities of Φ match those of the non-backtracking random walk that is stopped at the root, and thus, have completed the proof of the proposition. □

Having obtained the above result, the proceeding corollary regarding the non-backtracking frog model on $\mathbb{T}_{3,2}$ follows as an almost immediate consequence.

Corollary 4.1.3. *There exists a coupling between the non-backtracking and original frog models on* $\mathbb{T}_{3,2}$ *where the path of each non-backtracking frog is a subset of the path of the corresponding frog in the original model.*

Proof. First recalling how the process Φ was constructed using Υ, we can see that the collection of vertices landed on for an instance of Φ is a subset of the collection of vertices landed on for the corresponding instance of Υ. Likewise, since the process Υ is just a (potentially) truncated version of an unbiased random walk on $\mathbb{T}_{3,2}$, it follows from Proposition 4.1.2 that the non-backtracking random walk on $\mathbb{T}_{3,2}$ that terminates upon hitting the root can be coupled with the unbiased random walk on $\mathbb{T}_{3,2}$ so that the path traversed in the non-backtracking case is a subset of the path traversed in the unbiased case. From here the entire non-backtracking frog model on $\mathbb{T}_{3,2}$ can be coupled with the original model by starting with the original, and defining a corresponding non-backtracking model for which the path of each activated frog is determined by the instance of Φ corresponding to the path traversed by the same frog in the original model. Using this coupling, we find that the path of each frog in the non-backtracking model is a subset of that of its counterpart in the original model. $\square$

4.1.2 Coupling the original and self-similar models

The self-similar frog model on $\mathbb{T}_{3,2}$ is obtained by refining the non-backtracking frog model through the addition of the following constraints: (i) Any frog that goes down an edge (i.e. travels away from the root) from an even to an odd level, where that edge has already been traveled along by another frog, is immediately stopped. If multiple frogs go down a previously untraveled edge simultaneously then all but one are stopped. (ii) The same rule applies for frogs traveling down an edge from an odd to an even level *except* that a node on an even level can have up to two frogs land on it without being stopped (the frog originating at its parent node along with whichever frog activated the frog at its parent node) *provided* that the frog residing at the sibbling of the node in question has yet to be activated (see Figure 4.2 below).

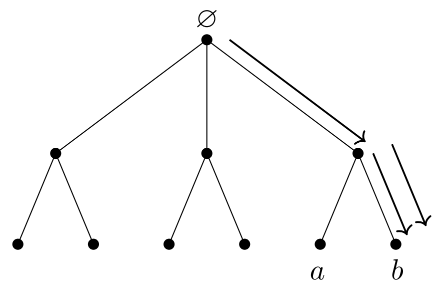

Figure 4.2: A depiction of a scenario in which a node on an even level (node b) has two frogs land on it without being stopped. Note that in order for such an event to accord with the specifications of the self-similar model, the sibling node (labeled a in the figure) cannot yet have been landed on by an active frog.

Since the frogs in the self-similar model defined above conduct truncated non-backtracking random walks stopped at the root, this yields a natural coupling between the self-similar and non-backtracking frog models in which the frogs in the self-similar model follow paths which are subsets of the paths followed by the corresponding non-backtracking frogs. Composing this coupling with the coupling described in the proof of Corollary 4.1.3 then gives a coupling between the self-similar and original frog models that also possesses this property. Letting V and Z represent the number of frogs that hit the root in the self-similar and original models respectively, we obtain the following proposition.

Proposition 4.1.4. *There exists a coupling between the self-similar and original frog models on $\mathbb{T}_{3,2}$ in which V is dominated by Z.*

Armed with this result, we now find that to prove Theorem 4.1.1 it suffices to prove recurrence of the self-similar frog model (i.e. that $\mathbb{P}(V = \infty) = 1$).

4.1.3 Constructing the operator $\mathcal{A}$

Let $f(x) := \mathbb{E}[x^V]$ be the generating function for V. Establishing that $\mathbb{P}(V = \infty) = 1$ will involve showing that $f(x)$ is a fixed point for an operator $\mathcal{A}$. This will be done by introducing operators $\mathcal{L}$ and $\mathcal{H}$. We can initially think of all three operators as acting on $C^0([0,1])$ (though we'll restrict our focus to a much smaller class of functions later on).

To start, define the random variable V_c to be the number of frogs (in the self-similar frog model) originating from the sub-tree rooted at c (see Figure 4.1 in subsection 4.1.1), which hit b (conditioned on the frog at c being activated). Letting $\mathbb{T}_{3,2}(c)$ represent the sub-tree rooted at c, we find that if we ignore frogs originating from outside $\{b\} \cup \mathbb{T}_{3,2}(c)$ which are stopped at b or c after the frog at c has been activated (this can be done since these frogs do not activate any other frogs in $\{b\} \cup \mathbb{T}_{3,2}(c)$), then the self-similar frog model restricted to $\{b\} \cup \mathbb{T}_{3,2}(c)$ (following the activation of the frog at c) looks exactly like the self-similar frog model on $\mathbb{T}_{3,2}$ following the initial step taken by the frog originating at the root. From this it then follows that V and V_c have the same distribution, and therefore that V_c also has f as its probability generating function.

Next define the random variable V_b to be the number of frogs originating from the sub-tree rooted at b (see Figure 4.1 again), which hit a (conditioned on the frog at b being activated by exactly one frog from the pair consisting of the frog starting at the root and the frog starting at a). Now letting $l(x)$ represent the probability generating function of V_b, we present the lemma below relating the functions $l(x)$ and $f(x)$ via the following operator.

Definition 4.1.5. $\mathcal{L}g(x) := \frac{x+3}{4} g(\frac{x+2}{3})^3 + 2 \cdot \frac{x+2}{4} \Big(g(\frac{x+1}{3})^2 - g(\frac{x+2}{3}) g(\frac{x+1}{3})^2 \Big) + \frac{x+1}{4} \Big(g(\frac{x}{3}) - 2g(\frac{x+1}{3}) g(\frac{x}{3}) - g(\frac{x+2}{3})^2 g(\frac{x}{3}) + 2g(\frac{x+2}{3}) g(\frac{x+1}{3}) g(\frac{x}{3}) \Big)$.

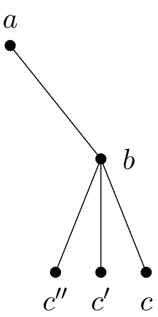

Figure 4.3: The right side of the first three levels of the subtree of $\mathbb{T}_{3,2}$ rooted at a.

Lemma 4.1.6. $l(x) = \mathcal{L}f(x)$.

Now the operator $\mathcal{H}$ will be introduced, along with another important lemma. In the lemma, $h(x)$ will refer to $\mathbb{E}[x^{V_b'}]$, where V_b' is the random variable representing the number of frogs originating from the sub-tree rooted at b which hit a (see Figures 4.1 and 4.3), conditioned on vertex b being hit by both the frog starting at the root and the frog that started at a.

Definition 4.1.7. $\mathcal{H}g(x) := \frac{1}{3}\mathcal{L}g(x) + \frac{x+3}{6}g(\frac{x+2}{3})^3 + \frac{x+2}{6}\Big(g(\frac{x+1}{3})^2 - g(\frac{x+2}{3})g(\frac{x+1}{3})^2\Big)$.

Lemma 4.1.8. $h(x) = \mathcal{H}f(x)$

Next we define $\mathcal{A}$ and state the main result of this section, following which are the proofs of our two lemmas.

Definition 4.1.9. $\mathcal{A}g(x) := \frac{x}{3}\mathcal{L}[g](\frac{x}{2}) + \frac{x+1}{3}\Big(\mathcal{L}[g](\frac{x+1}{2})\Big)^2 - \frac{x}{3}\mathcal{L}[g](\frac{x+1}{2})\mathcal{L}[g](\frac{x}{2}) + \frac{1}{3}\mathcal{H}[g](\frac{x}{2}) + \frac{1}{3}\mathcal{L}[g](\frac{x+1}{2})\mathcal{H}[g](\frac{x+1}{2}) - \frac{1}{3}\mathcal{L}[g](\frac{x+1}{2})\mathcal{H}[g](\frac{x}{2})$.

Remark 1. Note the brackets in expressions of the form $\mathcal{L}[g](\frac{x}{2})$ above, which are there to indicate that the expression is to be interpreted as the value of the function $\mathcal{L}g$ at $\frac{x}{2}$.

Theorem 4.1.10. $\mathcal{A}f = f$.

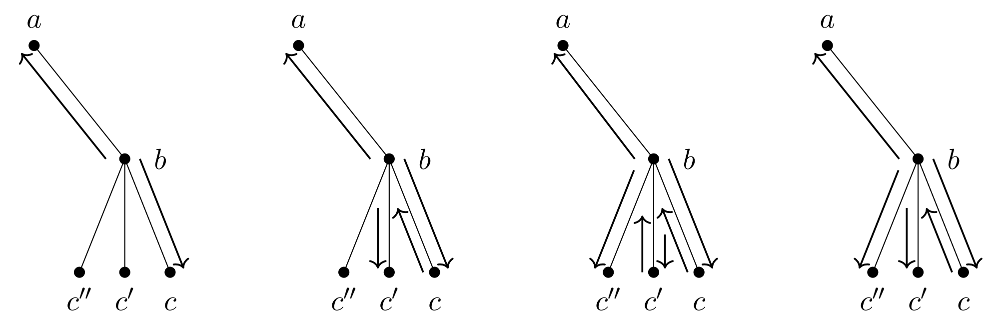

Figure 4.4: Diagrams representing the four events (from left to right) A_1, A_2, A_3, and A_4.

Proof of Lemma 4.1.6. Observe Figure 4.3 on the previous page, which shows the relevant portion of $\mathbb{T}_{3,2}$. Since we are conditioning on the frog at b being activated by either the frog from a or the frog from the root (but not both), it follows from property (ii) of the self-similar model (see beginning of subsection 4.1.2) that no additional frogs can enter the sub-tree rooted at b (meaning any such frogs are stopped at b). Hence, once the frog beginning at b is activated, we are starting with two active frogs there where one of them (we'll call it #1) can go in any of the four available directions and the other (call it #2) must travel away from vertex a. Letting A represent the event that #1 goes to a, $l(x)$ can then be expressed as $l(x) = \mathbb{E}[x^{V_b}] = \mathbb{E}[x^{V_b}; A] + \mathbb{E}[x^{V_b}; A^c]$.

Now A is split up into the four separate events A_1, A_2, A_3, and A_4 (see Figure 4.4 above) as follows: A_1 represents having the sub-tree activated by #2 fail to activate either of its two sibling sub-trees (represented by c' and c'' in leftmost figure); A_2 represents the sub-tree activated by #2 activating exactly one of its sibling sub-trees, which itself fails to activate the other sibling; A_3 represents the sub-tree activated by #2 activating exactly one of its sibling sub-trees, which itself activates the other sibling; and A_4 represents the sub-tree activated by #2 activating both of its sibling sub-trees.

The next step is to evaluate $\mathbb{E}[x^{V_b}; A_i]$ for each i as follows:

$$\mathbb{E}[x^{V_b}; A_1] = \frac{x}{4}\sum_{k=0}^{\infty}\mathbb{P}(V_c = k)\Big(\frac{1}{3}\Big)^k x^k = \frac{x}{4} f\Big(\frac{x}{3}\Big) \tag{4.1.1}$$

$$\begin{aligned}\mathbb{E}[x^{V_b}; A_2] &= \frac{x}{4}\sum_{k=1}^{\infty}\mathbb{P}(V_c = k)\sum_{j=0}^{k-1}\Big(\frac{1}{3}\Big)^j\Big(\frac{2}{3}\Big)^{k-j}\binom{k}{j}x^j\cdot 2\cdot\Big(\frac{1}{2}\Big)^{k-j} \\ &\cdot\Bigg(\sum_{i=0}^{\infty}\mathbb{P}(V_c = i)\Big(\frac{1}{3}\Big)^i\sum_{l=0}^{i}\binom{i}{l}x^l\Bigg) \\ &= \frac{x}{2}\sum_{k=1}^{\infty}\mathbb{P}(V_c = k)\sum_{j=0}^{k-1}\Big(\frac{x}{3}\Big)^j\Big(\frac{1}{3}\Big)^{k-j}\binom{k}{j}\sum_{i=0}^{\infty}\mathbb{P}(V_c = i)\Big(\frac{x+1}{3}\Big)^i \\ &= \frac{x}{2}f\Big(\frac{x+1}{3}\Big)\sum_{k=1}^{\infty}\mathbb{P}(V_c = k)\sum_{j=0}^{k-1}\Big(\frac{x}{3}\Big)^j\Big(\frac{1}{3}\Big)^{k-j}\binom{k}{j} \\ &= \frac{x}{2}f\Big(\frac{x+1}{3}\Big)\sum_{k=1}^{\infty}\mathbb{P}(V_c = k)\Big(\Big(\frac{x+1}{3}\Big)^k - \Big(\frac{x}{3}\Big)^k\Big) \\ &= \frac{x}{2}f\Big(\frac{x+1}{3}\Big)\Big(f\Big(\frac{x+1}{3}\Big) - f\Big(\frac{x}{3}\Big)\Big)\end{aligned} \tag{4.1.2}$$

$$\begin{aligned}\mathbb{E}[x^{V_b}; A_3] &= \frac{x}{4}\sum_{k=1}^{\infty}\mathbb{P}(V_c = k)\sum_{j=0}^{k-1}\Big(\frac{1}{3}\Big)^j\Big(\frac{2}{3}\Big)^{k-j}\binom{k}{j}x^j\cdot 2\cdot\Big(\frac{1}{2}\Big)^{k-j}\cdot\sum_{i=1}^{\infty}\mathbb{P}(V_c = i) \\ &\sum_{l=0}^{i-1}\Big(\frac{1}{3}\Big)^l\Big(\frac{2}{3}\Big)^{i-l}\binom{i}{l}x^l\Big(1-\Big(\frac{1}{2}\Big)^{i-l}\Big)\sum_{m=0}^{\infty}\mathbb{P}(V_c = m)\sum_{n=0}^{m}\Big(\frac{1}{3}\Big)^n\Big(\frac{2}{3}\Big)^{m-n}\binom{m}{n}x^n \\ &= \frac{x}{2}\sum_{k=1}^{\infty}\mathbb{P}(V_c = k)\sum_{j=0}^{k-1}\Big(\frac{x}{3}\Big)^j\Big(\frac{1}{3}\Big)^{k-j}\binom{k}{j}\sum_{i=1}^{\infty}\mathbb{P}(V_c = i) \\ &\quad\sum_{l=0}^{i-1}\Big(\frac{x}{3}\Big)^l\Big(\frac{2}{3}\Big)^{i-l}\binom{i}{l}\Big(1-\Big(\frac{1}{2}\Big)^{i-l}\Big)f\Big(\frac{x+2}{3}\Big) \\ &= \frac{x}{2}f\Big(\frac{x+2}{3}\Big)\sum_{k=1}^{\infty}\mathbb{P}(V_c = k)\Big(\Big(\frac{x+1}{3}\Big)^k - \Big(\frac{x}{3}\Big)^k\Big) \\ &\cdot\Bigg(\sum_{i=1}^{\infty}\mathbb{P}(V_c = i)\Big(\Big(\frac{x+2}{3}\Big)^i - \Big(\frac{x+1}{3}\Big)^i\Big)\Bigg) \\ &= \frac{x}{2}f\Big(\frac{x+2}{3}\Big)\Big(f\Big(\frac{x+1}{3}\Big) - f\Big(\frac{x}{3}\Big)\Big)\Big(f\Big(\frac{x+2}{3}\Big) - f\Big(\frac{x+1}{3}\Big)\Big)\end{aligned} \tag{4.1.3}$$

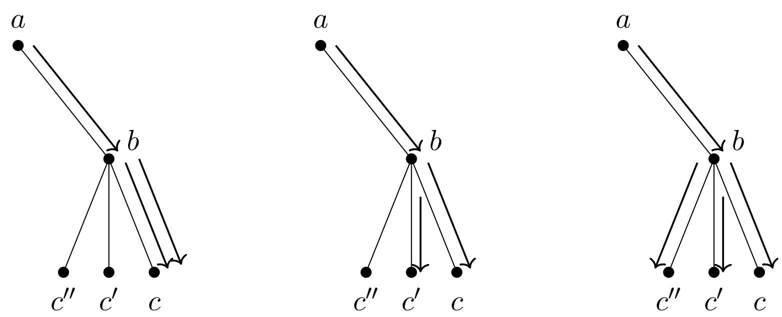

Figure 4.5: Illustrations representing the three events (from left to right) $B_1,\ B_2$, and B_3.

$$\mathbb{E}[x^{V_b}; A_4] = \frac{x}{4}\sum_{k=2}^{\infty}\mathbb{P}(V_c = k)\sum_{j=0}^{k-2}\Big(\frac{1}{3}\Big)^j\Big(\frac{2}{3}\Big)^{k-j}\binom{k}{j}x^j\Big(1-2\Big(\frac{1}{2}\Big)^{k-j}\Big) \tag{4.1.4}$$

$$\cdot\left(\sum_{i=0}^{\infty}\mathbb{P}(V_c = i)\sum_{l=0}^{i}\Big(\frac{1}{3}\Big)^l\Big(\frac{2}{3}\Big)^{i-l}\binom{i}{l}x^l\right)^2$$

$$= \frac{x}{4}\sum_{k=2}^{\infty}\mathbb{P}(V_c = k)\Big(\Big(\frac{x+2}{3}\Big)^k - 2\Big(\frac{x+1}{3}\Big)^k + \Big(\frac{x}{3}\Big)^k\Big)$$

$$\cdot\left(\sum_{i=0}^{\infty}\mathbb{P}(V_c = i)\Big(\frac{x+2}{3}\Big)^i\right)^2$$

$$= \frac{x}{4}f\Big(\frac{x+2}{3}\Big)^2\Big(f\Big(\frac{x+2}{3}\Big) - 2f\Big(\frac{x+1}{3}\Big) + f\Big(\frac{x}{3}\Big)\Big)$$

Having obtained expressions for the A_i's, we now split up A^c into the three separate events $B_1,\ B_2$, and B_3 (see Figure 4.5 above) in the following way: B_1 represents having #1 and #2 activate the same sub-tree; B_2 represents #1 and #2 activating different sub-trees (represented by c and c' in middle figure above), neither of which activates the third sub-tree; and B_3 represents #1 and #2 activating different sub-trees, which then activate the third sub-tree. Next the expression $\mathbb{E}[x^{V_b}; B_i]$ is evaluated for each i as follows:

$$\mathbb{E}[x^{V_b}; B_1] = \frac{1}{x}\mathbb{E}[x^{V_b}; A] \tag{4.1.5}$$

(this follows from the fact that $\mathbb{P}(A) = \mathbb{P}(B_1)$ and $\big(V_b - 1\big)|A = V_b|B_1$)

$$\mathbb{E}[x^{V_b}; B_2] = \frac{1}{2}\Big(\sum_{k=0}^{\infty}\mathbb{P}(V_c = k)\sum_{j=0}^{k}\Big(\frac{1}{3}\Big)^k\binom{k}{j}x^j\Big)^2 = \frac{1}{2}\Big(\sum_{k=0}^{\infty}\mathbb{P}(V_c = k)\Big(\frac{x+1}{3}\Big)^k\Big)^2 \quad (4.1.6)$$

$$= \frac{1}{2}f\Big(\frac{x+1}{3}\Big)^2$$

$$\mathbb{E}[x^{V_b}; B_3] = \frac{1}{2}\sum_{k_1+k_2\geq 1}\mathbb{P}(V_c = k_1)\mathbb{P}(V_c = k_2) \quad (4.1.7)$$

$$\cdot\left(\sum_{j=0}^{k_1+k_2-1}\Big(\frac{1}{3}\Big)^j\Big(\frac{2}{3}\Big)^{k_1+k_2-j}x^j\binom{k_1+k_2}{j}\Big(1-\Big(\frac{1}{2}\Big)^{k_1+k_2-j}\Big)\right)$$

$$\cdot\left(\sum_{i=0}^{\infty}\mathbb{P}(V_c = i)\sum_{l=0}^{i}\Big(\frac{1}{3}\Big)^l\Big(\frac{2}{3}\Big)^{i-l}\binom{i}{l}x^l\right)$$

$$= \frac{1}{2}\sum_{k_1+k_2\geq 1}\mathbb{P}(V_c = k_1)\mathbb{P}(V_c = k_2)\Big(\Big(\frac{x+2}{3}\Big)^{k_1+k_2} - \Big(\frac{x+1}{3}\Big)^{k_1+k_2}\Big)$$

$$\cdot\left(\sum_{i=0}^{\infty}\mathbb{P}(V_c = i)\Big(\frac{x+2}{3}\Big)^i\right)$$

$$= \frac{1}{2}f\Big(\frac{x+2}{3}\Big)\Big(f\Big(\frac{x+2}{3}\Big)^2 - f\Big(\frac{x+1}{3}\Big)^2\Big)$$

Using the calculations from (4.1.1)-(4.1.7) we now find that

$$l(x) = \mathbb{E}[x^{V_b}] = \sum_{i=1}^{4}\mathbb{E}[x^{V_b}; A_i] + \sum_{i=1}^{3}\mathbb{E}[x^{V_b}; B_i] \quad (4.1.8)$$

$$= \left(1+\frac{1}{x}\right)\left(\frac{x}{4}f\Big(\frac{x}{3}\Big) + \frac{x}{2}f\Big(\frac{x+1}{3}\Big)^2 - \frac{x}{2}f\Big(\frac{x+1}{3}\Big)f\Big(\frac{x}{3}\Big) + \frac{x}{2}f\Big(\frac{x+2}{3}\Big)^2 f\Big(\frac{x+1}{3}\Big)\right.$$

$$- \frac{x}{2}f\Big(\frac{x+2}{3}\Big)^2 f\Big(\frac{x}{3}\Big) - \frac{x}{2}f\Big(\frac{x+2}{3}\Big)f\Big(\frac{x+1}{3}\Big)^2 + \frac{x}{2}f\Big(\frac{x+2}{3}\Big)f\Big(\frac{x+1}{3}\Big)f\Big(\frac{x}{3}\Big)$$

$$\left.+ \frac{x}{4}f\Big(\frac{x+2}{3}\Big)^3 - \frac{x}{2}f\Big(\frac{x+2}{3}\Big)^2 f\Big(\frac{x+1}{3}\Big) + \frac{x}{4}f\Big(\frac{x+2}{3}\Big)^2 f\Big(\frac{x}{3}\Big)\right)$$

$$+ \frac{1}{2}f\Big(\frac{x+1}{3}\Big)^2 + \frac{1}{2}f\Big(\frac{x+2}{3}\Big)^3 - \frac{1}{2}f\Big(\frac{x+2}{3}\Big)f\Big(\frac{x+1}{3}\Big)^2$$

$$= \frac{x+3}{4} f\Big(\frac{x+2}{3}\Big)^3 + 2 \cdot \frac{x+2}{4}\left(f\Big(\frac{x+1}{3}\Big)^2 - f\Big(\frac{x+2}{3}\Big) f\Big(\frac{x+1}{3}\Big)^2\right)$$

$$+ \frac{x+1}{4}\left(f\Big(\frac{x}{3}\Big) - 2f\Big(\frac{x+1}{3}\Big) f\Big(\frac{x}{3}\Big) - f\Big(\frac{x+2}{3}\Big)^2 f\Big(\frac{x}{3}\Big) + 2f\Big(\frac{x+2}{3}\Big) f\Big(\frac{x+1}{3}\Big) f\Big(\frac{x}{3}\Big)\right)$$

$$= \mathcal{L}f(x)$$

Hence, the proof of Lemma 4.1.6 is complete. □

Proof of Lemma 4.1.8. The scenario under consideration (see Figure 4.3 again) begins with three active frogs at vertex b, where one (call it #1) is free to go in any of the four available directions, and the other two (call them the #2 frogs) can go in any of the three directions away from the root. Letting A_0 represent the event that the two #2 frogs travel to the same node from b (call this node c), $h(x)$ can be expressed as $\mathbb{E}[x^{V_b'}] = \mathbb{E}[x^{V_b'}; A_0] + \mathbb{E}[x^{V_b'}; A_0^c]$. If the event A_0 occurs, since one of the two #2 frogs is stopped at c, it follows that $V_b'|A_0$ has the same distribution as V_b. Hence, this implies that $\mathbb{E}[x^{V_b'}; A_0] = \mathbb{P}(A_0)\mathbb{E}[x^{V_b'}|A_0] = \mathbb{P}(A_0)\mathbb{E}[x^{V_b}] = \frac{1}{3}\mathcal{L}f(x)$.

Turning next to the event A_0^c, it will be split up into the events C_1, C_2, and C_3 (see Figure 4.6 below) as follows: C_1 represents having the #2 frogs go to different nodes and the #1 frog go to a; C_2 represents the #2 frogs going to different nodes and the #1 frog going to the same node as one of the #2 frogs; and C_3 represents the #2 frogs going to different nodes and the #1 frog going to the third sibling node. Evaluating $\mathbb{E}[x^{V_b'}; C_i]$ for each i now gives the following:

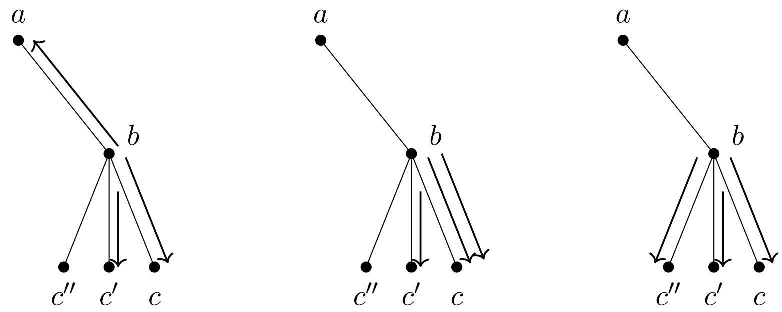

Figure 4.6: Illustrations representing the three events (from left to right) C_1, C_2, and C_3.

$$\begin{aligned}\mathbb{E}[x^{V_b'}; C_1] &= \frac{x}{4} \cdot \frac{2}{3} \cdot \mathbb{E}[x^{V_b} | B_2 \cup B_3] \\ &= \frac{x}{6} \cdot \frac{\left(\frac{1}{2} f\left(\frac{x+1}{3}\right)^2 + \frac{1}{2} f\left(\frac{x+2}{3}\right)^3 - \frac{1}{2} f\left(\frac{x+2}{3}\right) f\left(\frac{x+1}{3}\right)^2\right)}{1/2} \\ &= \frac{x}{6}\left(f\left(\frac{x+1}{3}\right)^2 + f\left(\frac{x+2}{3}\right)^3 - f\left(\frac{x+2}{3}\right) f\left(\frac{x+1}{3}\right)^2\right)\end{aligned} \tag{4.1.9}$$

$$\begin{aligned}\mathbb{E}[x^{V_b'}; C_2] &= \frac{1}{3} \mathbb{E}[x^{V_b} | B_2 \cup B_3] \\ &= \frac{1}{3}\left(f\left(\frac{x+1}{3}\right)^2 + f\left(\frac{x+2}{3}\right)^3 - f\left(\frac{x+2}{3}\right) f\left(\frac{x+1}{3}\right)^2\right)\end{aligned} \tag{4.1.10}$$

$$\begin{aligned}\mathbb{E}[x^{V_b'}; C_3] &= \frac{1}{6}\left(\sum_{k=0}^{\infty} \mathbb{P}(V_c = k) \sum_{j=0}^{k} \left(\frac{1}{3}\right)^j \left(\frac{2}{3}\right)^{k-j} \binom{k}{j} x^j\right)^3 \\ &= \frac{1}{6} f\left(\frac{x+2}{3}\right)^3\end{aligned} \tag{4.1.11}$$

Adding the expressions (4.1.9)-(4.1.11) to our expression for $\mathbb{E}[x^{V_b'}; A_0]$ then gives

$$\begin{aligned}h(x) = \mathbb{E}[x^{V_b'}] &= \frac{1}{3}\mathcal{L}f(x) + \frac{x+3}{6} f\left(\frac{x+2}{3}\right)^3 + \frac{x+2}{6}\left(f\left(\frac{x+1}{3}\right)^2 - f\left(\frac{x+2}{3}\right) f\left(\frac{x+1}{3}\right)^2\right) \\ &= \mathcal{H}f(x)\end{aligned}$$

Hence, the proof is complete. □

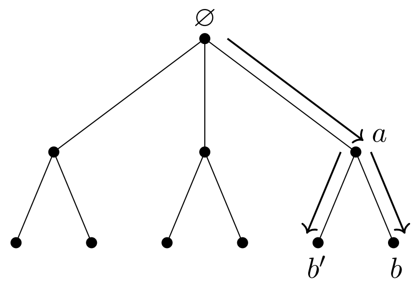

Figure 4.7: A representation of D_1, defined as the event in which the frog coming from the root and the frog at the first vertex it hits (labelled a in the figure above) go to different children of a.

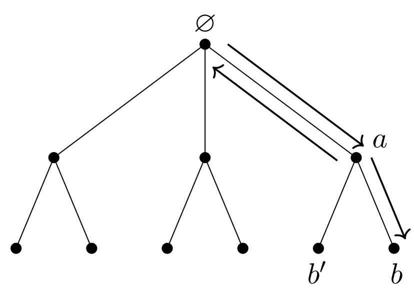

Figure 4.8: A representation of D_2, defined as the event in which the frog at the first vertex hit, upon being activated, returns to the root.

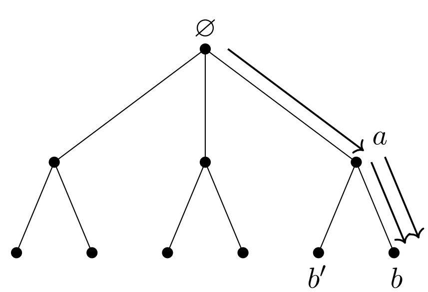

Figure 4.9: A representation of D_3, defined as the event in which the frog coming from the root and the frog coming from a (where a once again represents the first vertex landed on) go to the same child of a.

With Lemmas 4.1.6 and 4.1.8 established, the proof of Theorem 4.1.10 can now be presented.

Proof of Theorem 4.1.10. Begin by separating the collection of possible outcomes into the three events D_1, D_2, and D_3 (see Figures 4.7, 4.8, and 4.9 above). Next we compute $\mathbb{E}[x^V; D_i]$ for each i beginning with $i = 1$.

$$\mathbb{E}[x^V; D_1] = \frac{1}{3}\Big(\sum_{k=0}^{\infty} \mathbb{P}(V_b = k) \sum_{j=0}^{k} \Big(\frac{1}{2}\Big)^k x^j \binom{k}{j}\Big)^2 = \frac{1}{3}\Big(\mathcal{L}[f]\Big(\frac{x+1}{2}\Big)\Big)^2 \tag{4.1.12}$$

(where above we use the fact, shown in (4.1.8), that $\mathbb{E}[x^{V_b}] = \mathcal{L}f(x)$). D_2 can be separated into the two events $D_2^{(1)}$ and $D_2^{(2)}$ as follows: $D_2^{(1)}$ represents having all frogs that go to a from the sub-tree rooted at b then travel to the root; and $D_2^{(2)}$ represents having at least one frog that travels to a from the sub-tree rooted at b then go to b' (i.e. $D_2/D_2^{(1)}$). Computing $\mathbb{E}[x^V; D_2^{(i)}]$ for $i = 1, 2$ now gives

$$\mathbb{E}[x^V; D_2^{(1)}] = \frac{x}{3}\sum_{k=0}^{\infty} \mathbb{P}(V_b = k)\Big(\frac{1}{2}\Big)^k x^k = \frac{x}{3}\mathcal{L}[f]\Big(\frac{x}{2}\Big) \tag{4.1.13}$$

$$\begin{aligned} \mathbb{E}[x^V; D_2^{(2)}] &= \frac{x}{3}\sum_{k=1}^{\infty} \mathbb{P}(V_b = k) \sum_{j=0}^{k-1} \Big(\frac{1}{2}\Big)^k \binom{k}{j} x^j \mathcal{L}[f]\Big(\frac{x+1}{2}\Big) \\ &= \frac{x}{3}\mathcal{L}[f]\Big(\frac{x+1}{2}\Big)\sum_{k=1}^{\infty} \mathbb{P}(V_b = k)\Big(\Big(\frac{x+1}{2}\Big)^k - \Big(\frac{x}{2}\Big)^k\Big) \\ &= \frac{x}{3}\mathcal{L}[f]\Big(\frac{x+1}{2}\Big)\Big(\mathcal{L}[f]\Big(\frac{x+1}{2}\Big) - \mathcal{L}[f]\Big(\frac{x}{2}\Big)\Big) \end{aligned} \tag{4.1.14}$$

Moving on to D_3, it can also be broken up into two separate events in the following way: $D_3^{(1)}$ represents having all frogs that go to a from the sub-tree rooted at b then travel to the root; and $D_3^{(2)}$ represents having at least one frog that travels to a from the sub-tree rooted at b then go to b' (note the only difference between these two events and the events

$D_2^{(1)}$ and $D_2^{(2)}$ respectively is the behavior of the frog starting at a; as seen in Figures 4.8 and 4.9). Computing $\mathbb{E}[x^V; D_3^{(i)}]$ for $i = 1, 2$ gives

$$\mathbb{E}[x^V; D_3^{(1)}] = \frac{1}{3}\sum_{k=0}^{\infty} \mathbb{P}(V_b' = k)\Big(\frac{1}{2}\Big)^k x^k = \frac{1}{3}\mathcal{H}[f]\Big(\frac{x}{2}\Big) \tag{4.1.15}$$

$$\begin{aligned}\mathbb{E}[x^V; D_3^{(2)}] &= \frac{1}{3}\sum_{k=1}^{\infty} \mathbb{P}(V_b' = k)\sum_{j=0}^{k-1}\Big(\frac{1}{2}\Big)^k \binom{k}{j} x^j \mathcal{L}[f]\Big(\frac{x+1}{2}\Big) \\ &= \frac{1}{3}\mathcal{L}[f]\Big(\frac{x+1}{2}\Big)\sum_{k=1}^{\infty}\mathbb{P}(V_b' = k)\Big(\Big(\frac{x+1}{2}\Big)^k - \Big(\frac{x}{2}\Big)^k\Big) \\ &= \frac{1}{3}\mathcal{L}[f]\Big(\frac{x+1}{2}\Big)\Big(\mathcal{H}[f]\Big(\frac{x+1}{2}\Big) - \mathcal{H}[f]\Big(\frac{x}{2}\Big)\Big)\end{aligned} \tag{4.1.16}$$

Now adding together the expressions (4.1.12)-(4.1.16) gives

$$\begin{aligned} f(x) = \mathbb{E}[x^V] &= \sum_{i=1}^{3}\mathbb{E}[x^V; D_i] \\ &= \frac{1}{3}\Big(\mathcal{L}[f]\Big(\frac{x+1}{2}\Big)\Big)^2 + \frac{x}{3}\mathcal{L}[f]\Big(\frac{x}{2}\Big) + \frac{x}{3}\mathcal{L}[f]\Big(\frac{x+1}{2}\Big)\Big(\mathcal{L}[f]\Big(\frac{x+1}{2}\Big) - \mathcal{L}[f]\Big(\frac{x}{2}\Big)\Big) \\ &+ \frac{1}{3}\mathcal{H}[f]\Big(\frac{x}{2}\Big) + \frac{1}{3}\mathcal{L}[f]\Big(\frac{x+1}{2}\Big)\Big(\mathcal{H}[f]\Big(\frac{x+1}{2}\Big) - \mathcal{H}[f]\Big(\frac{x}{2}\Big)\Big) \\ &= \frac{x}{3}\mathcal{L}[f]\Big(\frac{x}{2}\Big) + \frac{x+1}{3}\Big(\mathcal{L}[f]\Big(\frac{x+1}{2}\Big)\Big)^2 - \frac{x}{3}\mathcal{L}[f]\Big(\frac{x+1}{2}\Big)\mathcal{L}[f]\Big(\frac{x}{2}\Big) + \frac{1}{3}\mathcal{H}[f]\Big(\frac{x}{2}\Big) \\ &+ \frac{1}{3}\mathcal{L}[f]\Big(\frac{x+1}{2}\Big)\mathcal{H}[f]\Big(\frac{x+1}{2}\Big) - \frac{1}{3}\mathcal{L}[f]\Big(\frac{x+1}{2}\Big)\mathcal{H}[f]\Big(\frac{x}{2}\Big) = \mathcal{A}f(x)\end{aligned}$$

Hence, the proof of Theorem 4.1.10 is complete. □

4.1.4 Monotonicity of $\mathcal{A}$

In order to prove Theorem 4.1.1 (i.e. show that $\mathbb{P}(V = \infty) = 1$) it suffices to show that $f(x) = 0$ on $[0, 1)$. With the proof of Theorem 4.1.10 now complete, this task is reduced to showing that $\mathcal{A}^n f(x) \to 0$ as $n \to \infty$ $\forall$ $x \in [0, 1)$. The first major step involved in accomplishing this will be to prove the following proposition.

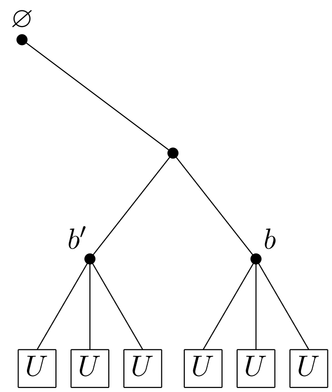

Figure 4.10: A depiction of the construction used to show that $\mathcal{AS} \subseteq \mathcal{S}$.

Proposition 4.1.11. *Define $\mathcal{S}$ to be the space of all probability generating functions (on $[0,1]$) associated with probability distributions on $\{0,1,\ldots\} \cup \{\infty\}$. Let $g_1, g_2 \in \mathcal{S}$ with $g_1 \geq g_2$ on $[0,1]$. Then $\mathcal{A}g_1 \geq \mathcal{A}g_2$ on $[0,1]$.*

The proof of 4.1.11 will require the lemma below.

Lemma 4.1.12. *$\mathcal{LS} \subseteq \mathcal{S}$, $\mathcal{HS} \subseteq \mathcal{S}$, and $\mathcal{AS} \subseteq \mathcal{S}$.*

Proof. Begin by defining the following model: Start with a single active frog at the root and sleeping frogs at the other three nodes (see Figure 4.10 above). The frog at the root performs a non-backtracking random walk that is stopped upon hitting any one of the six boxes, and any time an active frog hits a vertex with a sleeping frog, that frog is activated and begins performing its own non-backtracking random walk that is stopped upon hitting either the root or one of the boxes. In addition, the first time a box is hit by a frog, it releases frogs which also perform non-backtracking random walks that are stopped upon hitting either the root or another box.

The number of frogs released by the different boxes, conditioned on being hit, are i.i.d. random variables with distribution U. Furthermore, the model obeys property (ii) with respect to the nodes b and b' (see beginning of subsection 4.1.2). Now let $\mathcal{A}^*U$ represent the distribution of the number of frogs that hit the root in this model. It then follows that $\mathcal{A}^*\tilde{V}_c = \tilde{V}$ (where $\tilde{V}$ and $\tilde{V}_c$ represent the distributions of V and V_c). Now recall that the proof of Theorem 4.1.10 involved calculating the generating function of V (denoted as $f(x)$) in terms of the generating function of V_c (also denoted as $f(x)$ on account of our recognition that V and V_c share the same distribution) and showing that V has generating function $\mathcal{A}f$ (i.e. $\mathcal{A}f$ is the generating function associated with the distribution $\mathcal{A}^*\tilde{V}_c$). Since the derivation of this formula was carried out purely symbolically (meaning without taking into account the particular properties of V_c or its generating function f), this means that for any probability distribution U (concentrated on $\{1, 2, \ldots\} \cup \{\infty\}$) with generating function η, the generating function of the distribution $\mathcal{A}^*U$ is $\mathcal{A}\eta$. Hence, it follows that $\mathcal{AS} \subseteq \mathcal{S}$.

The proofs of $\mathcal{LS} \subseteq \mathcal{S}$ and $\mathcal{HS} \subseteq \mathcal{S}$ are very similar to the proof of $\mathcal{AS} \subseteq \mathcal{S}$, so some of the details will therefore be omitted. In both cases we define a model using the diagram below (see Figure 4.11). For $\mathcal{L}$, begin with two active frogs at vertex b, one of which must go in one of the three downward directions, while the other is free to go in any of the four available directions. Active frogs are to perform non-backtracking random walks which stop upon hitting either a or any of the boxes. The first time a box is hit by an active frog, it releases active frogs according to the distribution U. The numbers of frogs released by the different boxes (conditioned on being hit) are independent. Letting $\mathcal{L}^*U$ represent the distribution of the number of frogs that hit a, we find (by a similar argument to the one used for $\mathcal{A}^*$) that the generating function of $\mathcal{L}^*U$ is $\mathcal{L}\eta$ (where η once again represents the generating function associated with the distribution U). From this it follows that $\mathcal{LS} \subseteq \mathcal{S}$. Furthermore, using a model which differs from this one only in that a single additional

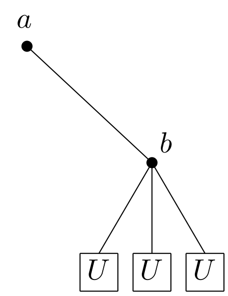

Figure 4.11: A depiction of the construction used to show that $\mathcal{HS} \subseteq \mathcal{S}$ and $\mathcal{LS} \subseteq \mathcal{S}$.

active frog that can go in any of the three downward directions is positioned at b, we also find that $\mathcal{H}^*U$ has generating function $\mathcal{H}\eta$, from which it follows that $\mathcal{HS} \subseteq \mathcal{S}$. Hence, the proof is complete. $\square$

Proof of Proposition 4.1.11. The first step will be to show that $\mathcal{L}g_1(x) \geq \mathcal{L}g_2(x)$ on $[0,1]$. Letting $F_t(x) = tg_1(x) + (1-t)g_2(x)$, it will suffice to show that $\frac{\partial(\mathcal{L}F_t(x))}{\partial t} \geq 0 \ \forall\ x, t \in [0,1]$. Using the formula for $\mathcal{L}$ (see Definition 4.1.5) along with the fact that $\frac{\partial F_t(x)}{\partial t} = g_1(x) - g_2(x)$, then gives the following equalities:

$$\begin{aligned}
\frac{\partial(\mathcal{L}F_t(x))}{\partial t} &= 3 \cdot \frac{x+3}{4} F_t\Big(\frac{x+2}{3}\Big)^2 \Big(g_1\Big(\frac{x+2}{3}\Big) - g_2\Big(\frac{x+2}{3}\Big)\Big) \\
&+ 4 \cdot \frac{x+2}{4} F_t\Big(\frac{x+1}{3}\Big)\Big(g_1\Big(\frac{x+1}{3}\Big) - g_2\Big(\frac{x+1}{3}\Big)\Big) \\
&- 2 \cdot \frac{x+2}{4} F_t\Big(\frac{x+1}{3}\Big)^2 \Big(g_1\Big(\frac{x+2}{3}\Big) - g_2\Big(\frac{x+2}{3}\Big)\Big) \\
&- 4 \cdot \frac{x+2}{4} F_t\Big(\frac{x+2}{3}\Big) F_t\Big(\frac{x+1}{3}\Big)\Big(g_1\Big(\frac{x+1}{3}\Big) - g_2\Big(\frac{x+1}{3}\Big)\Big) \\
&- 2 \cdot \frac{x+1}{4} F_t\Big(\frac{x+2}{3}\Big) F_t\Big(\frac{x}{3}\Big)\Big(g_1\Big(\frac{x+2}{3}\Big) - g_2\Big(\frac{x+2}{3}\Big)\Big) \\
&- \frac{x+1}{4} F_t\Big(\frac{x+2}{3}\Big)^2 \Big(g_1\Big(\frac{x}{3}\Big) - g_2\Big(\frac{x}{3}\Big)\Big) \\
&- 2 \cdot \frac{x+1}{4} F_t\Big(\frac{x+1}{3}\Big)\Big(g_1\Big(\frac{x}{3}\Big) - g_2\Big(\frac{x}{3}\Big)\Big) \\
&- 2 \cdot \frac{x+1}{4} F_t\Big(\frac{x}{3}\Big)\Big(g_1\Big(\frac{x+1}{3}\Big) - g_2\Big(\frac{x+1}{3}\Big)\Big)
\end{aligned}$$

$$+ 2 \cdot \frac{x+1}{4} F_t\Big(\frac{x+2}{3}\Big) F_t\Big(\frac{x+1}{3}\Big)\Big(g_1\Big(\frac{x}{3}\Big) - g_2\Big(\frac{x}{3}\Big)\Big)$$
$$+ 2 \cdot \frac{x+1}{4} F_t\Big(\frac{x+2}{3}\Big) F_t\Big(\frac{x}{3}\Big)\Big(g_1\Big(\frac{x+1}{3}\Big) - g_2\Big(\frac{x+1}{3}\Big)\Big)$$
$$+ 2 \cdot \frac{x+1}{4} F_t\Big(\frac{x+1}{3}\Big) F_t\Big(\frac{x}{3}\Big)\Big(g_1\Big(\frac{x+2}{3}\Big) - g_2\Big(\frac{x+2}{3}\Big)\Big)$$
$$+ \frac{x+1}{4}\Big(g_1\Big(\frac{x}{3}\Big) - g_2\Big(\frac{x}{3}\Big)\Big)$$

$$= \Big[\Big(2 \cdot \frac{x+3}{4} F_t\Big(\frac{x+2}{3}\Big)^2 - 2 \cdot \frac{x+2}{4} F_t\Big(\frac{x+1}{3}\Big)^2\Big)$$
$$+ \Big(\frac{x+3}{4} F_t\Big(\frac{x+2}{3}\Big)^2 - \frac{x+1}{2} F_t\Big(\frac{x+2}{3}\Big) F_t\Big(\frac{x}{3}\Big)\Big)$$
$$+ \Big(\frac{x+1}{2} F_t\Big(\frac{x+1}{3}\Big) F_t\Big(\frac{x}{3}\Big)\Big)\Big]\Big(g_1\Big(\frac{x+2}{3}\Big) - g_2\Big(\frac{x+2}{3}\Big)\Big)$$
$$+ \Big[\Big(1 - F_t\Big(\frac{x+2}{3}\Big)\Big)\Big((x+2) F_t\Big(\frac{x+1}{3}\Big) - \frac{x+1}{2} F_t\Big(\frac{x}{3}\Big)\Big)\Big]\Big(g_1\Big(\frac{x+1}{3}\Big) - g_2\Big(\frac{x+1}{3}\Big)\Big)$$
$$+ \Big[\frac{x+1}{4}\Big(1 + F_t\Big(\frac{x+2}{3}\Big) - 2F_t\Big(\frac{x+1}{3}\Big)\Big)\Big(1 - F_t\Big(\frac{x+2}{3}\Big)\Big)\Big]\Big(g_1\Big(\frac{x}{3}\Big) - g_2\Big(\frac{x}{3}\Big)\Big)$$

Since F_t is a convex combination of the probability generating functions g_1 and g_2, this means $F_t \in \mathcal{S}$ (for any $t \in [0,1]$). It follows that $0 \le F_t \le 1$ on $[0,1]$ and that F_t is increasing on $[0,1]$ (w.r.t. x). This then implies that each of the three terms inside the first set of brackets above is non-negative. Likewise, it also follows that the expressions inside the second and third sets of brackets are non-negative. Coupling this with the fact that $g_1 \ge g_2$, it can then be concluded that $\frac{\partial(\mathcal{L}F_t(x))}{\partial t} \ge 0 \ \forall \ x, t \in [0,1]$, from which it follows that $\mathcal{L}g_1 \ge \mathcal{L}g_2$ on $[0,1]$.

It is also necessary to establish that $\mathcal{H}g_1 \ge \mathcal{H}g_2$ on $[0,1]$. Recalling the formula for $\mathcal{H}$ (see Definition 4.1.7) and using the fact, established above, that $\mathcal{L}g_1 \ge \mathcal{L}g_2$, this task amounts to showing that $\mathcal{G}g_1 \ge \mathcal{G}g_2$ (where $\mathcal{G}g(x) = \frac{x+3}{6} g(\frac{x+2}{3})^3 + \frac{x+2}{6}(g(\frac{x+1}{3})^2 - g(\frac{x+2}{3})g(\frac{x+1}{3})^2))$.

Once again letting $F_t(x) = tg_1(x) + (1-t)g_2(x)$, we find that

$$\begin{aligned}\frac{\partial(\mathcal{G}F_t(x))}{\partial t} &= \frac{x+3}{6}\cdot 3F_t\Big(\frac{x+2}{3}\Big)^2\Big(g_1\Big(\frac{x+2}{3}\Big) - g_2\Big(\frac{x+2}{3}\Big)\Big) \\ &+ \frac{x+2}{6}\cdot 2F_t\Big(\frac{x+1}{3}\Big)\Big(g_1\Big(\frac{x+1}{3}\Big) - g_2\Big(\frac{x+1}{3}\Big)\Big) \\ &- \frac{x+2}{6}F_t\Big(\frac{x+1}{3}\Big)^2\Big(g_1\Big(\frac{x+2}{3}\Big) - g_2\Big(\frac{x+2}{3}\Big)\Big) \\ &- \frac{x+2}{6}\cdot 2F_t\Big(\frac{x+2}{3}\Big)F_t\Big(\frac{x+1}{3}\Big)\Big(g_1\Big(\frac{x+1}{3}\Big) - g_2\Big(\frac{x+1}{3}\Big)\Big) \\ &= \Big[3\cdot\frac{x+3}{6}F_t\Big(\frac{x+2}{3}\Big)^2 - \frac{x+2}{6}F_t\Big(\frac{x+1}{3}\Big)^2\Big]\Big(g_1\Big(\frac{x+2}{3}\Big) - g_2\Big(\frac{x+2}{3}\Big)\Big) \\ &+ \Big[\frac{x+2}{3}F_t\Big(\frac{x+1}{3}\Big) - \frac{x+2}{3}F_t\Big(\frac{x+1}{3}\Big)F_t\Big(\frac{x+2}{3}\Big)\Big]\Big(g_1\Big(\frac{x+1}{3}\Big) - g_2\Big(\frac{x+1}{3}\Big)\Big)\end{aligned}$$

It then follows from the three facts –(i) $0 \le F_t \le 1$, (ii) F_t is increasing with respect to x, and (iii) $g_1 \ge g_2$ – that both terms in the above sum are non-negative, which means

$$\frac{\partial(\mathcal{G}F_t(x))}{\partial t} \ge 0 \;\forall\; x, t \in [0,1] \implies \mathcal{G}g_1 \ge \mathcal{G}g_2 \implies \mathcal{H}g_1 \ge \mathcal{H}g_2$$

as desired.

Having established the monotonicity of $\mathcal{L}$ and $\mathcal{H}$ on $\mathcal{S}$, we are now ready to prove the proposition. To start, define $\tilde{\mathcal{A}}$ to be an operator on $\mathcal{S}\times\mathcal{S}$ where

$$\begin{aligned}\tilde{\mathcal{A}}[f_1, f_2](x) = \frac{1}{3}f_1\Big(\frac{x+1}{2}\Big)^2 + \frac{x}{3}f_1\Big(\frac{x}{2}\Big) + \frac{x}{3}f_1\Big(\frac{x+1}{2}\Big)\Big(f_1\Big(\frac{x+1}{2}\Big) - f_1\Big(\frac{x}{2}\Big)\Big) \\ + \frac{1}{3}f_2\Big(\frac{x}{2}\Big) + \frac{1}{3}f_1\Big(\frac{x+1}{2}\Big)\Big(f_2\Big(\frac{x+1}{2}\Big) - f_2\Big(\frac{x}{2}\Big)\Big)\end{aligned}$$

Noting that $\mathcal{A}g(x) = \tilde{\mathcal{A}}[\mathcal{L}g, \mathcal{H}g](x)$ and that $\mathcal{L}\mathcal{S} \subseteq \mathcal{S}$, $\mathcal{H}\mathcal{S} \subseteq \mathcal{S}$, $\mathcal{L}g_1 \ge \mathcal{L}g_2$, and $\mathcal{H}g_1 \ge \mathcal{H}g_2$, it suffices to show that if $H_1, H_2, G_1, G_2 \in \mathcal{S}$ with $H_1 \ge G_1$ and $H_2 \ge G_2$, then the following inequality holds:

$$\tilde{\mathcal{A}}[H_1, H_2](x) \ge \tilde{\mathcal{A}}[G_1, G_2](x) \tag{4.1.17}$$

Defining $F_t^{(i)} = tH_i + (1-t)G_i$ (for $i = 1, 2$), if it can be established that

$$\frac{\partial(\tilde{\mathcal{A}}[F_t^{(1)}, F_t^{(2)}](x))}{\partial t} \geq 0 \tag{4.1.18}$$

$\forall\ t, x \in [0, 1]$, then (4.1.17) will follow. Now writing out the formula for the left side of (4.1.18) gives the following expression:

$$\begin{aligned}
&\frac{2}{3}F_t^{(1)}\Big(\frac{x+1}{2}\Big)\Big(H_1\Big(\frac{x+1}{2}\Big) - G_1\Big(\frac{x+1}{2}\Big)\Big) + \frac{x}{3}\Big(H_1\Big(\frac{x}{2}\Big) - G_1\Big(\frac{x}{2}\Big)\Big)\\
&+ \frac{2x}{3}F_t^{(1)}\Big(\frac{x+1}{2}\Big)\Big(H_1\Big(\frac{x+1}{2}\Big) - G_1\Big(\frac{x+1}{2}\Big)\Big) - \frac{x}{3}F_t^{(1)}\Big(\frac{x+1}{2}\Big)\Big(H_1\Big(\frac{x}{2}\Big) - G_1\Big(\frac{x}{2}\Big)\Big)\\
&- \frac{x}{3}F_t^{(1)}\Big(\frac{x}{2}\Big)\Big(H_1\Big(\frac{x+1}{2}\Big) - G_1\Big(\frac{x+1}{2}\Big)\Big) + \frac{1}{3}\Big(H_2\Big(\frac{x}{2}\Big) - G_2\Big(\frac{x}{2}\Big)\Big)\\
&+ \frac{1}{3}F_t^{(1)}\Big(\frac{x+1}{2}\Big)\Big(H_2\Big(\frac{x+1}{2}\Big) - G_2\Big(\frac{x+1}{2}\Big)\Big) + \frac{1}{3}F_t^{(2)}\Big(\frac{x+1}{2}\Big)\Big(H_1\Big(\frac{x+1}{2}\Big) - G_1\Big(\frac{x+1}{2}\Big)\Big)\\
&- \frac{1}{3}F_t^{(1)}\Big(\frac{x+1}{2}\Big)\Big(H_2\Big(\frac{x}{2}\Big) - G_2\Big(\frac{x}{2}\Big)\Big) - \frac{1}{3}F_t^{(2)}\Big(\frac{x}{2}\Big)\Big(H_1\Big(\frac{x+1}{2}\Big) - G_1\Big(\frac{x+1}{2}\Big)\Big)
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{3}\Bigg(\Big[\Big(2\cdot(x+1)F_t^{(1)}\Big(\frac{x+1}{2}\Big) - xF_t^{(1)}\Big(\frac{x}{2}\Big)\Big) + \Big(F_t^{(2)}\Big(\frac{x+1}{2}\Big) - F_t^{(2)}\Big(\frac{x}{2}\Big)\Big)\Big]\\
&\Big(H_1\Big(\frac{x+1}{2}\Big) - G_1\Big(\frac{x+1}{2}\Big)\Big) + \Big[x - xF_t^{(1)}\Big(\frac{x+1}{2}\Big)\Big]\Big(H_1\Big(\frac{x}{2}\Big) - G_1\Big(\frac{x}{2}\Big)\Big)\\
&+ \Big[F_t^{(1)}\Big(\frac{x+1}{2}\Big)\Big]\Big(H_2\Big(\frac{x+1}{2}\Big) - G_2\Big(\frac{x+1}{2}\Big)\Big) + \Big[1 - F_t^{(1)}\Big(\frac{x+1}{2}\Big)\Big]\Big(H_2\Big(\frac{x}{2}\Big) - G_2\Big(\frac{x}{2}\Big)\Big)\Bigg)
\end{aligned}$$

Now noting that $F_t^{(1)}, F_t^{(2)} \in \mathcal{S}$ (implying they are increasing and between 0 and 1), and recalling that $H_i \geq G_i$ for $i = 1, 2$, we see that (4.1.18) follows. This then implies (4.1.17), which implies $\mathcal{A}g_1 \geq \mathcal{A}g_2$. Hence, the proof of the proposition is complete. $\square$

4.1.5 Completing the proof of Theorem 4.1.1

Having established that $\mathcal{A}$ is monotone, it follows that $\mathcal{A}^n f \leq \mathcal{A}^n 1\ \forall\ n \geq 1$. Hence, to show that the expression on the left goes to 0, it suffices to show that $\mathcal{A}^n 1 \to 0$ on $[0, 1)$. This will be achieved by employing a method referred to in [6] as Poisson thinning. Specifically, it involves establishing the existence of a sequence $0 = a_0 < a_1 < a_2 < \dots$ (diverging to infinity) such that $\mathcal{A}^n 1 \le e^{a_n(x-1)}$ (the probability generating function for Poiss(a_n)) for all $n \ge 0$. The existence of this sequence is established in two parts. First, in Proposition 4.1.13 it is shown that $\forall\ a \ge 15$, $\mathcal{A}[e^{a(x-1)}] \le e^{(a+\epsilon)(x-1)}$ on $[0,1]$ (where $\epsilon = \frac{1}{20}$). It then follows from a simple induction argument which relies on the monotonicity of $\mathcal{A}$ established in Proposition 4.1.11, that $\mathcal{A}^n[e^{a(x-1)}] \le e^{(a+n\epsilon)(x-1)}\ \forall\ n \ge 1$. From this point, establishing the existence of the sequence $\{a_n\}$ reduces to establishing the existence of a finite sequence $0 = a_0 < a_1 < \cdots < a_N$ (where $a_N \ge 15$) such that $\mathcal{A}^n 1 \le e^{a_n(x-1)}$ on $[0,1]$ $\forall\ n$ with $0 \le n \le N$. This is accomplished (with the help of a Python program) in Proposition 4.1.15, where we inductively construct a sequence $0 = a_0 < a_1 < \cdots < a_N$ satisfying the above constraints. Along with Proposition 4.1.13, this will then establish the existence of $\{a_n\}$. The result $\mathcal{A}^n 1 \to 0$ on $[0,1)$ follows immediately, which then implies $\mathcal{A}^n f \to 0$ on $[0,1)$. As explained at the beginning of the previous section, this is then sufficient for establishing Theorem 4.1.1.

Proposition 4.1.13. *If $a \ge 15$ then $\mathcal{A}[e^{a(x-1)}] \le e^{(a+\frac{1}{20})(x-1)}$ on $[0,1]$.*

Proof. The first step will be to define a simple expression $\Psi(x,a)$ to serve as an upper bound on $\mathcal{A}[e^{a(x-1)}]$ (for $a \ge 15$). To start, note that

$$\begin{aligned}\mathcal{A}[g](x) = {} & \frac{x}{3}\mathcal{L}[g]\Big(\frac{x}{2}\Big) + \frac{x+1}{3}\Big(\mathcal{L}[g]\Big(\frac{x+1}{2}\Big)\Big)^2 - \frac{x}{3}\mathcal{L}[g]\Big(\frac{x+1}{2}\Big)\mathcal{L}[g]\Big(\frac{x}{2}\Big) \\ & + \frac{1}{3}\mathcal{H}[g]\Big(\frac{x}{2}\Big) + \frac{1}{3}\mathcal{L}[g]\Big(\frac{x+1}{2}\Big)\mathcal{H}[g]\Big(\frac{x+1}{2}\Big) - \frac{1}{3}\mathcal{L}[g]\Big(\frac{x+1}{2}\Big)\mathcal{H}[g]\Big(\frac{x}{2}\Big) \\ \le {} & \frac{x}{3}\mathcal{L}[g]\Big(\frac{x}{2}\Big) + \frac{x+1}{3}\Big(\mathcal{L}[g]\Big(\frac{x+1}{2}\Big)\Big)^2 + \frac{1}{3}\mathcal{H}[g]\Big(\frac{x}{2}\Big) + \frac{1}{3}\mathcal{L}[g]\Big(\frac{x+1}{2}\Big)\mathcal{H}[g]\Big(\frac{x+1}{2}\Big)\end{aligned} \tag{4.1.19}$$

$\forall\ g \in \mathcal{S}$. To bound the larger expression in (4.1.19) above (for $g(x) = e^{a(x-1)}$) we'll first obtain upper bounds for $\mathcal{L}[e^{a(x-1)}]$ and $\mathcal{H}[e^{a(x-1)}]$ as follows:

$$\mathcal{L}[e^{a(x-1)}] = \frac{x+3}{4}e^{a(x-1)} + 2\cdot\frac{x+2}{4}\Big(e^{\frac{2a}{3}(x-2)} - e^{a(x-\frac{5}{3})}\Big) + \frac{x+1}{4}\Big(e^{\frac{a}{3}(x-3)} - 2e^{\frac{2a}{3}(x-\frac{5}{2})} - e^{a(x-\frac{5}{3})} + 2e^{a(x-2)}\Big)$$

Observing that for all $x\in[0,1]$, $2\cdot\frac{x+2}{4}e^{a(x-\frac{5}{3})} \geq e^{-\frac{a}{3}}e^{\frac{2a}{3}(x-2)}$, $2\cdot\frac{x+1}{4}e^{\frac{2a}{3}(x-\frac{5}{2})} \geq \frac{1}{2}e^{-\frac{a}{3}}e^{\frac{2a}{3}(x-2)}$, and $\frac{x+1}{4}e^{a(x-\frac{5}{3})} \geq \frac{1}{4}e^{-\frac{a}{3}}e^{\frac{2a}{3}(x-2)}$, along with the fact that $2\cdot\frac{x+2}{4}e^{\frac{2a}{3}(x-2)} \leq \frac{3}{2}e^{\frac{2a}{3}(x-1)}$ and $2\cdot\frac{x+1}{4}e^{a(x-2)} \leq e^{-\frac{a}{3}}e^{\frac{2a}{3}(x-2)}$, we find that if we make the given substitutions in the expression for $\mathcal{L}[e^{a(x-1)}]$ above, it gives

$$\mathcal{L}[e^{a(x-1)}] \leq \frac{x+3}{4}e^{a(x-1)} + \frac{x+1}{4}e^{\frac{a}{3}(x-3)} + ce^{\frac{2a}{3}(x-2)}$$

(where $c = \frac{3}{2} - \frac{3}{4}e^{-\frac{a}{3}}$). The above upper bound on $\mathcal{L}[e^{a(x-1)}]$ will be denoted as $l_a(x)$. Now noting that

$$\mathcal{H}[e^{a(x-1)}] = \frac{x+3}{4}e^{a(x-1)} + 2\cdot\frac{x+2}{6}\Big(e^{\frac{2a}{3}(x-2)} - e^{a(x-\frac{5}{3})}\Big) + \frac{x+1}{12}\Big(e^{\frac{a}{3}(x-3)} - 2e^{\frac{2a}{3}(x-\frac{5}{2})} - e^{a(x-\frac{5}{3})} + 2e^{a(x-2)}\Big)$$

applying a similar set of inequalities then gives the bound

$$\mathcal{H}[e^{a(x-1)}] \leq \frac{x+3}{4}e^{a(x-1)} + \frac{x+1}{12}e^{\frac{a}{3}(x-3)} + de^{\frac{2a}{3}(x-2)}$$

(where $d = 1 - \frac{7}{12}e^{-\frac{a}{3}}$). This upper bound on $\mathcal{H}[e^{a(x-1)}]$ will be denoted as $h_a(x)$.

Combining the above bounds with (4.1.19) we obtain the inequality

$$\mathcal{A}[e^{a(x-1)}] \leq \frac{x}{3}l_a\Big(\frac{x}{2}\Big) + \frac{x+1}{3}l_a\Big(\frac{x+1}{2}\Big)^2 + \frac{1}{3}h_a\Big(\frac{x}{2}\Big) + \frac{1}{3}l_a\Big(\frac{x+1}{2}\Big)h_a\Big(\frac{x+1}{2}\Big)$$

Writing out this full expression gives the following:

$$\begin{aligned}
\mathcal{A}[e^{a(x-1)}] \le \frac{x}{3}\Big(&\frac{x+6}{8}e^{\frac{a}{2}(x-2)} + \frac{x+2}{8}e^{\frac{a}{6}(x-6)} + ce^{\frac{a}{3}(x-4)}\Big) \\
&+ \frac{x+1}{3}\Big(\Big(\frac{x+7}{8}\Big)^2 e^{a(x-1)} + \Big(\frac{x+3}{8}\Big)^2 e^{\frac{a}{3}(x-5)} + c^2 e^{\frac{2a}{3}(x-3)} \\
&+ 2\cdot\frac{x+7}{8}\cdot\frac{x+3}{8}e^{\frac{2a}{3}(x-2)} + 2\cdot\frac{x+7}{8}\cdot ce^{\frac{5a}{6}(x-\frac{9}{5})} + 2\cdot\frac{x+3}{8}\cdot ce^{\frac{a}{2}(x-\frac{11}{3})}\Big) \\
&+ \frac{1}{3}\Big(\frac{x+6}{8}e^{\frac{a}{2}(x-2)} + \frac{x+2}{24}e^{\frac{a}{6}(x-6)} + de^{\frac{a}{3}(x-4)}\Big) \\
&+ \frac{1}{3}\Big(\Big(\frac{x+7}{8}\Big)^2 e^{a(x-1)} + \frac{x+3}{8}\cdot\frac{x+3}{24}e^{\frac{a}{3}(x-5)} + cde^{\frac{2a}{3}(x-3)} \\
&+ \frac{4}{3}\cdot\frac{x+7}{8}\cdot\frac{x+3}{8}e^{\frac{2a}{3}(x-2)} + (c+d)\frac{x+7}{8}e^{\frac{5a}{6}(x-\frac{9}{5})} + (\frac{c}{3}+d)\frac{x+3}{8}e^{\frac{a}{2}(x-\frac{11}{3})}\Big) \\
\\
&= \frac{x+2}{3}\Big(\frac{x+7}{8}\Big)^2 e^{a(x-1)} + \frac{x+1}{3}\cdot\frac{x+6}{8}e^{\frac{a}{2}(x-2)} + \frac{x+\frac{1}{3}}{3}\cdot\frac{x+2}{8}e^{\frac{a}{6}(x-6)} \\
&+ e^{\frac{2a}{3}(x-2)}\Big(\frac{x}{3}\cdot ce^{-\frac{a}{3}x} + \frac{x+1}{3}\cdot\Big(\frac{x+3}{8}\Big)^2 e^{-\frac{a}{3}(x+1)} + \frac{x+1}{3}\cdot c^2 e^{-\frac{2a}{3}} \\
&+ 2\cdot\frac{x+1}{3}\cdot\frac{x+7}{8}\cdot\frac{x+3}{8} + 2\cdot\frac{x+1}{3}\cdot\frac{x+7}{8}\cdot ce^{\frac{a}{6}(x-1)} \\
&+ 2\cdot\frac{x+1}{3}\cdot\frac{x+3}{8}\cdot ce^{-\frac{a}{6}(x+3)} + \frac{d}{3}e^{-\frac{a}{3}x} + \Big(\frac{x+3}{24}\Big)^2 e^{-\frac{a}{3}(x+1)} + \frac{c}{3}de^{-\frac{2a}{3}} \\
&+ \frac{4}{9}\cdot\frac{x+7}{8}\cdot\frac{x+3}{8} + (c+d)\frac{x+7}{24}e^{\frac{a}{6}(x-1)} + \frac{1}{3}\Big(\frac{c}{3}+d\Big)\frac{x+3}{8}e^{-\frac{a}{6}(x+3)}\Big)
\end{aligned}$$

An upper bound for the long expression in parentheses above can be obtained by replacing x with 1 wherever it is part of an increasing expression (such as $\frac{x}{3}$ or e^{ax}) and replacing it with 0 wherever it is part of a decreasing expression. After simplifying, this gives the following inequality:

$$\begin{aligned}
\mathcal{A}[e^{a(x-1)}] \le &\frac{x+2}{3}\Big(\frac{x+7}{8}\Big)^2 e^{a(x-1)} + \frac{x+1}{3}\cdot\frac{x+6}{8}e^{\frac{a}{2}(x-2)} + \frac{x+\frac{1}{3}}{3}\cdot\frac{x+2}{8}e^{\frac{a}{6}(x-6)} \\
&+ \Big(\frac{41}{9} - \frac{61}{36}e^{-\frac{a}{3}} + \frac{5}{4}e^{-\frac{a}{2}} + 2e^{-\frac{2a}{3}} - \frac{23}{36}e^{-\frac{5a}{6}} - \frac{49}{24}e^{-a} + \frac{25}{48}e^{-\frac{4a}{3}}\Big)e^{\frac{2a}{3}(x-2)}
\end{aligned}$$

Note that for $a \ge 3$ the following string of inequalities holds

$$\frac{41}{9} - \frac{61}{36}e^{-\frac{a}{3}} + \frac{5}{4}e^{-\frac{a}{2}} + 2e^{-\frac{2a}{3}} - \frac{23}{36}e^{-\frac{5a}{6}} - \frac{49}{24}e^{-a} + \frac{25}{48}e^{-\frac{4a}{3}} \le \frac{41}{9} + e^{-\frac{a}{3}}\Big(\frac{5}{4}e^{-\frac{a}{6}} + 2e^{-\frac{a}{3}} - \frac{61}{36}\Big) \le \frac{41}{9}$$

Hence, we now finally define $\Psi(x,a)$ to be

$$\Psi(x,a) = \frac{x+2}{3}\Big(\frac{x+7}{8}\Big)^2 e^{a(x-1)} + \frac{x+1}{3}\cdot\frac{x+6}{8}e^{\frac{a}{2}(x-2)} + \frac{x+\frac{1}{3}}{3}\cdot\frac{x+2}{8}e^{\frac{a}{6}(x-6)} + \frac{41}{9}e^{\frac{2a}{3}(x-2)}$$

From the above computations, it follows that $\mathcal{A}[e^{a(x-1)}] \leq \Psi(x,a)$ on $[0,1]$ for $a \geq 15$ as desired (though as we saw above, having $a \geq 3$ is sufficient for this inequality to hold).

Now that $\Psi(x,a)$ has been defined, we'll proceed to prove the proposition by splitting up the interval $[0,1]$ into four parts, and showing that the inequality stated in the proposition holds for all x in each one of them.

(i) $x \in [1-c(a),1]$ (where $c(a) = a^{-\frac{9}{4}}$).

Since $\mathcal{A}[e^{a(x-1)}]$ is a convex function of x (this follows from it being a probability generating function), this means that for any $c \in [0,1]$ we have $\mathcal{A}[e^{a(x-1)}] \leq \mathcal{A}[e^{a(c-1)}] + \Big(1 - \mathcal{A}[e^{a(c-1)}]\Big)\Big(\frac{x-c}{1-c}\Big)$ $\forall\ x \in [c,1]$. Using the fact that $\mathcal{A}[e^{a(x-1)}] \leq \Psi(x,a)$ (for $a \geq 15$), it follows that $\mathcal{A}[e^{a(x-1)}] \leq \Psi(c,a) + \Big(1 - \Psi(c,a)\Big)\Big(\frac{x-c}{1-c}\Big)$ on $[c,1]$. Noting that $e^{(a+\frac{1}{20})(x-1)}$ is itself a convex function of x that has derivative $a+\frac{1}{20}$ at $x = 1$, it follows that $e^{(a+\frac{1}{20})(x-1)} \geq 1-(a+\frac{1}{20})(1-x)$ on $[0,1]$. Putting these last two observations together, we find that if we can establish

$$\Psi(1-c(a),a) \leq 1-(a+\frac{1}{20})(1-(1-c(a))) \tag{4.1.20}$$

then it will follow that

$$\begin{aligned}\mathcal{A}[e^{a(x-1)}] &\leq 1-\Big(a+\frac{1}{20}\Big)\Big(1-(1-c(a))\Big) + \Big(a+\frac{1}{20}\Big)\Big(1-(1-c(a))\Big)\Big(\frac{x-(1-c(a))}{1-(1-c(a))}\Big)\\ &= 1-\Big(a+\frac{1}{20}\Big)\Big(1-x\Big) \leq e^{(a+\frac{1}{20})(x-1)}\end{aligned}$$

for all $x \in [1-c(a),1]$.

Now using the formula for Ψ, we get the string of inequalities

$$\begin{aligned}\Psi(1-c(a),a) &\le \Big(1-\frac{c(a)}{3}\Big)\Big(1-\frac{c(a)}{8}\Big)^2 e^{-ac(a)} + \frac{7}{12}e^{-\frac{a}{2}} + \frac{1}{6}e^{-\frac{5a}{6}} + \frac{41}{9}e^{-\frac{2a}{3}} \\ &\le e^{-(a+\frac{7}{12})c(a)} + \frac{7}{12}e^{-\frac{a}{2}} + \frac{85}{18}e^{-\frac{2a}{3}} \\ &\le 1-\Big(a+\frac{7}{12}\Big)c(a) + \frac{13}{24}a^2c(a)^2 + \frac{7}{12}e^{-\frac{a}{2}} + \frac{85}{18}e^{-\frac{2a}{3}}\end{aligned}$$

(where the last inequality follows from the fact that $e^{-x} \le 1-x+\frac{x^2}{2}$ for $x \in [0,1]$, and the fact that $\Big(a+\frac{7}{12}\Big)^2 \le \frac{13}{12}a^2$ for $a \ge 15$). Plugging $c(a) = a^{-\frac{9}{4}}$ into the above expression then gives

$$\Psi(1-c(a),a) \le 1-\Big(a+\frac{7}{12}\Big)c(a) + \Big(\frac{13}{24}a^{-\frac{1}{4}} + \frac{7}{12}a^{\frac{9}{4}}e^{-\frac{a}{2}} + \frac{85}{18}a^{\frac{9}{4}}e^{-\frac{2a}{3}}\Big)c(a)$$

Now to establish (4.1.20) it just needs to be shown that

$$\frac{13}{24}a^{-\frac{1}{4}} + \frac{7}{12}a^{\frac{9}{4}}e^{-\frac{a}{2}} + \frac{85}{18}a^{\frac{9}{4}}e^{-\frac{2a}{3}} \le \frac{7}{12} - \frac{1}{20} \tag{4.1.21}$$

for $a \ge 15$. So observe the string of inequalities below (which holds for $a \ge \frac{9}{2}$), where the left side is equal to the derivative of the left side of (4.1.21).

$$\begin{aligned}&-\frac{13}{96}a^{-\frac{5}{4}} + \frac{9}{4}a^{\frac{5}{4}}\Big(\frac{7}{12}e^{-\frac{a}{2}} + \frac{85}{18}e^{-\frac{2a}{3}}\Big) - a^{\frac{9}{4}}\Big(\frac{1}{2}\cdot\frac{7}{12}e^{-\frac{a}{2}} + \frac{2}{3}\cdot\frac{85}{18}e^{-\frac{2a}{3}}\Big) \\ &< \Big(\frac{9}{4}a^{\frac{5}{4}} - \frac{1}{2}a^{\frac{9}{4}}\Big)\Big(\frac{7}{12}e^{-\frac{a}{2}} + \frac{85}{18}e^{-\frac{2a}{3}}\Big) < 0\end{aligned}$$

Combining this with the fact that the left side of (4.1.21) equals $.513 < \frac{7}{12} - \frac{1}{20}$ at $a = 15$, we find that (4.1.21) does indeed hold for $a \ge 15$ which, as was shown, implies that $\mathcal{A}[e^{a(x-1)}] \le e^{(a+\frac{1}{20})(x-1)}$ on $[1-c(a),1]$.

(ii) $x \in [\frac{1}{2}, 1-c(a))$.

Denoting $e^{-a(x-1)}\Psi(x,a)$ as $Q(x,a)$ (for $a \ge 15$), it suffices to show that $Q(x,a) \le e^{\frac{1}{20}(x-1)}$ on $[\frac{1}{2}, 1-c(a))$. Since we saw in (i) that $\Psi(1-c(a),a) \le 1-\Big(a+\frac{1}{20}\Big)c(a) \le e^{(a+\frac{1}{20})((1-c(a))-1)}$, it follows that $Q(1-c(a),a) \le e^{\frac{1}{20}((1-c(a))-1)}$, which implies that to prove $Q(x,a) \le e^{\frac{1}{20}(x-1)}$, it suffices to prove that the right side of

$$\frac{\partial\left(e^{\frac{1}{20}(x-1)}\right)}{\partial x} \le \frac{1}{20} \le \frac{\partial Q(x,a)}{\partial x}$$

holds on $[\frac{1}{2}, 1-c(a))$. Computing the formula for the expression on the right, we get

$$\begin{aligned}\frac{\partial Q(x,a)}{\partial x} &= \frac{1}{3}\Big(\frac{x+7}{8}\Big)^2 + \frac{1}{4}\cdot\frac{x+2}{3}\cdot\frac{x+7}{8} + \frac{1}{3}\cdot\frac{x+6}{8}e^{-\frac{a}{2}x} + \frac{1}{8}\cdot\frac{x+1}{3}e^{-\frac{a}{2}x} \\ &\quad - \frac{a}{2}\cdot\frac{x+1}{3}\cdot\frac{x+6}{8}e^{-\frac{a}{2}x} + \frac{1}{3}\cdot\frac{x+2}{8}e^{-\frac{5a}{6}x} + \frac{1}{8}\cdot\frac{x+\frac{1}{3}}{3}e^{-\frac{5a}{6}x} \\ &\quad - \frac{5a}{6}\cdot\frac{x+\frac{1}{3}}{3}\cdot\frac{x+2}{8}e^{-\frac{5a}{6}x} - \frac{a}{3}\cdot\frac{41}{9}e^{-\frac{a}{3}(x+1)} \\ &\ge \frac{1}{3}\Big(\frac{x+7}{8}\Big)^2 + \frac{1}{4}\cdot\frac{x+2}{3}\cdot\frac{x+7}{8} - \frac{a}{2}\cdot\frac{x+1}{3}\cdot\frac{x+6}{8}e^{-\frac{a}{2}x} \\ &\quad - \frac{5a}{6}\cdot\frac{x+\frac{1}{3}}{3}\cdot\frac{x+2}{8}e^{-\frac{5a}{6}x} - \frac{a}{3}\cdot\frac{41}{9}e^{-\frac{a}{3}(x+1)}\end{aligned}$$

Plugging in $x = \frac{1}{2}$ for the exponential functions and the polynomial expressions that follow a plus sign, and $x = 1$ for the polynomial expressions that follow a minus sign, we find that the expression on the right side of the inequality is greater than or equal to

$$\frac{1}{3}\Big(\frac{15}{16}\Big)^2 + \frac{1}{4}\cdot\frac{5}{6}\cdot\frac{15}{16} - \frac{a}{2}\cdot\frac{2}{3}\cdot\frac{7}{8}e^{-\frac{a}{4}} - \frac{5a}{6}\cdot\frac{4}{9}\cdot\frac{3}{8}e^{-\frac{5a}{12}} - \frac{a}{3}\cdot\frac{41}{9}e^{-\frac{a}{2}}$$

on $[\frac{1}{2}, 1-c(a))$. Simplifying, and using the string of inequalities above, gives

$$\frac{\partial Q(x,a)}{\partial x} \ge \frac{125}{256} - \frac{7a}{24}e^{-\frac{a}{4}} - \frac{5a}{36}e^{-\frac{5a}{12}} - \frac{41a}{27}e^{-\frac{a}{2}} \tag{4.1.22}$$

on this interval. If we differentiate this expression with respect to a we get

$$\left(\frac{a}{4}-1\right)\cdot\frac{7}{24}e^{-\frac{a}{4}} + \left(\frac{5a}{12}-1\right)\cdot\frac{5}{36}e^{-\frac{5a}{12}} + \left(\frac{a}{2}-1\right)\cdot\frac{41}{27}e^{-\frac{a}{2}} \ge 0$$

(recall we're assuming $a \ge 15$). Coupling this with the fact that the expression on the right side of (4.1.22), when evaluated at $a = 15$, is equal to $.369 > \frac{1}{20}$, we indeed find that

$\frac{\partial Q(x,a)}{\partial x} \geq \frac{1}{20}$ on $[\frac{1}{2}, 1-c(a))$ for $a \geq 15$. As was shown, this implies that $Q(x,a) \leq e^{\frac{1}{20}(x-1)}$, which implies $\mathcal{A}[e^{a(x-1)}] \leq e^{(a+\frac{1}{20})(x-1)}$ on $[\frac{1}{2}, 1-c(a))$ for $a \geq 15$ as desired.

(iii) $x \in [\frac{1}{8}, \frac{1}{2})$.

Once again it suffices to show that $Q(x,a) \leq e^{\frac{1}{20}(x-1)}$ (this time on $[\frac{1}{8}, \frac{1}{2})$). Taking the formula for $Q(x,a) = e^{-a(x-1)}\Psi(x,a)$ and substituting $\frac{1}{2}$ for x when it is part of a polynomial function, and $\frac{1}{8}$ when it is part of an exponential expression (with negative exponent), we find that

$$Q(x,a) \leq \frac{375}{512} + \frac{13}{32}e^{-\frac{a}{16}} + \frac{25}{288}e^{-\frac{5a}{48}} + \frac{41}{9}e^{-\frac{3a}{8}}$$

for $x \in [\frac{1}{8}, \frac{1}{2})$. Since the expression on the right is a decreasing function of a, plugging in $a = 15$ shows that

$$Q(x,a) \leq \frac{375}{512} + \frac{13}{32}e^{-\frac{15}{16}} + \frac{25}{288}e^{-\frac{25}{16}} + \frac{41}{9}e^{-\frac{45}{8}} \approx .926 < e^{\frac{1}{20}(\frac{1}{8}-1)} \leq e^{\frac{1}{20}(x-1)}$$

on $[\frac{1}{8}, \frac{1}{2})$ for $a \geq 15$, thus giving the desired inequality.

(iv) $x \in [0, \frac{1}{8})$.

Using the exact same method that was used in (iii), but plugging in 0 and $\frac{1}{8}$ in place of $\frac{1}{8}$ and $\frac{1}{2}$ respectively, we find that

$$Q(x,a) \leq \frac{17}{24}\Big(\frac{57}{64}\Big)^2 + \frac{3}{8} \cdot \frac{49}{64} + \frac{11}{72} \cdot \frac{17}{64} + \frac{41}{9}e^{-5} \approx .9203 < e^{-\frac{1}{20}} \leq e^{\frac{1}{20}(x-1)}$$

on $[0, \frac{1}{8})$ for $a \geq 15$, once again yielding the desired inequality.

Combining parts (i)-(iv) we find that $\mathcal{A}[e^{a(x-1)}] \leq e^{(a+\frac{1}{20})(x-1)}$ does hold on $[0,1]$ for $a \geq 15$, thus completing the proof of the proposition. □

Corollary 4.1.14. *If $a \geq 15$ and $n \geq 1$ then $\mathcal{A}^n[e^{a(x-1)}] \leq e^{(a+n\epsilon)(x-1)}$ (where $\epsilon = \frac{1}{20}$).*

Proof. We know from the previous result that the statement holds for $n = 1$. Now assume it holds for some $n \geq 1$. Then by the monotonicity of $\mathcal{A}$ on $\mathcal{S}$ (established in Proposition 4.1.11), along with Proposition 4.1.13, it follows that

$$\mathcal{A}^{n+1}[e^{a(x-1)}] = \mathcal{A}\Big[\mathcal{A}^n[e^{a(x-1)}]\Big] \leq \mathcal{A}[e^{(a+n\epsilon)(x-1)}] \leq e^{(a+(n+1)\epsilon)(x-1)}$$

on $[0,1]$. By induction we then find that $\mathcal{A}^n[e^{a(x-1)}] \leq e^{(a+n\epsilon)(x-1)}$ on $[0,1]$ for all $n \geq 1$. $\square$

Having proven Proposition 4.1.13 and it's corollary, our last significant task is to establish the following result.

Proposition 4.1.15. *There exists a finite sequence $0 = a_0 < a_1 < \cdots < a_N$ (with $a_N \geq 15$) such that $\mathcal{A}^n 1 \leq e^{a_n(x-1)}$ on $[0,1]$ for all n with $0 \leq n \leq N$.*

The proof of Proposition 4.1.15 will make use of the following lemma.

Lemma 4.1.16. *Let f_1 and f_2 be convex increasing functions on $[0,1]$ where f_1 is differentiable and $f_1(1) = f_2(1)$. Suppose there is a finite sequence $1 = c_0 > c_1 > \cdots > c_n = 0$ that satisfies*

$$f_2(c_{j+1}) \leq f_1(c_j) - (c_j - c_{j+1}) f_1'(c_j) \tag{4.1.23}$$

for all j with $0 \leq j < n$. Then $f_1(x) \geq f_2(x)\ \forall\ x \in [0,1]$.

Proof. Assume $f_1(c_j) \geq f_2(c_j)$ for some $j < n$. We know by the convexity (and differentiability) of f_1 that $f_1(t) \geq f_1(c_j) - f_1'(c_j)(c_j - t)$ for $t \in [c_{j+1}, c_j]$. By the convexity of f_2 it follows that

$$f_2(t) \leq f_2(c_j) - \frac{f_2(c_j) - f_2(c_{j+1})}{c_j - c_{j+1}}(c_j - t) \leq f_1(c_j) - f_1'(c_j)(c_j - t) \leq f_1(t)$$

for $t \in [c_{j+1}, c_j]$ (where the middle inequality follows from $f_1(c_j) \geq f_2(c_j)$, (4.1.23), and the fact that both functions are linear). Since $f_1(1) \geq f_2(1)$, it follows by induction that $f_1(t) \geq f_2(t)\ \forall\ t \in [0,1]$. $\square$

Proof of Proposition 4.1.15. Let $u \geq 0$, $a > 0$, and $c_i = \frac{256-i}{256}$ for $0 \leq i \leq 256$. Recalling that $\mathcal{A}[e^{u(x-1)}]$ is a probability generating function (implying it is increasing and convex on $[0,1]$) and noting that $e^{(u+a)(x-1)}$ is increasing, convex, and differentiable on $[0,1]$, along with the fact that the two functions both equal 1 at $x = 1$, we find that if (4.1.23) holds for each i with $0 \leq i < 256$ (where $f_1(x) = e^{(u+a)(x-1)}$ and $f_2(x) = \mathcal{A}[e^{u(x-1)}]$), then it will follow from Lemma 4.1.16 that $\mathcal{A}[e^{u(x-1)}] \leq e^{(u+a)(x-1)}$ on $[0,1]$. Now observe the attached Python program. For each pass through the while loop it checks to see if (4.1.23) holds (at each c_i) for $a = \frac{1}{16}$, $f_1(x) = e^{(u+a)(x-1)}$, and $f_2(x) = \mathcal{A}[e^{u(x-1)}]$. If (4.1.23) does hold at each c_i then u is increased by $\frac{1}{16}$ and we repeat the process with the new values of u, f_1, and f_2. If not, a is set to $\frac{1}{32}$ and it tests to see if (4.1.23) holds for each i for this value of a. If so, u is increased by $\frac{1}{32}$ and the process is repeated for the new u, f_1, and f_2 (again starting with $a = \frac{1}{16}$). If not, it tests again with $a = \frac{3}{256}$. If (4.1.23) holds at each c_i then the process repeats with u, f_1, and f_2 adjusted accordingly. If not, then the while loop terminates. The loop keeps running until either it terminates (as described above) because (4.1.23) fails to hold at some c_i for a equal to each of the three specified values ($\frac{1}{16}$, $\frac{1}{32}$, and $\frac{3}{256}$), or because $m = 341$ (i.e. we've passed through the loop 340 times). In order to ensure that the program does not return a false negative (as a result of rounding) when evaluating the inequality inside the loop, interval arithmetic is employed (see https://en.wikipedia.org/wiki/Interval_arithmetic for a definition) so that, for each a, u, i combination that is considered, the loop only fails to break if A (an interval containing the precise value of $f_1(c_j) - (c_j - c_{j+1})f_1'(c_j))$ lies entirely to the right of B (an interval containing the precise value of $f_2(c_{j+1})$). At the end, the program prints the final values of m and u. Upon running the program you will find that these values are 341 and 15.203125 respectively (the program prints the current value of m as it runs, and should take about eight minutes to finish).

Now for $0 \le n \le 340$ let a_n represent the value taken by u following the nth pass through the loop. Hence, $0 = a_0 < a_1 < \cdots < a_{340} = 15.203125$ and $a_{j+1} - a_j \in \left\{\frac{1}{16}, \frac{1}{32}, \frac{3}{256}\right\}$ for each $0 \le j < 340$. Furthermore, since the program output indicates that 340 passes through the loop were completed, this implies that (4.1.23) holds (at each c_i for $0 \le i < 256$) for each $0 \le j \le 340$ (where $f_1(x) = e^{a_{j+1}(x-1)}$ and $f_2(x) = \mathcal{A}[e^{a_j(x-1)}]$). By Lemma 4.1.16, this implies that $\mathcal{A}[e^{a_j(x-1)}] \le e^{a_{j+1}(x-1)}$ on $[0,1]$ for every $0 \le j < 340$. It then follows from the same induction argument that was used to prove Corollary 4.1.14 that $\mathcal{A}^n 1 \le e^{a_n(x-1)}$ for every n with $0 \le n \le 340$. Hence, we find that the a_n terms satisfy the conditions given in the statement of the proposition. Hence, the proof is complete. □

With Proposition 4.1.15 established, the proof of Theorem 4.1.1 can now be completed.

Proof of Theorem 4.1.1. Proposition 4.1.15 and Corollary 4.1.14 together indicate that on $[0,1)$, $\mathcal{A}^n 1 \to 0$ as $n \to \infty$. Since the monotonicity of $\mathcal{A}$ implies that $\mathcal{A}^n f \le \mathcal{A}^n 1 \;\forall\; n \ge 0$, it follows that $\mathcal{A}^n f \to 0$ on $[0,1)$ as $n \to \infty$. Since f is known to be a fixed point of $\mathcal{A}$, this then means that $f(x) = 0$, which implies that $\mathbb{P}(V = \infty) = 1$. Recalling from Proposition 4.1.4 that V (the number of times the root is hit in the self-similar model on $\mathbb{T}_{3,2}$) is dominated by Z (the number of times it is hit in the original model on $\mathbb{T}_{3,2}$), it follows that $\mathbb{P}(Z = \infty) = 1$. Thus we find that the frog model on $\mathbb{T}_{3,2}$ is indeed recurrent.

Hence, the proof of Theorem 4.1.1 is complete. □

Appendix A

```
from mpmath import *

u=mpi(0)

def h(x):
    return iv.exp(u*(x-1))

def f_1(x,a):
    return h(x)*iv.exp(a*(x-1))

def h_1(x,y,a):
    return (1-y*(u+a))*f_1(x,a)

def L_3(f):
    def g(x):
```

```
        return ((x+3)/4 * f((x+2)/3)**3 + (x+2)/2 * ( f((x+1)/3)**2
               -f((x+2)/3)*f((x+1)/3)**2 ) - (x+1)/4 * (f((x+2)/3)**2
               * f(x/3) + 2*f((x+1)/3)*f(x/3) - 2*f((x+2)/3)*f((x+1)/3)
               *f(x/3) - f(x/3)))
    return g


def H_3(f):
    def g(x):
        return ((x+3)/4 * f((x+2)/3)**3 + (x+2)/3 * ( f((x+1)/3)**2
            - f((x+2)/3)*f((x+1)/3)**2 ) - (x+1)/12 * (f((x+2)/3)**2
            * f(x/3) + 2*f((x+1)/3)*f(x/3) - 2*f((x+2)/3)*f((x+1)/3)
            *f(x/3) - f(x/3)))
    return g


def G_1(f):
    def G_1f(x):
        a=L_3(f)(x/2)
        b=L_3(f)((x+1)/2)
        c=H_3(f)(x/2)
        d=H_3(f)((x+1)/2)

        return (x/3)*a+((x+1)/3)*b**2-(x/3)*a*b+(1/3)*c+(1/3)*b*d-(1/3)*b*c
    return G_1f
```

```
m=1


while m < 341:

    f_2=G_1(h)

    for a in map(mpi, [1/16,1/32,3/256]):

        for j in range(256):

            A = f_2( mpi(j/256) )

            B = h_1( mpi(j+1)/256, mpi(1/256), a)

            if A.b>B.a:

                break

        else: break

    else: break


    print(m)

    m=m+1

    u+=a

print(m, u)
```

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