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WHAT IS THE PROBABILITY OF INTERSECTING THE SET OF BROWNIAN DOUBLE POINTS?

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We give potential theoretic estimates for the probability that a set A contains a double point of planar Brownian motion run for unit time. Unlike the probability for A to intersect the range of a Markov process, this cannot be estimated by a capacity of the set A. Instead, we introduce the notion of a capacity with respect to two gauge functions simultaneously. We also give a polar decomposition of A into a set that never intersects the set of Brownian double points and a set for which intersection with the set of Brownian double points is the same as intersection with the Brownian path.

1. Introduction. Let A be a compact subset of the $\frac{1}{3}$-unit disk in the plane. For fifty years it has been known that A intersects the path of a Brownian motion with positive probability if and only if A has positive Newtonian capacity. In fact, the Newtonian (logarithmic) capacity gives an estimate, up to a constant factor, the probability that A is hit by a Brownian motion started, say, from the point $(1,0)$ and run for a fixed time. The estimate is of course stronger than the dichotomous result, and moreover, it turns out to be important when examining properties of intersections with random sets; see, for example, the simple Cantor-type random fractal shown in Peres [5] to be "intersection-equivalent" to the Brownian motion; see also the remark after Theorem 2.3.

Similar results are known for much more general Markov processes. Let $G(x,y)$ denote the Green function for a transient Markov process. The capacity, $\mathrm{Cap}_K(A)$ of a set A, with respect to a kernel K is defined to be the

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reciprocal of the infimum of energies

$$\mathcal{E}_K(\mu) := \int\int K(x,y)\,d\mu(x)\,d\mu(y)$$

as μ ranges over probability measures supported on A. In a wide variety of cases it is known that the range of the process intersects A with positive probability if and only if A has positive capacity with respect to the Green kernel. The same is true of any of a number of related kernels, and choosing the Martin kernel $M(x,y) = G(x,y)/G(\rho,y)$ with respect to any starting point ρ (see, e.g., Benjamini, Pemantle and Peres [1]) leads to the estimate

$$\tfrac{1}{2}\,\mathrm{Cap}_M(A) \le \mathbf{P}_\rho(\text{the process intersects } A) \le \mathrm{Cap}_M(A).$$

We are chiefly interested in the set $\mathcal{D}$ of double points of a planar Brownian motion. We work on a probability space $(\Omega, \{\mathcal{F}_t\}, \mathbf{P})$ on which are defined two independent Brownian motions, B_t and $\tilde{B}_t$, both started from the point $\rho := (1,0)$. The notation $\mathbf{P}_x$ (or $\mathbf{P}_{x,y}$) will be used when a different starting point (or points) is required. Let $\tau_* = \inf\{t : |B_t| = 3\}$ be the exit time of B_t from the disk $\{|x| \le 3\}$. Formally, then,

$$\mathcal{D} := \{x : B_r = B_s = x \text{ for some } 0 < r < s < \tau_*\}.$$

The choice to start at ρ, stop at τ_*, and choose sets inside the $\frac{1}{3}$-unit disk are conveniences that make the Martin and Green kernel both comparable to $|\log|x-y||$.

The random set $\mathcal{D}$ is not the range of any Markov process, but we may still ask about the probability for the random set $\mathcal{D}$ to intersect a fixed set A. A closely related random set to $\mathcal{D}$ is the intersection of two independent Brownian motions, denoted here by

$$\mathcal{I} := \{x : B_r = \tilde{B}_s = x \text{ for some } 0 < r < \tau_*, 0 < s < \tilde{\tau_*}\},$$

where $\tilde{\tau_*} = \inf\{t : |\tilde{B}_t| \ge 3\}$. Fitzsimmons and Salisbury [3] showed, for a subset A of the $\frac{1}{3}$-unit disk, that $\mathbf{P}(\mathcal{I} \cap A \ne \varnothing)$ may be estimated up to a constant factor by $\mathrm{Cap}_L(A)$ where $L(x,y) = (\log|x-y|)^2$. In general, they show that taking intersections of random sets multiplies the kernels in the capacity tests; see also Salisbury [6] and Peres [5]. The set $\mathcal{D}$ may be written as a countable union of the sets of ε-separated double points (we use a time separation of ε^2 so that ε may be thought of as a small spatial unit):

$$\mathcal{D}_\varepsilon := \{x : B_r = B_s = x \text{ for some } 0 < r < r + \varepsilon^2 \le s < \tau_*\}.$$

It is not hard to see that each random set $\mathcal{D}_\varepsilon$ behaves similarly to the set $\mathcal{I}$, but with an increasingly poor constant. In other words,

$$c_\varepsilon\,\mathrm{Cap}_L(A) \le \mathbf{P}(\mathcal{D}_\varepsilon \cap A \ne \varnothing) \le C_\varepsilon\,\mathrm{Cap}_L(A),$$

but the constant C_ε goes to infinity as ε goes to zero. Since the property of having zero capacity is closed under countable unions, we again have the dichotomous criterion

$$\mathbf{P}(\mathcal{D} \cap A \neq \varnothing) = 0 \Leftrightarrow \mathrm{Cap}_L(A) = 0 \tag{1.1}$$

for $L(x,y) = (\log|x-y|)^2$. No estimate follows, however.

An example helps to explain this shortcoming. Fix an $\alpha \in (1/2, 1)$ and let A_n be nested subsets of the line segment $A_0 := [-1/2, 1/2] \times \{0\}$ such that A_n is made of 2^n intervals of length $2^{-2^{\alpha n}}$, with each of the 2^n intervals of A_n containing exactly two intervals of A_{n+1} situated at the opposite ends of the interval of A_n. The intersection, denoted A, is a Cantor set for which, if $K(x,y) = |\log|x-y||$ and $L(x,y) = \log^2|x-y|$, then

$$\mathrm{Cap}_K(A) > 0 = \mathrm{Cap}_L(A).$$

For each set A_n, a Brownian motion that hits the set will immediately after have a double point in the set. Thus,

$$\mathbf{P}(\mathcal{D} \cap A_n \neq \varnothing) := p_n,$$

where p_n decreases as $n \to \infty$ to a positive number, estimated by $\mathrm{Cap}_K(A)$. On the other hand, since $\mathrm{Cap}_L(A) = 0$, we know that $\mathcal{D}$ is almost surely disjoint from A.

From this we see that the probability of A intersecting $\mathcal{D}$ is not continuous as A decreases to a given compact set, and therefore, that this probability cannot be uniformly estimated by Cap_K for any K, since Cap_K is a Choquet capacity, and must be continuous with respect to this kind of limit. On the other hand, since the probability that A_n intersects $\mathcal{D}_\varepsilon$ is estimated by the Choquet capacity $\mathrm{Cap}_L(A_n)$ which goes to zero as $n \to \infty$, we see that these estimates are indeed getting worse and worse as $n \to \infty$ for fixed ε, and are only good when $\varepsilon \to 0$ as some function of n.

We remark that such behavior is possible only because $\mathcal{D}$ is not a closed set. Indeed, if X is a random closed set and $\{Y_n\}$ are closed sets decreasing to Y, then the events $\{X \cap Y_n \neq \varnothing\}$ decrease to the event $\{X \cap Y \neq \varnothing\}$, whence

$$\mathbf{P}(X \cap Y_n \neq \varnothing) \downarrow \mathbf{P}(X \cap Y \neq \varnothing). \tag{1.2}$$

The goal of this note is to provide a useful estimate for $\mathbf{P}(\mathcal{D} \cap A \neq \varnothing)$. We have just seen that it cannot be of the form Cap_K for some kernel, K. Instead, we must introduce the notion of a capacity with respect to two different kernels, which we denote $\mathrm{Cap}_{f\to g}$. We go about this two different ways. The first approach is to show that $\mathrm{Cap}_{f\to g}$ gives estimates on probabilities of intersection with $\mathcal{D}_\varepsilon$ which are uniform in ε and thus allow passage to the limit. This relies on the result of Fitzsimmons and Salisbury (or Peres), so is less self-contained, but yields as a by-product the estimates for $\varepsilon > 0$ which may be considered interesting in themselves. The second is a softer and more elementary argument, which produces a sort of polar decomposition of the set A but is less useful for computing. Section 2 states our results, Section 3 contains proofs of the estimates and Section 4 contains the proof of the decomposition result.

2. Results. Since Brownian motion is isotropic, we will restrict attention to kernels $K(x,y) = f(|x-y|)$ that depend only on $|x-y|$. When K has this form, we write $\mathcal{E}_f$ and Cap_f instead of $\mathcal{E}_K$ and Cap_K. Let f and g be functions from $\mathbb{R}^+$ to $\mathbb{R}^+$ going to infinity at zero, with $f \le g$. Let h_ε denote the function on $\mathbb{R}^+$ defined by

$$h_\varepsilon(x) = \begin{cases} f(x), & \text{if } x \ge \varepsilon, \\ g(x) \cdot \dfrac{f(\varepsilon)}{g(\varepsilon)}, & \text{if } x < \varepsilon. \end{cases}$$

Let Cap_ε denote $\mathrm{Cap}_{h_\varepsilon}$. The following result defines the hybrid capacity $\mathrm{Cap}_{f\to g}$ as a limit and also characterizes it as "Cap_f measured only at places where Cap_g is positive."

PROPOSITION 2.1. *The limit* $\lim_{\varepsilon\to 0} \mathrm{Cap}_\varepsilon(A)$ *exists. Denoting this limit by* $\mathrm{Cap}_{f\to g}(A)$, *we have*

$$\mathrm{Cap}_{f\to g}(A) = [\inf\{\mathcal{E}_f(\mu) : \mathcal{E}_g(\mu) < \infty \textit{ and } \mu(A) = 1\}]^{-1}. \tag{2.1}$$

PROOF. If $\mathrm{Cap}_g(A) = 0$, then both sides of (2.1) are clearly zero, so assume that $\mathrm{Cap}_g(A) > 0$. For each ε, let μ_ε be a probability measure on A that minimizes $\mathcal{E}_{h_\varepsilon}$, so that $\mathrm{Cap}_{h_\varepsilon}(A) = \mathcal{E}_{h_\varepsilon}(\mu_\varepsilon)$. Since $f \le h_\varepsilon$ for all ε, we have

$$\mathcal{E}_{h_\varepsilon}(\mu_\varepsilon) \ge \mathcal{E}_f(\mu_\varepsilon).$$

Observe that each μ_ε has finite g-energy and take the infimum on the left-hand side and the supremum on the right-hand side, then invert, to see that

$$\sup_\varepsilon \mathrm{Cap}_\varepsilon(A) \le [\inf\{\mathcal{E}_f(\mu) : \mathcal{E}_g(\mu) < \infty \text{ and } \mu(A) = 1\}]^{-1}.$$

On the other hand, if μ is any measure of finite g-energy, then by choice of μ_ε, we know that

$$\mathcal{E}_{h_\varepsilon}(\mu_\varepsilon) \le \mathcal{E}_{h_\varepsilon}(\mu).$$

As $\varepsilon \to 0$, dominated convergence shows that the right-hand side of this converges to $\mathcal{E}_f(\mu)$, and hence, that

$$\liminf_{\varepsilon\to 0} \mathrm{Cap}_\varepsilon(A) \ge [\inf\{\mathcal{E}_f(\mu) : \mathcal{E}_g(\mu) < \infty \text{ and } \mu(A) = 1\}]^{-1},$$

which finishes the proof. $\square$

REMARK. The infimum in (2.1) need not be achieved. For example, if A is a small disk, $f(x) = |\log x|$, and $g(x) = x^{-\alpha}$ for any $\alpha \in [1,2)$, then the infimum of logarithmic energies of probability measures on A is equal to the log-energy of normalized one-dimensional Lebesgue measure on the boundary of the disk, and is strictly less than the logarithmic energy of any measure of finite g-energy.

This proposition is our only general result on hybrid capacities. For the remainder of the paper, f will always be $|\log \varepsilon|$ and g will always be $\log^2 \varepsilon$, so the notation h_ε will be unambiguous. [We have also found the notation easier to read if we use $\log|x-y|/\log\varepsilon$ rather that $|\log|x-y||/|\log\varepsilon|$ or $\log(1/|x-y|)/\log(1/\varepsilon)$ whenever the signs cancel.] Our main interest in Cap_ε is that it gives the estimate on the probability of an intersection with $\mathcal{D}_\varepsilon$.

THEOREM 2.2 (Estimates for intersecting $\mathcal{D}_\varepsilon$). *Let $f(x) = |\log x|$ and $g(x) = \log^2 x$. There are constants c and C such that, for any $\varepsilon > 0$ and any closed subset A of disk $\{x : |x| \le 1/3\}$,*

$$c\,\mathrm{Cap}_\varepsilon(A) \le \mathbf{P}(\mathcal{D}_\varepsilon \cap A \ne \varnothing) \le C\,\mathrm{Cap}_\varepsilon(A).$$

Since $\mathrm{Cap}_\varepsilon \uparrow \mathrm{Cap}_{f\to g}$ and $\mathcal{D} = \bigcup \mathcal{D}_\varepsilon$, our first main result follows as an immediate corollary.

THEOREM 2.3 (Two-gauge capacity estimate). *For the same constants c and C, and the same f and g,*

$$c\,\mathrm{Cap}_{f\to g}(A) \le \mathbf{P}(\mathcal{D} \cap A \ne \varnothing) \le C\,\mathrm{Cap}_{f\to g}(A).$$

REMARK. Suppose the set A is a bi-Hölder image of some set S for which the intersection probabilities with $\mathcal{D}$ are known. Since the logarithm of the distance between two points in a small disk changes by a bounded factor under such a map, the Newtonian and $\log^2$ capacities change only by a bounded factor, so the probability of A intersecting $\mathcal{D}$ is estimated by the probability of S intersecting $\mathcal{D}$. This is more than can be concluded from the dichotomy (1.1).

The characterization of $\mathrm{Cap}_{f\to g}$ in Proposition 2.1 suggests an explanation for the two-gauge capacity result. The probability of intersection with $\mathcal{D}$ is estimated by $\mathrm{Cap}_{\log}$ "at places of finite $\log^2$-energy," so perhaps the operative mechanism is that one must eliminate certain "thin" places that can never contain Brownian double points, leaving a "core set," such that if and when Brownian motion hits the core set, immediately there will be a Brownian double point in the core set. This turns out to be true.

THEOREM 2.4 (Polar decomposition). *Any compact subset A of the plane not containing $(1,0)$ may be written as a union $A = A_1 \cup A_2$, such that (1) the set A_1 is almost surely disjoint from $\mathcal{D}$, and (2), on the event that the hitting time τ_2 of A_2 is finite, then for any $\varepsilon > 0$, with probability 1, Brownian motion stopped at time $\tau_2 + \varepsilon$ has a double point in A_2.*

It follows from this that

$$\mathbf{P}(\mathcal{D} \cap A \neq \varnothing) = \mathbf{P}(\text{Brownian motion hits } A_2),$$

which is estimated up to a constant factor by $\mathrm{Cap}_{\log}(A_2)$, and, in fact, is equal to the Martin capacity of A_2. Thus, this decomposition is in some ways stronger than Theorem 2.3; it is, in principle, less useful for computation because A_2 must first be computed, though, in practice, usually $A_2 = A$ or is empty. We remark that $\mathrm{Cap}_{\log}(A_2)$ is a different estimate from $\mathrm{Cap}_{\log\to\log^2}(A)$, if harmonic measure on A_2 has infinite $\log^2$-energy.

3. Proof of estimates for intersecting $\mathcal{D}$. Fix $\varepsilon \in (0, 1/3)$ and any $\delta < \varepsilon/2$. Let x and y be points in the quarter unit disk with $|x - y| > 3\delta$ and denote by D_x and D_y the balls of radius δ centered at x and y, respectively. The key estimates for applying potential theoretic methods are the first and second moment estimates, as given in the following lemma. The notation $\asymp$ denotes equivalence up to a constant multiple.

LEMMA 3.1. *Let $H(A) = H(A, \varepsilon)$ denote the event $\{\mathcal{D}_\varepsilon \cap A \neq \varnothing\}$:*

$$\mathbf{P}(H(D_x)) \asymp \frac{|\log \varepsilon|}{\log^2 \delta}. \tag{3.1}$$

Letting $\mathbf{P}_\xi$ denote probabilities with respect to a Brownian motion started at the point $\xi \notin D_x$, we have, in general,

$$\mathbf{P}_\xi(H(D_x)) \asymp \frac{\log \varepsilon \log |\xi - x|}{\log^2 \delta}. \tag{3.2}$$

The probabilities for double points simultaneously occurring in two balls are given as follows. When $|x - y| \geq \varepsilon$,

$$\mathbf{P}(H(D_x) \cap H(D_y)) \asymp \frac{|\log |x - y|| \cdot \log^2 \varepsilon}{\log^4 \delta}. \tag{3.3}$$

When $|x - y| < \varepsilon$,

$$\mathbf{P}(H(D_x) \cap H(D_y)) \asymp \frac{|\log \varepsilon| \cdot \log^2 |x - y|}{\log^4 \delta}. \tag{3.4}$$

PROOF. Let τ be the hitting time on D_x. For $H(D_x)$ to occur, it is necessary that $\tau < \infty$ and that the Brownian motion hit D_x after time $\tau + \varepsilon^2$. Denoting this event by G, use the Markov property at time τ and $\tau + \varepsilon^2$ and average over the position at time $\tau + \varepsilon^2$ to see that

$$\mathbf{P}(G) \asymp \frac{1}{|\log \delta|} \frac{\log \varepsilon}{\log \delta}.$$

On the other hand, conditioning on the position at time τ and at the return time to D_x, it is easy to bound $\mathbf{P}(H(D_x) \,|\, G)$ away from zero, since this is the probability that a Brownian path and a Brownian bridge, each started on the boundary of a ball of radius δ and run for time greater than δ^2, intersect inside the ball. This establishes (3.1). When starting at a point ξ near x instead of at the point $(1,0)$, the probability of the event $\{\tau < \infty\}$ is $\log|\xi - x| / \log \delta$ rather than $1/|\log \delta|$, which gives the estimate in (3.2).

To establish the other two estimates, we consider possible sequences of visits, two to each ball, with the correct time separations. Let $H_1(x,y)$ denote the event that there exist times $0 < r < r + \varepsilon^2 \le s < t < t + \varepsilon^2 \le u < \tau_*$ such that $B_r \in D_x$, $B_s \in D_x$, $B_t \in D_y$ and $B_u \in D_y$. Let $H_2(x,y)$ denote the event that there exist times $0 < r < s < t < u < \tau_*$ such that $r + \varepsilon^2 \le t, s + \varepsilon^2 \le u$, $B_r \in D_x$, $B_s \in D_y$, $B_t \in D_x$ and $B_u \in D_y$. Let $H_3(x,y)$ denote the event that there exist times $0 < r < s < s + \varepsilon^2 \le t < u < \tau_*$ such that $B_r \in D_x$, $B_s \in D_y$, $B_t \in D_y$ and $B_u \in D_x$. The estimate

$$\mathbf{P}(H(D_x) \cap H(D_y)) \asymp \mathbf{P}(H_1(x,y)) + \mathbf{P}(H_2(x,y)) + \mathbf{P}(H_3(x,y)) \tag{3.5}$$

follows from the same considerations: that for $j = 1,2,3$, $\mathbf{P}(H(D_x) \cap H(D_y) \,|\, H_j(x,y))$ is bounded away from zero; that the same holds when x and y are switched; that $\mathbf{P}(H_j(x,y)) \asymp \mathbf{P}(H_j(y,x))$; and that $H(D_x) \cap H(D_y)$ entails either $H_j(x,y)$ or $H_j(y,x)$ for some j. The estimates (3.3) and (3.4) will then follow from

$$\mathbf{P}(H_1(x,y)) \asymp \frac{|\log|x-y|| \cdot \log^2 \varepsilon}{\log^4 \delta}, \tag{3.6}$$

$$\mathbf{P}(H_3(x,y)) \asymp \frac{\log^2|x-y| \cdot |\log \varepsilon|}{\log^4 \delta}, \tag{3.7}$$

$$\mathbf{P}(H_2(x,y)) = O(\mathbf{P}(H_1(x,y)) + \mathbf{P}(H_3(x,y))). \tag{3.8}$$

The Markov property gives a direct estimate of $\mathbf{P}(H_1(x,y))$. In particular, we may take r to be the hitting time of D_x, s to be the next time after $r + \varepsilon^2$ that D_x is hit, and so forth. The probability of hitting D_x is $\asymp 1/|\log \delta|$. Given that $B_r \in D_x$, the probability that $B_s \in D_x$ for some $s \ge r + \varepsilon^2$ is $\asymp |\log \varepsilon| / |\log \delta|$. Given that, the probability of subsequently hitting D_y is $\asymp |\log|x-y|| / |\log \delta|$, and given such a hit at time t, the probability of $B_u \in D_y$ for some $u \ge t + \varepsilon^2$ is $\asymp |\log \varepsilon| / |\log \delta|$. Multiplying these together produces the estimate (3.6). Similarly, $\mathbf{P}(H_3(x,y))$ is the product of four factors, respectively comparable to $1/|\log\delta|$, $\log|x-y|/\log\delta$, $\log\varepsilon/\log\delta$ and $\log|x-y|/\log\delta$, proving (3.7).

In the case $|x-y|\geq\varepsilon$, the bound $\mathbf{P}(H_2(x,y))=O(\frac{\log^3|x-y|}{\log^4\delta})$ is good enough to imply (3.8) and follows in the same manner from the Markov property at the hitting time of D_x, the next hit of D_y, the next hit of D_x and the next hit on D_y. In the case $|x-y|\leq\varepsilon$, define an event $H_2'\subseteq H_2$ by additionally requiring $t\geq s+\varepsilon^2/2$. Let $H_2''=H_2\setminus H_2'$. The Markov property gives

$$\mathbf{P}(H_2')=O\biggl(\frac{1}{|\log\delta|}\frac{\log|x-y|}{\log\delta}\frac{\log\varepsilon}{\log\delta}\frac{\log|x-y|}{\log\delta}\biggr)\asymp\mathbf{P}(H_3). \tag{3.9}$$

Finally, to estimate $\mathbf{P}(H_2'')$, observe that H_2'' entails both $s\geq r+\varepsilon^2/2$ and $u\geq t+\varepsilon^2/2$. The Markov property then gives

$$\mathbf{P}(H_2'')=O\biggl(\frac{1}{|\log\delta|}\frac{\log\varepsilon}{\log\delta}\frac{\log|x-y|}{\log\delta}\frac{\log\varepsilon}{\log\delta}\biggr)\asymp\mathbf{P}(H_1) \tag{3.10}$$

and adding (3.9) to (3.10) establishes (3.8) and the lemma. □

3.1. *Proof of the first inequality of Theorem 2.2.* The first inequality follows from Lemma 3.1 by standard methods. We give the details, since it is a little unusual to discretize space in only part of the argument (composing the set A of lattice squares, but not discretizing the double point process itself). For the remainder of the argument, ε and A are fixed.

Let μ be any probability measure on A; we need to show that $\mathbf{P}(H(A))\geq c\mathcal{E}_{h_\varepsilon}(\mu)^{-1}$. The closed set A may be written as a decreasing intersection over finer and finer grids of finite unions of lattice squares. According to (1.2), we may therefore assume that A is a finite union of lattice squares of width $\delta<\varepsilon$. Index the rows and columns of the grid, and let $\mathcal{B}$ denote the subcollection of squares where both coordinates are even. Let $\mathcal{B}'$ denote the collection of inscribed disks of $\mathcal{B}$. Then some translation $\mathcal{B}''$ of $\mathcal{B}'$ has μ-measure at least $1/8$ (since space may be covered by 8 translates of the set of disks centered at points with both coordinates even). Define a random variable

$$X:=\sum_{S\in\mathcal{B}''}\frac{\log^2\delta}{|\log\varepsilon|}\mu(S)\mathbf{1}_{H(S)}.$$

By the first estimate in Lemma 3.1, the expectation of each $(\log^2\delta/|\log\varepsilon|)\mathbf{1}_{H(S)}$ is bounded above and below by some constants c_1 and c_2. Thus, $c_1/8\leq\mathbf{E}X\leq c_2$.

The second moment of X is computed as

$$\mathbf{E}X^2=\frac{\log^4\delta}{\log^2\varepsilon}\sum_{S,T\in\mathcal{B}''}\mu(S)\mu(T)\mathbf{E}\mathbf{1}_{H(S)\cap H(T)}. \tag{3.11}$$

By estimates (3.3) and (3.4) of Lemma 3.1, when $S \neq T$,

$$\mathbf{E}\frac{\log^4 \delta}{\log^2 \varepsilon}\mathbf{1}_{H(S)\cap H(T)} \tag{3.12}$$

is bounded between constant multiples of $h_\varepsilon(|x-y|)$, where x and y are the centers of S and T. Since S and T are separated by δ, this is bounded between $c_3 h_\varepsilon(|x-y|)$ and $c_4 h_\varepsilon(|x-y|)$ for any $x \in S$ and $y \in T$. Thus, letting U denote the union of $\mathcal{B}''$, the sum of the off-diagonal terms of (3.11) is estimated by

$$\begin{aligned} c_3 \int h_\varepsilon(x,y)\mathbf{1}_{|x-y|>\delta}\, d\mu(x)\, d\mu(y) &\leq \frac{\log^4 \delta}{\log^2 \varepsilon} \sum_{S,T\in\mathcal{B}''} \mu(S)\mu(T)\mathbf{1}_{H(S)\cap H(T)}\mathbf{1}_{S\neq T} \\ &\leq c_4 \int h_\varepsilon(x,y)\mathbf{1}_{|x-y|>\delta}\, d\mu(x)\, d\mu(y). \end{aligned}$$

The diagonal terms sum to exactly $\mathbf{E}X$, so we see that

$$\mathbf{E}X^2 \leq \mathbf{E}X + c_4\mathcal{E}_{h_\varepsilon}(\mu).$$

The second moment inequality $\mathbf{P}(X>0) \geq (\mathbf{E}X)^2/\mathbf{E}X^2$ now implies that

$$\mathbf{P}(X>0) \geq \frac{c_1^2}{64(c_2 + c_4\mathcal{E}_{h_\varepsilon}(\mu))}.$$

Since $X>0$ implies the existence of an ε-separated double point in A, we have proved the first inequality with $c = c_1^3/(64(8c_2 + c_1c_4))$.

3.2. *Proof of the second inequality of Theorem 2.2.* The following two propositions represent most of the work in finishing the proof of Theorem 2.2.

PROPOSITION 3.2. *If A has diameter at most ε, then*

$$\mathbf{P}(\mathcal{D}_\varepsilon \cap A \neq \varnothing) \asymp |\log \varepsilon|\mathbf{P}(\mathcal{I} \cap A \neq \varnothing).$$

PROPOSITION 3.3 (Capacity criterion for $\mathcal{I}$). *For any A in the $\frac{1}{3}$-unit disk,*

$$\mathbf{P}(\mathcal{I} \cap A \neq \varnothing) \asymp \mathrm{Cap}_{\log^2}(A).$$

The second of these two propositions is proved in Peres [5] but also follows from the methods of Fitzsimmons and Salisbury [3] if one upgrades to a quantitative estimate by observing that the Green kernel is comparable to the Martin kernel (see Benjamini, Pemantle and Peres [1]).

The $\geq$-half of Proposition 3.2 follows from Proposition 3.3 and the first inequality in Theorem 2.2. Specifically, on a set of diameter at most ε, we have $h_\varepsilon(x,y) = \log^2|x-y|/|\log\varepsilon|$, and therefore,

$$\begin{aligned}
\mathbf{P}(\mathcal{D}_\varepsilon \cap A \neq \varnothing) &\geq c\,\mathrm{Cap}_\varepsilon(A) && \text{(first half of Theorem 2.2)}\\
&= c|\log\varepsilon|\,\mathrm{Cap}_{\log^2}(A)\\
&\asymp |\log\varepsilon|\mathbf{P}(\mathcal{I}\cap A \neq \varnothing) && \text{(Proposition 3.3).}
\end{aligned}$$

Among the two propositions, what is left to prove is the $\leq$-half of Proposition 3.2, namely,

$$\mathbf{P}(\mathcal{D}_\varepsilon \cap A \neq \varnothing) \leq c|\log\varepsilon|\mathbf{P}(\mathcal{I}\cap A \neq \varnothing). \tag{3.13}$$

To prove this, the following corollary of Proposition 3.3 will be useful.

COROLLARY 3.4. *Let A be a subset of the disk of radius $\varepsilon/2$ centered at the origin. Let σ and $\tilde{\sigma}$ denote the respective hitting times of B_t and $\tilde{B}_t$ on the circle $\{|x| = 2\varepsilon\}$. Let z denote the point $(\varepsilon, 0)$ and let*

$$\begin{aligned}
p &= \mathbf{P}_{z,z}(A \cap B[0,\sigma] \cap \tilde{B}[0,\tilde{\sigma}] \neq \varnothing),\\
p' &= \mathbf{P}_{z,z}(A \cap B[0,\tau_*] \cap \tilde{B}[0,\tilde{\tau_*}] \neq \varnothing)
\end{aligned}$$

be the probabilities of two independent Brownian motions starting at $(\varepsilon,0)$ intersecting in A when stopped at $\{|x| = 2\varepsilon\}$ or $\{|x| = 3\}$ respectively. Then

$$p' \asymp (p \cdot \log^2\varepsilon) \wedge 1$$

and, consequently,

$$\mathbf{P}(\mathcal{I}\cap A \neq \varnothing) \asymp p \wedge \frac{1}{\log^2\varepsilon}.$$

PROOF. If $|x|, |y| \leq \varepsilon/2$, then the Green function for Brownian motion stopped when it exits the disk of radius R satisfying

$$G_R(x,y) \asymp \log\frac{R}{|x-y|} \tag{3.14}$$

uniformly in R for $R \geq 2\varepsilon$. This follows, for instance, from $G_R(0,y) = \log(R/|y|)$ by applying a bi-Lipshitz map. Applying (3.14) to $R = 2\varepsilon$ gives

$$M_{2\varepsilon}(x,y) = \frac{G_{2\varepsilon}(x,y)}{G_{2\varepsilon}(z,y)} \asymp \frac{\log(2\varepsilon/|x-y|)}{\log(2\varepsilon/|z-y|)} \asymp \log\frac{2\varepsilon}{|x-y|}.$$

Applying (3.14) to $R=3$ then gives

$$\begin{aligned} M_3(x,y) &\asymp \frac{\log(3/|x-y|)}{\log(3/|z-y|)} \\ &\asymp \frac{\log(2\varepsilon/|x-y|)+\log(3/(2\varepsilon))}{\log(3/\varepsilon)} \\ &\asymp 1+\frac{M_2(x,y)}{|\log\varepsilon|}. \end{aligned}$$

It follows that $\mathrm{Cap}_{M_3^2} \asymp 1\wedge(\log^2\varepsilon\cdot\mathrm{Cap}_{M_{2\varepsilon}^2})$. The first assertion of the corollary follows from this, and the second from the first and conditioning both Brownian motions to hit $D_{2\varepsilon}$. ☐

Proof of the $\le$-half of Proposition 3.2. For $z=(\varepsilon,0)$, we will show that

$$\mathbf{P}_z(\mathcal{D}_\varepsilon\cap A\neq\varnothing)\le cp\log^2\varepsilon\wedge 1. \tag{3.15}$$

This suffices, since, by the Markov property,

$$\begin{aligned} \mathbf{P}(\mathcal{D}_\varepsilon\cap A\neq\varnothing) &\asymp \frac{1}{|\log\varepsilon|}\mathbf{P}_z(\mathcal{D}_\varepsilon\cap A\neq\varnothing) \\ &\le (cp|\log\varepsilon|)\wedge\frac{1}{|\log\varepsilon|} \qquad \text{[consequence of (3.15)]} \\ &\asymp c|\log\varepsilon|\mathbf{P}(\mathcal{I}\cap A\neq\varnothing) \qquad \text{(by Corollary 3.4)}, \end{aligned}$$

establishing (3.13) and the proposition.

To prove (3.15), let $\sigma_1<\tau_1<\sigma_2<\tau_2\cdots$ be the alternating sequence of hitting times of ∂D_ε and $\partial D_{2\varepsilon}$:

$$\sigma_{n+1}=\inf\{t>\tau_n : |B_t|=\varepsilon\}$$

and

$$\tau_{n+1}=\inf\{t>\sigma_{n+1} : |B_t|=2\varepsilon\}.$$

We call the path segments $\{B_s : \sigma_j\le s\le\tau_j\}$ sojourns. The left-hand side of (3.15) is bounded above by the sum

$$\sum_{i,j\ge 1}\mathbf{P}(\sigma_i\le\tau_*,\sigma_j\le\tau_*,B[\sigma_i,\tau_i]\cap B[\sigma_j,\tau_j]\cap A\neq\varnothing) \tag{3.16}$$

of probabilities that sojourns i and j exist and intersect inside A. Since

$$\mathbf{P}(\tau_*<\sigma_{n+1}\mid\mathcal{F}_{\tau_n})=\frac{\log 2}{\log(3/\varepsilon)}\asymp\frac{1}{|\log\varepsilon|},$$

the Markov property shows that the number of sojourns is geometrically distributed with mean $\log(3/\varepsilon)/\log 2 \asymp 1/|\log\varepsilon|$. For distinct sojourns, the Harnack principle again implies that the probability of their intersecting in A is at most a constant multiple of p, and this is still true when conditioned on the number of sojourns. The expected number of pairs of sojourns is estimated by $\log^2\varepsilon$, hence, we have a contribution of $O(p\cdot\log^2\varepsilon)$ to the right-hand side of (3.16) from terms with $i\neq j$.

To finish, we need to estimate the probability of an ε-separated intersection in A within a single sojourn. For $0\le i\le j-2$, let G_{ij} denote the event

$$\{B_r=B_s\in S\cap A \text{ for some } r\in[i\varepsilon^2/2,(i+1)\varepsilon^2/2] \text{ and } s\in[j\varepsilon^2/2,(j+1)\varepsilon^2/2]\}.$$

Let $t_j=(j-\frac{1}{2})\frac{\varepsilon^2}{2}$ and let σ be the hitting time of $\{|x|=2\varepsilon\}$. We apply the Markov property at time σ_n to estimate the summand in (3.16) with $i=j=n$, then sum over n. This bounds the contributions to (3.16) from off-diagonal terms by

$$\sum_n \mathbf{P}_z(\sigma_n<\tau_*)\sum_{i<j}\mathbf{P}_{B(\sigma_n)}(G_{ij},t_j<\sigma).$$

The sum over n is $O(|\log\varepsilon|)$ and the sum over $0\le i<j$ of $\mathbf{P}_{B(\sigma_n)}(t_j<\sigma)$ is $O(1)$ (e.g., this is at most the sum of j times the probability that $\{|x|=2\varepsilon\}$ is not hit by time $j\varepsilon^2/2$, which is at most the expected square of the time for a Brownian motion to reach $\{|x|=4\}$). We will be done, therefore, when we have shown that

$$\sup_{0\le i\le j-2,|z|=\varepsilon}\mathbf{P}_z(G_{ij}\mid\mathcal{F}_{t_j})\le cp \tag{3.17}$$

on the event $\{t_j<\sigma\}$ (actually, an upper bound of $cp|\log\varepsilon|$ would suffice).

This is more or less obvious from the Markov property, but we go ahead and spell out the details. Let ω_i denote the ith sub-sojourn defined by $\omega_i(s)=\omega(s+i\varepsilon^2/2)$ for $0\le s\le\varepsilon^2/2$. Let μ_{ij} denote the conditional law of ω_i under $\mathbf{P}_z$ given $\mathcal{F}_{t_j}$ and μ denote the P_z-law of ω on the interval $[\varepsilon^2/2,\varepsilon^2]$. The quantity p is estimated by the probability of two independent draws from μ intersecting inside A; conditioning on $\mathcal{F}_{t_j}$ makes ω_i and ω_j independent, so (3.17) follows if we can show that

$$\frac{d\mu_{ij}}{d\mu}\le C\mathbf{1}_{t_j<\sigma} \tag{3.18}$$

when $0\le i\le j-2$ or $i=j$. For $i=j$, μ_{jj} and μ are Wiener measure from starting points with comparable densities. For $1\le i\le j-2$, we use the Markov property to write

$$\mu_{ij}=\int\mu^{xy}_{\varepsilon^2/2}\mathbf{1}_H\,d\pi_{ij}(x,y),$$

$$\mu=\int\mu^{xy}_{\varepsilon^2/2}\,d\pi(x,y),$$

where μ_t^{xy} is the law of a Brownian bridge from x to y in time t, H is the event that the path remains inside the ball of radius 2ε, and π_{ij} and π are mixing measures. By Bayes' rule and the Markov property,

$$\frac{\pi_{i,j}(x,y)}{\pi(x,y)} = \frac{1}{Z}\,\mu_{x,y}(H)\mu^{zx}_{i\varepsilon^2/2}(H)\mu^{y,B(t_j)}_{(t_j-i-1)\varepsilon^2/2}(H),$$

where Z is the normalizing constant gotten by integrating the product of the three probabilities on the right-hand side against $\pi(x,y)$. The probabilities are all at most 1, so all we need is that Z is at least $c>0$. By Brownian scaling, we see that the three probabilities are at least a constant when $|x|,|y|<\varepsilon$ and, since π gives positive measure to this set, the verification for $1\le i\le j-2$ is complete. Finally, for $i=0$, we compare to μ' instead of μ, where μ' is the $\mathbf{P}_z$-law of ω on $[0,\varepsilon^2/2]$. This establishes (3.17) and, hence, (3.15) and the remainder of Proposition 3.2. $\square$

PROOF OF THE SECOND INEQUALITY IN THEOREM 2.2. Let

$$\tau = \tau_\varepsilon = \inf\{t : B_t \in A \text{ and } B_s = B_t \text{ for some } s \le t-\varepsilon^2\}$$

be the first time that a point of A is hit by the Brownian motion and has previously been hit at a time at least ε^2 in the past; thus, $\mathbf{P}(\tau\le\tau_*) = \mathbf{P}(\mathcal{D}_\varepsilon\cap A\ne\varnothing)$. The second inequality in Theorem 2.2 is equivalent to the existence of a measure ν on A whose mass is equal to $\mathbf{P}(\tau<\tau_*)$ and whose energy is at most a constant multiple of this [normalizing ν to be a probability measure gives an energy of $C\mathbf{P}(\mathcal{D}_\varepsilon\cap A\ne\varnothing)^{-1}$, thereby witnessing the inequality].

To construct ν, partition the plane into a grid of squares of side $\varepsilon/3$. For each square S in the grid, let ν_S be a probability measure of minimal $\log^2$-energy on $S\cap A$. By Proposition 3.3,

$$\mathcal{E}_{h_\varepsilon}(\nu_S) = \frac{1}{|\log\varepsilon|}\mathcal{E}_{\log^2}(\nu_S) \le C\mathbf{P}(\mathcal{I}\cap S\cap A\ne\varnothing)^{-1}$$

and so by Proposition 3.2, for a different constant,

$$\mathcal{E}_{h_\varepsilon}(\nu_S) \le C\mathbf{P}(\mathcal{D}_\varepsilon\cap S\cap A\ne\varnothing)^{-1}. \tag{3.19}$$

Let

$$\nu := \sum_S \mathbf{P}(B_\tau\in S, \tau<\tau_*)\nu_S.$$

Clearly, we have constructed ν so that $\|\nu\| = \mathbf{P}(\tau<\tau_*)$. It remains to show that $\mathcal{E}_{h_\varepsilon}(\nu)\le c\mathbf{P}(\tau<\tau_*)$. We will tally separately the contributions to the energy from pairs (x,y) at distances at least ε and at most ε, showing

$$\int h_\varepsilon(x,y)\mathbf{1}_{|x-y|\ge\varepsilon}\,d\nu(x)\,d\nu(y) \le C\|\nu\| \tag{3.20}$$

and

$$\int h_\varepsilon(x,y)\mathbf{1}_{|x-y|\le\varepsilon}\,d\nu(x)\,d\nu(y) \le C\|\nu\|. \tag{3.21}$$

For the bound (3.20) on the first piece, observe that points separated by ε are in nonadjacent squares S and S', and that the value of h at any $x \in S$ and $y \in S'$ is estimated by the $|\log|x_* - y_*||$ for any $x_* \in S, y_* \in S'$. Therefore, on the event $\{B_\tau \in S\}$, we may replace $x \in S$ by B_τ to obtain

$$\begin{aligned}
&\int h_\varepsilon(x,y)\mathbf{1}_{|x-y|\ge\varepsilon}\,d\nu(x)\,d\nu(y) \\
&\quad \le C\sum_{S'}\mathbf{P}(B_\tau \in S', \tau < \tau_*) \\
&\qquad\qquad \times \int d\nu_{S'}(y)\Bigg[\sum_{S \text{ not adjacent to } S'} \mathbf{E}(\mathbf{1}_{B_\tau\in S,\tau<\tau_*}|\log|B_\tau - y||)\Bigg] \\
&\quad \le \|\nu\|\sup_y V(y),
\end{aligned}$$

where

$$V(y) = \mathbf{E}(|\log|B_\tau - y||\mathbf{1}_G)$$

is the logarithmic potential at y of the subprobability law of B_τ restricted to the event $G := \{\tau < \tau_*, |B_\tau - y| \ge \varepsilon)\}$.

To see that $V(y)$ is bounded, fix y and observe that the probability that $\mathcal{D}_\varepsilon$ intersects the δ-ball D_y is at least equal to the probability that it does so after time τ has been reached. Throwing away those paths where B_τ is within ε of y, we have, by the Markov property and (3.2),

$$\mathbf{P}(\mathcal{D}_\varepsilon \cap D_y \ne \varnothing) \ge c\mathbf{E}\frac{\log\varepsilon\log|X-y|}{\log^2\delta}\mathbf{1}_G.$$

On the other hand, by (3.1),

$$\mathbf{P}(\mathcal{D}_\varepsilon \cap D_y \ne \varnothing) \asymp \frac{|\log\varepsilon|}{\log^2\delta}.$$

It follows that $\mathbf{E}|\log|X-y||\,\mathbf{1}_G \le c^{-1}$, which is the desired bound on the first piece.

For the bound (3.21) on the second piece, begin with the well known trick of reducing to the diagonal:

$$\int h_\varepsilon(x,y)\mathbf{1}_{|x-y|\le\varepsilon}\,d\nu(x)\,d\nu(y) \le C\sum_S \mathbf{P}(B_\tau \in S, \tau < \tau_*)^2\mathcal{E}_{h_\varepsilon}(\nu_S). \tag{3.22}$$

One way to see this is to observe that, while $|x-y| \le \varepsilon$ and $x \in S$ does not force $y \in S$, it does force y to be in one of 49 nearby squares. The function $\log^2|x-y|/|\log\varepsilon|$ is positive definite, so one may use the Cauchy–Schwarz inequality to conclude (3.22). In fact, (3.22) holds when h_ε is not positive definite but only assumed to be monotone; see Pemantle and Peres [4], equation 11 for details.

Finally, since $\mathbf{P}(B_\tau \in S, \tau < \tau_*) \le \mathbf{P}(\mathcal{D}_\varepsilon \cap S \cap A \neq \varnothing)$, we see from (3.19) that

$$\mathbf{P}(B_\tau \in S, \tau < \tau_*)^2 \mathcal{E}_{h_\varepsilon}(\nu_S) \le C\mathbf{P}(B_\tau \in S, \tau < \tau_*).$$

Summing over S bounds the right-hand side of (3.22) by $\mathbf{P}(\tau < \tau_*) = \|\nu\|$, establishing (3.21) and finishing the proof of Theorem 2.2. □

4. Proof of Theorem 2.4. There are two obvious choices for the set A_2. The first is the set $\mathcal{P}$ of points x such that a Brownian motion started at x and run for any positive time almost surely has a double point in A. Call such a point an *immediate point*. The second choice would be the set $\mathcal{R}$ of regular points of A with respect to the potential of the least-energy measure for the kernel $K(x,y) = \log^2|x-y|$. [A *regular point* x for the potential $\int K(x,y)\,d\nu(y)$ of a measure ν is one where the potential reaches its maximum value.] If $\mathcal{P} = \mathcal{R}$, then Theorem 2.4 has a very short proof:

> Let $A_2 = \mathcal{P} = \mathcal{R}$. It is well known (see Proposition 4.3 below) that the nonregular points $A_1 := A \setminus \mathcal{R}$ must have zero K-capacity, and thus, using the intersection criterion from Fitzsimmons and Salisbury [3], cannot intersect $\mathcal{D}$. This is property (1) required by the theorem. But property (2) in the Theorem is satisfied by definition of $\mathcal{P}$, noting that by what we just proved, having a double point in A is the same as having a double point in A_2.

Embarrassingly, we do not know whether $\mathcal{R} = \mathcal{P}$. We can, however, establish something close, namely, Lemma 4.1, which will be enough to prove the theorem. The apparent obstacle to proving the equality of $\mathcal{P}$ and $\mathcal{R}$ is their different nature: $\mathcal{P}$ is defined probabilistically and the definition is inherently local, while $\mathcal{R}$ is defined analytically and its definition is at first glance nonlocal. Accordingly, we define an analytic version of $\mathcal{P}$ and a localized version of $\mathcal{R}$ as follows.

Fix the closed set A and let ξ be a point of A. Let f be any decreasing continuous function from $\mathbb{R}^+$ to $\mathbb{R}^+$ going to infinity at 0, and let M_ξ denote the f-Martin kernel at ξ:

$$M_\xi(x,y) := \frac{f(|x-y|)}{f(|\xi-y|)}.$$

We say that A has nonvanishing local Martin capacity (NLMC) at ξ if and only if

$$\lim_{\varepsilon\to 0} \mathrm{Cap}_{M_\xi}(A \cap \{y : |y-\xi| < \varepsilon\}) > 0.$$

Let $\mathcal{P}'$ denote the set of points with NLMC. The relation to $\mathcal{P}$ will be clarified shortly.

Call a point $\xi \in A$ *strongly regular* if and only if the f-capacity of $A \cap \{y : |y-\xi| < \varepsilon\}$ is nonzero for every ε, and ξ is a regular point for the potential of the least f-energy measure on each such set. Let $\mathcal{R}'$ denote the set of strongly regular points.

LEMMA 4.1 (Strongly regular implies NLMC for any gauge). *For any A and f as above, the inclusion $\mathcal{R}' \subseteq \mathcal{P}'$ holds.*

PROOF. Fix $\xi \in \mathcal{R}'$. Given any ball D containing ξ, let ν_D denote the measure minimizing the f-energy and let Φ_D denote its potential:

$$\Phi_D(x) = \int f(|x-y|)\, d\nu_D(y).$$

By assumption, $\Phi_D(\xi)$ is equal to the maximum value of Φ_D. It is well known that the maximum value is attained on a set of full measure; standard references such as Carleson [2] state unnecessary assumptions on f, so we include the proof (Proposition 4.3 below). It follows that

$$\mathcal{E}_f(\nu_D) = \Phi_D(\xi).$$

Define a new measure ρ_D, which is a probability measure, by

$$\frac{d\rho_D}{d\nu_D}(y) = \frac{f(|\xi-y|)}{\Phi_D(\xi)}.$$

The potential of this new measure with respect to the Martin kernel M_ξ at a point x is computed to be

$$\frac{1}{\Phi_D(\xi)} \int M_\xi(x,y) f(|\xi-y|)\, d\nu_D(y) = \frac{1}{\Phi_D(\xi)} \int f(|x-y|)\, d\nu_D(y) = \frac{\Phi_D(x)}{\Phi_D(\xi)}.$$

Since ξ is regular for Φ_D, this is at most 1. Since the Martin potential is bounded by 1, the Martin energy $\mathcal{E}_{M_\xi}(\nu_D)$ of the probability measure ν_D is also at most 1, and we see that each ball D has Martin capacity at least 1. □

LEMMA 4.2 (NLMC points are immediate). *Let the closed set A have nonvanishing local Martin capacity at ξ for the* $\log^2$ *Martin gauge*

$$M_\xi(x,y) := \log^2|x-y| / \log^2|\xi-y|.$$

Then ξ is an immediate point.

REMARK. We first remark that if Ξ is the range of a transient Markov process with Green function G and M_ξ is the Martin kernel for the process started at ξ, then the implication holds in both directions: the set A has nonvanishing local M_ξ-capacity near ξ if and only if the process started from ξ almost surely intersects A in any positive time interval. This follows from the methods of Benjamini, Pemantle and Peres [1].

The set of double points is not the range of a Markov process, which makes proving a reverse implication tricky, but the direction in the lemma may still be obtained by applying the method of second moments. Recall that $H(\Lambda, \varepsilon)$ denotes the event that there is a double point in the set Λ with an ε^2 time separation.

PROOF OF LEMMA 4.2. Begin by observing it is enough to show

$$\mathbf{P}_\xi(\mathcal{D} \cap A \neq \varnothing) \geq c \operatorname{Cap}_{M_\xi}(A). \tag{4.1}$$

For, under the hypothesis of NLMC, this implies that

$$\inf_{\varepsilon > 0} \mathbf{P}_\xi[H(A \cap \{y : |y - \xi| < \varepsilon\}, \varepsilon)] > 0.$$

By Fatou's lemma,

$$\mathbf{P}_\xi\left[\limsup_{\varepsilon \to 0} H(A \cap \{y : |y - \xi| < \varepsilon\}, \varepsilon)\right] > 0,$$

whence, with positive probability, $\mathcal{D}$ intersects A in a set with ξ as a limit point. Since

$$\mathbf{P}_\xi(|B_t - \xi| < \varepsilon \text{ for some } t > s) \to 0$$

as $\varepsilon \to 0$ for any fixed s, it follows that a Brownian motion run from ξ for an arbitrarily short time has a double point in A with probability bounded away from zero. By Blumenthal's zero–one law, this probability must be 1, so ξ is an immediate point.

We will prove something slightly stronger than (4.1), replacing $\mathcal{D}$ in (4.1) by a subset akin to $\mathcal{D}_\varepsilon$ but where the value of ε depends on the distance to the point ξ:

$$\mathcal{D}_* = \{x : B_s = B_t \text{ for some } s, t < \tau \text{ with } |x - \xi|^2 \leq t - s \leq |x - \xi|\}.$$

(Recall we stop at the time τ that the Brownian motion exits a disk of radius 3.) Let $H_*(S)$ denote the event that $\mathcal{D}_*$ has nonempty intersection with S, and let S_x denote the disk of radius $\delta|x - \xi|$ centered at x. The relevant two-point correlation estimate we will prove is, for $|x - \xi| \leq |y - \xi|$,

$$\frac{\mathbf{P}_\xi[H_*(S_x) \cap H_*(S_y)]}{\mathbf{P}_\xi(H_*(S_x))\mathbf{P}_\xi(H_*(S_y))} \leq C M_\xi(x, y). \tag{4.2}$$

Assuming this, the proof is finished in the same manner as the proof of the lower bound in Theorem 2.2, as follows.

Let μ be any probability measure on A. Fix $1/4 > \delta > 0$, which will later be sent to zero. According to (1.2), we may assume A to be a finite disjoint union of squares of a lattice which has been subdivided so that squares at distance r from ξ have sides between δr and $3\delta r$; the Whitney decomposition of the complement of ξ forms such a subdivision. This contains the union of disks $\{S_x : x \in \mathcal{B}\}$ and, as before, we may choose $\mathcal{B}$ so no two disks are closer to each other than the radius of the smaller disk, while the union of the disks still has measure at least $c\mu(A)$. Define

$$X := \sum_{x\in\mathcal{B}} \frac{1}{\mathbf{P}(H_*(S_x))} \mu(S_x) \mathbf{1}_{H_*(S_x)}.$$

Then $\mathbf{E}X \geq c$ and by (4.2),

$$\mathbf{E}X^2 \leq 2C \sum_{S_x, S_y} \mu(S_x)\mu(S_y) M_\xi(x,y).$$

Here, instead of counting each pair twice, we have summed over (x,y) for which $|x-\xi| \leq |y-\xi|$ and then doubled. As in (3.12), for $x' \in S_x$ and $y' \in S_y$, we have $M_\xi(x',y') \asymp M_\xi(x,y)$, so we may apply the second moment method to obtain

$$\mathbf{P}(H_*(A)) \geq \frac{(\mathbf{E}X)^2}{\mathbf{E}X^2} \geq c^2(c + 2\mathcal{E}(\mu))^{-1}.$$

This is uniform in δ, so sending δ to zero proves (4.1). It remains to prove (4.2).

Given x and y and $\delta \leq 1/4$, observe that when $|x-\xi| < |y-\xi|^2$, then $\mathbf{P}_\xi$ makes $H_*(S_x)$ and $H_*(S_y)$ independent up to a constant factor which is independent of δ. To see this, compute the probabilities of hitting in various orders to find that the dominant term comes from hitting S_x twice before the Brownian motion reaches a disk of radius $|y-\xi|/2$; after this, the conditional probability of $H_*(S_y)$ is only a constant multiple of the unconditional probability. Independence up to a constant factor means a two-point correlation function bounded by a constant, whence (4.2) is satisfied.

In the complementary case, the ratio of $\log|x-\xi|$ to $\log|y-\xi|$ is bounded, so we may again compute the two-point correlation function as in the proof of Theorem 2.2. Recall from (3.2) of Lemma 3.1 that using $\mathbf{P}_\xi$ instead of $\mathbf{P}$ boosts the individual probabilities of $H(D_x)$ by a factor of $|\log|x-\xi||$. The same holds for $H_*(S_x)$. Thus,

$$\mathbf{P}_\xi(H_*(S_x)) \asymp \frac{\log^2|x-\xi|}{\log^2(\delta|x-\xi|)}.$$

The probability of $H_*(S_x) \cap H_*(S_y)$ is again computed by summing the probabilities of various scenarios, the likeliest of which (up to a constant factor) is a hit on S_x, then on S_y, then a time separation of at least $|x-\xi|^2$, then another hit on S_x and then on S_y. Multiplying this out gives

$$\frac{\log|x-\xi|}{\log(\delta|x-\xi|)} \cdot \frac{\log|x-y|}{\log(\delta|y-\xi|)} \cdot \frac{\log|x-\xi|}{\log(\delta|x-\xi|)} \cdot \frac{\log|x-y|}{\log(\delta|y-\xi|)},$$

which results in the estimate (4.2). □

For completeness' sake, as mentioned above, we repeat here the standard argument to show that the complement of the strongly regular points is a set of zero capacity.

PROPOSITION 4.3 ($\mathcal{R}^c$ has zero capacity in any gauge). *The set $\mathcal{R}^c$ of nonregular points of a set A for the minimizing measure with respect to any continuous gauge f has zero f-capacity (and, in particular, has zero minimizing measure). It follows from countable additivity that $\mathrm{Cap}_f(\mathcal{R}')^c = 0$ as well.*

PROOF. Assume to the contrary that $A \setminus \mathcal{R}$ has positive capacity. Let ν be a minimizing probability measure on A for $\mathcal{E}_f$. Then for some δ, the set $\{y \in A : \Phi_\nu(y) < (1-\delta)\mathcal{E}(\nu)\}$ has positive capacity, where $\Phi_\nu(y) := \int f(x,y)\,d\nu(x)$ is the f-potential of ν at y. Fix such a δ and let μ be a probability measure supported on this set with $\mathcal{E}_f(\mu) < \infty$. For $\varepsilon \in (0,1)$, consider the measure $\rho_\varepsilon := (1-\varepsilon)\nu + \varepsilon\mu$. Its energy is given by

$$(1-\varepsilon)^2\mathcal{E}_f(\nu) + \varepsilon^2\mathcal{E}_f(\mu) + 2\varepsilon(1-\varepsilon)\int\int f(x,y)\,d\mu(x)\,d\nu(y).$$

The double integral is equal to $\int \Phi_\nu(x)\,d\mu(x)$ and since this is at most $(1-\delta)\mathcal{E}_f(\nu)$ on the support of μ, the energy of ρ_ε is bounded above by

$$[(1-\varepsilon)^2 + 2\varepsilon(1-\varepsilon)(1-\delta)]\mathcal{E}_f(\nu) + \varepsilon^2\mathcal{E}_f(\mu).$$

Write this as $\mathcal{E}_f(\nu)(1 - 2\varepsilon\delta + \varepsilon^2 Q)$, where $Q = \mathcal{E}_f(\mu)/\mathcal{E}_f(\nu) + 2\delta - 1 < \infty$, and take the derivative at $\varepsilon = 0$ to see that $\mathcal{E}_f(\rho_\varepsilon) < \mathcal{E}_f(\nu)$ for small positive ε. This contradicts the minimality of $\mathcal{E}_f(\nu)$ and proves the proposition. □

Finally, we complete the proof of the decomposition as follows. Let A_2 be the set of strongly regular points of A. We have just seen that $A_1 := A \setminus A_2$ has zero capacity in the gauge $\log^2|x-y|$. By Fitzsimmons and Salisbury [3], this implies that A_1 is almost surely disjoint from the set of Brownian double points, which is property (1).

On the other hand, by Lemma 4.1, A has NLMC at each point of A_2, and by Lemma 4.2, all such points are immediate for A. Using the fact that A_1 has no double points again, we conclude that property (2) in the statement of Theorem 2.4 is satisfied.

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