Ergod. Th. & Dynam. Sys. (1995), **15**, 593–619
Printed in Great Britain

Ergodic theory on Galton–Watson trees: speed of random walk and dimension of harmonic measure

RUSSELL LYONS
Department of Mathematics, Indiana University, Bloomington, IN 47405-5701, USA
ROBIN PEMANTLE
Department of Mathematics, University of Wisconsin, Madison, WI 53706, USA
YUVAL PERES
Department of Statistics, University of California, Berkeley, CA 94720, USA

(*Received* 22 *July* 1993 *and revised* 17 *February* 1994)

Abstract. We consider simple random walk on the family tree T of a nondegenerate supercritical Galton–Watson branching process and show that the resulting harmonic measure has a.s. strictly smaller Hausdorff dimension than that of the whole boundary of T. Concretely, this implies that an exponentially small fraction of the nth level of T carries most of the harmonic measure. First-order asymptotics for the rate of escape, Green function and the Avez entropy of the random walk are also determined. Ergodic theory of the shift on the space of random walk paths on trees is the main tool; the key observation is that iterating the transformation induced from this shift to the subset of 'exit points' yields a nonintersecting path sampled from harmonic measure.

1. *Introduction*

Consider a supercritical Galton–Watson branching process with generating function $f(s) = \sum_{k=0}^{\infty} p_k s^k$, i.e., each individual has k offspring with probability p_k, and $m := f'(1) > 1$. Started with a single progenitor, this process yields a random infinite family tree T, called a *Galton–Watson tree*, on the event of nonextinction. See Figure 1 for an example. We are interested in the asymptotic properties of simple random walk on T and what they reveal about the structure of T. Recall that a particle performing simple random walk moves from a vertex x to a vertex y chosen uniformly among the neighbors of x (*including* the parent of x). For concreteness, start the simple random walk at the root (that is, the progenitor) of T. The fact that simple random walk on a Galton–Watson tree T is almost surely transient was first established by Grimmett and Kesten [**8**], but their long proof was not published. Criteria later developed for general trees, however, easily imply that simple random walk on a Galton–Watson tree T is almost surely transient ([**16**, Theorem 4.3 and Proposition 6.4]). Equivalently, the electrical conductance of T is almost surely positive when each edge has unit conductance. A self-contained proof of a stronger version of this fact for Galton–Watson trees is included here in Lemma 9.1. See Figure 2 for the distribution of the conductance when $f(s) = (s + s^2)/2$.

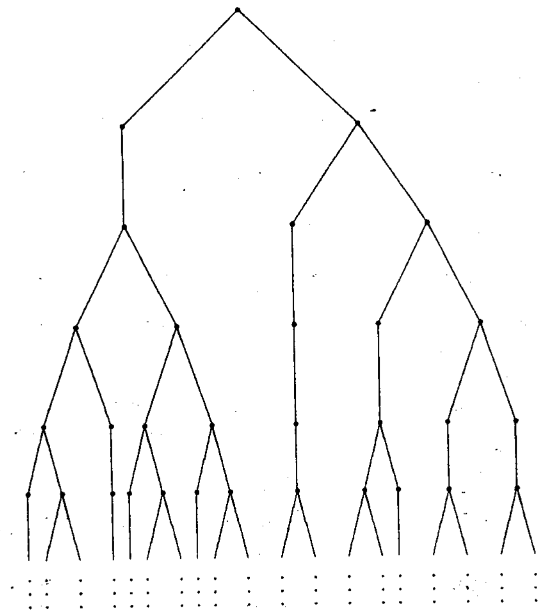

FIGURE 1. A typical Galton–Watson tree for $f(s) = (s + s^2)/2$.

More detailed study of random walk on Galton–Watson trees is aided by ergodic theory. While the trees themselves are completely inhomogeneous, we recover stationarity by considering Markov chains on the ensemble of trees.

The most basic question after transience concerns the rate of escape (or speed) of simple random walk. This is clearly related to the proportion of the time the walk spends at vertices of degree $k + 1$ for each k. Perhaps surprisingly, this asymptotic proportion is simply p_k; unlike the situation for finite graphs, there is no biasing in favor of vertices of large degree. As we show in Theorem 3.2, this means that the speed is

$$l := \sum_{k=1}^{\infty} p_k \frac{k-1}{k+1}.$$

One consequence of this is that simple random walk is slower on a *nondegenerate* Galton–Watson tree ($p_k < 1$ for all k) than on a regular tree of the same growth.

This settled, the main question which interests us is how the random irregularities which recur in a nondegenerate Galton–Watson tree T essentially confine the random walk to an exponentially smaller subtree of T. Transience of the random walk on T implies that the walking particle converges almost surely to a (random) boundary point,

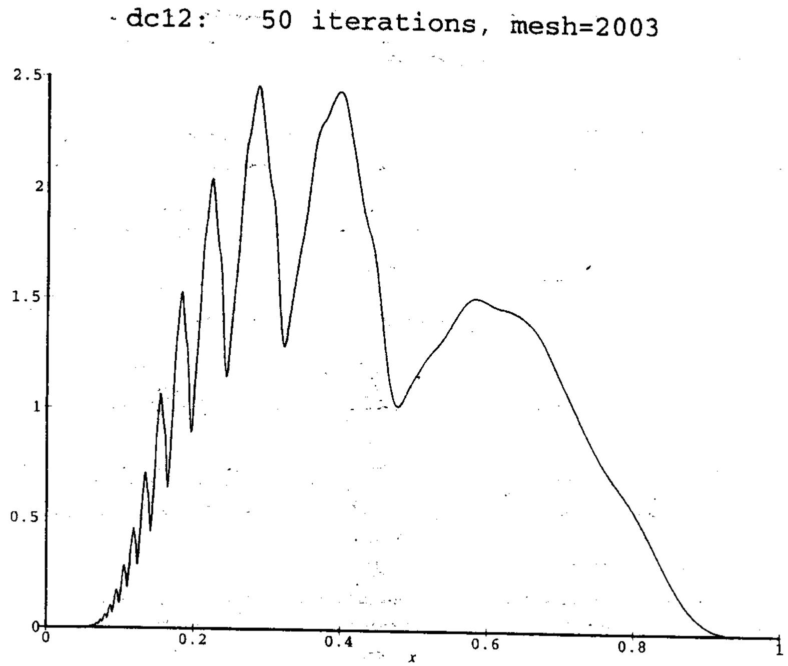

FIGURE 2. The apparent density of the conductance for $f(s) = (s + s^2)/2$.

i.e., an infinite ray of T (precise definitions are in §2) and the distribution of this random boundary point is called *harmonic measure*. The boundary ∂T has Hausdorff dimension $\log m$ in the natural metric (defined in §2). This follows from a result of Hawkes [**10**]; a simpler proof is in [**16**, Proposition 6.4]. Our main result compares this with the dimension of harmonic measure:

THEOREM 1.1. *The Hausdorff dimension of harmonic measure on the boundary of a nondegenerate Galton–Watson tree T is a.s. a constant $d < \log m = \dim(\partial T)$, i.e., there is a Borel subset of ∂T of full harmonic measure and dimension d.*

This result is established in a sharper form in Theorem 8.4.

With further work, Theorem 1.1 yields the following restriction on the range of random walk.

COROLLARY 1.2. *Fix a nondegenerate offspring distribution with mean m. Let d be as in Theorem 1.1. For any $\epsilon > 0$ and for almost every Galton–Watson tree T, there is a rooted subtree Γ of T of growth*

$$\lim_{n\to\infty} |\Gamma^n|^{1/n} = e^d < m$$

such that with probability $1-\epsilon$, the sample path of simple random walk on T is contained in Γ. (Here, $|\Gamma^n|$ is the cardinality of the nth level of Γ.)

See Theorem 9.9 for a restatement and proof.

This corollary gives a partial explanation for the low speed of simple random walk on a Galton–Watson tree: the walk is confined to a smaller subtree. Interpreted for the first N levels of T, Corollary 1.2 yields an asymptotic result about simple random walk on large finite trees for which the only available proof goes through ergodic theory on infinite trees.

The proof of Theorem 1.1 gives an abstract integral formula for the number d appearing in Theorem 1.1 and Corollary 1.2. This formula can be rewritten as follows:

$$d = \frac{1}{l} \int_{s=0}^{\infty} \int_{t=0}^{\infty} \frac{\log(1+s)}{1+s^{-1}+t^{-1}} \, dF(t) \, dF(s), \tag{1.1}$$

where F is the distribution function of the effective conductance from the root to infinity of a Galton–Watson tree. For example, this gives that $d \approx \log 1.47$ for the tree with $f(s) = (s+s^2)/2$.

Beyond the intrinsic interest of Galton–Watson trees, an additional motivation for our study of harmonic measure is the fundamental work of Makarov [**17**] on the dimension of harmonic measure for planar Brownian motion and the work of Kifer and Ledrappier [**14**] concerning the dimension of harmonic measure on the boundary of the universal cover of a compact surface of variable negative curvature.

The rest of the paper is organized as follows. Definitions, notation and a review of some useful facts from ergodic theory are in §2. In §3, we start by identifying the stationary measure for simple random walk on the space of trees. The resulting Markov process is ergodic and allows computation of the speed of simple random walk. This approach works even if $p_0 > 0$. We remark that Kesten [**13**] has analyzed simple random walk on a *critical* Galton–Watson tree conditioned on nonextinction, where the rate of escape is subdiffusive. In §4, we recall the relation between Hölder exponents and dimension of measures. Certain Markov chains on the space of trees (inspired by Furstenberg [**7**]) are discussed in §5. In §6, we define limit uniform measure, which is the analogue on the boundary of a Galton–Watson tree of Patterson measure, and compute its dimension, thus extending a theorem of Hawkes [**10**]. A general condition for dimension drop is given in §7 and applied to harmonic measure in §8, where Theorem 1.1 is proved. In §9, we derive asymptotics for the first-hitting probabilities, the Green function and the Avez entropy; Corollary 1.2 is proved there. Kaimanovich [**12**], extending work of Ledrappier [**15**], has established the relation (Theorem 9.7) between speed, Avez entropy and dimension of harmonic measure in a general setting. The paper ends with some unresolved questions in §10.

2. *Basic notation and definitions*

The following notation will be used throughout the paper. Trees will be unlabelled but rooted. This will be important for constructing stationary Markov chains on the space of trees. For a tree with no nontrivial graph-automorphisms, it still makes sense to refer to vertices of the tree. All of our trees will have no nontrivial graph-automorphisms. Write $\deg x$ for the degree of a vertex x in a tree. If we change the root of a tree T to a vertex $x \in T$, we denote the new rooted tree by MoveRoot(T, x). Given a tree T and a vertex

x in T, the subtree $T(x)$ rooted at x denotes the subgraph of T formed from those edges and vertices which become disconnected from the root of T when x is removed. This is considered as the descendant tree of x. A path $x_0, x_1, \ldots$ in T will be denoted $\vec{x}$, while a bi-infinite path $\ldots, x_{-1}, x_0, x_1, \ldots$ will be denoted $\overleftrightarrow{x}$. Similarly, a path $\ldots, x_{-1}, x_0$ will be denoted $\overleftarrow{x}$. Rays are special cases of singly-infinite paths, namely, ones which never backtrack. They will be denoted ξ, regardless of their direction. If ξ is a ray, the vertices along ξ will be denoted $\xi_0, \xi_1, \ldots$. The set of all rays emanating from the root (also known as infinite lines of descent, or ends) is called the *boundary* of T, denoted ∂T. A path $\vec{x}$ that passes through every vertex at most finitely many times intersects a unique ray $\xi \in \partial T$ infinitely often; we say that $\vec{x}$ *converges* to ξ and write $x_{+\infty} := \xi$. Similarly for a limit $x_{-\infty}$ of a path $\overleftarrow{x}$. The space of convergent paths $\vec{x}$ in T will be denoted $\vec{T}$; likewise, $\overleftarrow{T}$ denotes the convergent paths $\overleftarrow{x}$ and $\overleftrightarrow{T}$ denotes the paths $\overleftrightarrow{x}$ for which both $\vec{x}$ and $\overleftarrow{x}$ converge and have distinct limits. For disjoint trees $T_1, \ldots, T_k$, let $\bigvee_{i=1}^k T_i$ denote the tree formed by joining the roots of T_i by single edges to a new vertex, the new vertex being the root of the new tree.

For a vertex $x \in T$, let $|x|$ denote the distance from the root of T to x, i.e., the number of edges on the shortest path from the root of T to x. More generally, for two vertices $x, y \in T$, write $|x - y|$ for the distance from x to y in T. Let T^n be the set of vertices at distance n from the root of T. If $y \in T(x)$ and $|y| = |x| + 1$, we write $x \to y$; we think of y as a child of x. For distinct boundary points $\xi, \eta \in \partial T$, let $\xi \wedge \eta$ denote the furthest vertex from the root common to ξ and η. Define the metric

$$d(\xi, \eta) := e^{-|\xi \wedge \eta|} \quad (\xi \neq \eta)$$

on ∂T.

A function θ on the vertices of T is called a *flow* if $\theta \geq 0$ and for all $x \in T$,

$$\theta(x) = \sum_{x \to y} \theta(y).$$

These functions are in one-to-one correspondence with positive Borel measures μ on ∂T via

$$\theta(x) = \mu(\{\xi \in \partial T;\ x \in \xi\}).$$

For this reason, we identify flows on T and measures on ∂T.

A Galton–Watson process is determined by a probability distribution $\{p_0, p_1, p_2, \ldots\}$ on $\mathbf{N}$. Let the generation sizes be Z_n, so that Z_1 has the given distribution. Let the mean generation size be $m := \sum k p_k = \mathbf{E}[Z_1]$. We assume throughout that $1 < m < \infty$ and that all $p_k < 1$. The usual martingale $Z_n/m^n \to W$ will play an important role. Note that since our trees are unlabelled, the chance, say, that the family tree has two children of the root, one having one child and the other having three, is $2p_2p_1p_3$. Since various measures on the space of trees will need to be considered, we use GW to denote the standard measure on (family) trees given by a Galton–Watson process. Here, we regard the space of trees as being given the weak topology generated by finite subtrees from the root.

Formally, the space $\mathcal{T}$ of rooted unlabelled locally finite trees can be defined as follows. Let $\mathcal{T}_n$ be the space of rooted unlabelled finite trees of height n

with the discrete topology. There are natural maps from $\mathcal{T}_{n+1} \to \mathcal{T}_n$. Define $\mathcal{T}$ to be the inverse limit of $\mathcal{T}_n$. This is a Polish space.

We shall assume that $p_0 = 0$ unless stated otherwise. In particular, Galton–Watson trees a.s. have no nontrivial automorphisms. We call two measures *equivalent* if they are mutually absolutely continuous.

Given a measure-preserving transformation S of a measure space (X, μ) and a measurable set $A \subseteq X$ with $0 < \mu(A) < \infty$, we denote the *induced measure* on A by $\mu_A(C) := \mu(C)/\mu(A)$ for $C \subseteq A$. We also write $\mu(C \mid A)$ for $\mu_A(C)$ since it is a conditional measure. Define the *return time* to A by $n_A(x) := \inf\{n \geq 1;\ S^n x \in A\}$ for $x \in A$ and, if $n_A(x) < \infty$, the *return map* $S_A(x) := S^{n_A(x)}(x)$. The Poincaré recurrence theorem [**18**, p. 34] states that if $\mu(X) < \infty$, then $n_A(x) < \infty$ for a.e. $x \in A$. In this case, (A, μ_A, S_A) is a measure-preserving system [**18**, p. 39], called the *induced system.* If $\mu\left(X \setminus \bigcup_{n=1}^{\infty} S^{-n} A\right) = 0$, then (X, μ, S) is called a *(Kakutani) tower* over (A, μ_A, S_A). In this case, the Kac lemma [**18**, p. 46] gives that $\int_A n_A \, d\mu_A = \mu(X)/\mu(A)$; also, S is ergodic iff S_A is ergodic [**18**, p. 56].

3. *Speed of simple random walk*

As in the rest of this paper, we assume that $p_0 = 0$; however, towards the end of this section, we discuss what changes result when $p_0 > 0$. In order to analyze the speed of simple random walk, we need to find a stationary measure for the environment process, i.e., the tree as seen from the current vertex. This will be a fundamental tool as well for our analysis of harmonic measure. Now the root of a Galton–Watson tree is different from the other vertices since it has stochastically one fewer neighbor. To remedy this defect, we consider augmented Galton–Watson measure, AGW. This measure is defined just like GW except that the number of children of the root (only) has the law of $Z_1 + 1$; i.e., the root has $k+1$ children with probability p_k and these children all have independent standard Galton–Watson descendant trees. Consider the Markov chain which moves from a tree T to the tree MoveRoot(T, x) for a random neighbor x of the root of T. For fixed T, this chain is isomorphic to simple random walk on T. Write the transition probabilities as

$$\mathbf{p}_{\mathrm{SRW}}(T, T') = \begin{cases} 1/\deg(\mathrm{root}(T)), & \text{if } \exists x \in T\ |x| = 1 \ \&\ T' = \mathrm{MoveRoot}(T, x); \\ 0, & \text{otherwise.} \end{cases}$$

THEOREM 3.1. *The Markov chain with transition probabilities* $\mathbf{p}_{\mathrm{SRW}}$ *and initial distribution* AGW *is stationary and reversible.*

Proof. For Borel sets A, B of trees, write

$$\widehat{\mathbf{p}}_{\mathrm{SRW}}(A, B) := \int_A \mathbf{p}_{\mathrm{SRW}}(T, B)\, d\mathrm{AGW}(T).$$

We must show that $\widehat{\mathbf{p}}_{\mathrm{SRW}}(A, B) = \widehat{\mathbf{p}}_{\mathrm{SRW}}(B, A)$. Given disjoint trees T_1, T_2, define $[T_1 \bullet\!\!- T_2]$ to be the tree rooted at root(T_1) formed by joining root(T_1) and root(T_2) by an edge. Note that this is not a symmetric operation. For sets C, D, write

$$[C \bullet\!\!- D] := \{[T_1 \bullet\!\!- T_2];\ T_1 \in C,\ T_2 \in D\}.$$

Then it suffices to show that $\widehat{\mathbf{p}}_{\mathrm{SRW}}(A, B) = \widehat{\mathbf{p}}_{\mathrm{SRW}}(B, A)$ for sets of the form $A = [C \bullet\!\!- D]$, $B = [D \bullet\!\!- C]$ with C, D being disjoint Borel sets of trees since such sets generate the σ-field (up to sets of AGW-measure 0). Furthermore, for sets F_i, write

$$\bigvee F_i := \{\bigvee T_i;\ T_i \in F_i\}.$$

Then we may further assume that there are some k, l, disjoint C_i $(1 \le i \le k)$, disjoint D_j $(1 \le j \le l)$ such that

$$C = \bigvee_{i=1}^{k} C_i, \quad D = \bigvee_{j=1}^{l} D_j,$$

$$\emptyset = D \cap \bigcup_{i=1}^{k} C_i = C \cap \bigcup_{j=1}^{l} D_j$$

for the same reason.

Now we may calculate that

$$\begin{aligned}\mathrm{AGW}(A) = \mathrm{AGW}([C \bullet\!\!- D]) &= a_k (k+1)! \prod_{i=1}^{k} \mathrm{GW}(C_i)\ \mathrm{GW}(D)\\ &= (k+1)\mathrm{GW}(C)\mathrm{GW}(D).\end{aligned}$$

Also for all $T \in A$ and $T' \in B$, $\mathbf{p}_{\mathrm{SRW}}(T, T') = 1/(k+1)$. Therefore,

$$\widehat{\mathbf{p}}_{\mathrm{SRW}}(A, B) = \int_A \frac{1}{k+1}\, d\mathrm{AGW}(T) = \mathrm{GW}(C)\mathrm{GW}(D).$$

Likewise, $\widehat{\mathbf{p}}_{\mathrm{SRW}}(B, A) = \mathrm{GW}(D)\mathrm{GW}(C)$, whence the two are equal. □

We shall find it convenient to work with the bi-infinite path space (actually, path bundle over the space of trees) of simple random walk on Galton–Watson trees:

$$\mathrm{PathsInTrees} := \left\{(\overleftrightarrow{x}, T);\ \overleftrightarrow{x} \in \overleftrightarrow{T},\ x_0 = \mathrm{root}(T)\right\}.$$

Let S be the shift map:

$$(S\overleftrightarrow{x})_n := x_{n+1},$$

$$S(\overleftrightarrow{x}, T) := \left(S\overleftrightarrow{x}, \mathrm{MoveRoot}(T, x_1)\right).$$

Let SRW × AGW denote the measure on the path bundle associated with the Markov chain above, even though this is not a tensor product of measures. In §8, we shall see that the system (PathsInTrees, SRW × AGW, S) is a tower over an ergodic Markov chain, and hence is ergodic itself.

THEOREM 3.2. *The speed (rate of escape) of simple random walk is* SRW × AGW*-a.s.*

$$l := \lim_{n\to\infty} \frac{|x_n|}{n} = \mathbf{E}\left[\frac{Z_1 - 1}{Z_1 + 1}\right]. \tag{3.1}$$

Proof. Rather than calculate the speed as the rate of escape from the root of the tree, we shall calculate it as the rate of increase of the 'horodistance' (Busemann function) from a boundary point. In other words, given a boundary point $\xi \in \partial T$ and a vertex $x \in T$, let $[x, \xi]$ denote the ray from x to ξ. (More precisely, there is a unique one-to-one correspondence $\xi \mapsto [x, \xi]$ from $\partial T \to \partial\text{MoveRoot}(T, x)$ such that ξ and $[x, \xi]$ have infinitely many vertices in common.) Given two distinct vertices $x, y \in T$, define $x \wedge_\xi y$ to be the vertex where $[x, \xi]$ and $[y, \xi]$ meet. Let the signed distance from x to y as seen from ξ be $[y - x]_\xi := |y - x \wedge_\xi y| - |x - x \wedge_\xi y|$. Note that for any vertices x, y, z and any ray ξ, we have $[z - x]_\xi = [z - y]_\xi + [y - x]_\xi$.

Now, for $\overleftrightarrow{x} \in \overleftrightarrow{T}$, since $x_{+\infty} \neq x_{-\infty}$, there is a constant c such that for all sufficiently large n,

$$|x_n - x_0| = [x_n - x_0]_{x_{-\infty}} + c,$$

whence the speed is the limit

$$\lim_{n\to\infty} \frac{1}{n}[x_n - x_0]_{x_{-\infty}} = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} [x_{k+1} - x_k]_{x_{-\infty}}.$$

But these are averages of an ergodic stationary sequence, $\langle S^k[x_1 - x_0]_{x_{-\infty}} \rangle$, whence the ergodic theorem tells us that they converge a.s. to their mean

$$\int [x_1 - x_0]_{x_{-\infty}} \, d\text{SRW} \times \text{AGW}(\overleftrightarrow{x}, T). \tag{3.2}$$

To evaluate this expectation, consider a bi-infinite path $\overleftrightarrow{x} \in \overleftrightarrow{T}$ with $(\overleftrightarrow{x}, T)$ picked according to SRW $\times$ AGW. Since x_1 is uniformly distributed among the neighbors of x_0 given the number of such neighbors, $\deg(x_0)$, and given $x_{-\infty}$, the chance that $[x_1 - x_0]_{x_{-\infty}}$ is -1 is $1/\deg(x_0)$; otherwise, $[x_1 - x_0]_{x_{-\infty}} = +1$. Therefore the number (3.2) evaluates to

$$\int \frac{\deg(\text{root}(T)) - 2}{\deg(\text{root}(T))} \, d\text{AGW}(T),$$

which is the same as (3.1) since AGW gives the root one more edge than does GW. □

Remark. By Jensen's inequality, unless $Z_1 = m$ a.s., this is strictly smaller than $(m-1)/(m+1)$, the speed on the deterministic tree of the same growth rate when m is an integer. Since random walk on a random spherically symmetric tree is essentially the same as a special case of random walk in a random environment (RWRE) on the nonnegative integers, we may compare this slowing down with the fact that randomness also slows down random walk for the general RWRE on the integers [**20**].

Remark. The same result holds for simple random walk on GW-a.e. tree. To prove this intuitively clear fact, note that the AGW-law of $T \setminus T(x_{-1})$ is GW since x_{-1} is uniformly chosen from the neighbors of the root of T. Let A be the event that the walk remains in $T \setminus T(x_{-1})$:

$$\begin{aligned} A := &\left\{(\overleftrightarrow{x}, T) \in \text{PathsInTrees};\ \forall n > 0\ x_n \in T \setminus T(x_{-1})\right\} \\ = &\left\{(\overleftrightarrow{x}, T) \in \text{PathsInTrees};\ \vec{x} \subset T \setminus T(x_{-1})\right\} \end{aligned}$$

and B_k be the event that the walk returns to the root of T exactly k times:

$$B_k := \left\{(\vec{x}, T) \in \text{PathsInTrees};\ \left|\{i \geq 1;\ x_i = x_0\}\right| = k\right\}.$$

Then the (SRW × AGW | A, B_k)-law of $(\vec{x}, T \setminus T(x_{-1}))$ is equal to the (SRW × GW | B_k)-law of $(\vec{x}, T)$, whence the (SRW × AGW | A)-law of $(\vec{x}, T \setminus T(x_{-1}))$ is equivalent to the (SRW × GW)-law of $(\vec{x}, T)$. By the theorem, this implies that the speed of the latter is almost surely $\mathbf{E}[(Z_1 - 1)/(Z_1 + 1)]$.

Now we consider the case when $p_0 > 0$. As usual, let q be the probability of extinction of a Galton–Watson process. Let Nonextinction be the event of nonextinction of an AGW tree. It is easily seen that $\text{AGW}_{\text{Nonextinction}}$ is SRW-invariant. (In fact, AGW is still invariant and Nonextinction is an invariant subset of trees.) The $\text{AGW}_{\text{Nonextinction}}$-distribution of the degree of the root is seen to be

$$\begin{aligned}\text{AGW}(\deg x_0 = k+1 \mid \text{Nonextinction}) &= \frac{\text{AGW}(\text{Nonextinction} \mid \deg x_0 = k+1)}{\text{AGW}(\text{Nonextinction})} p_k \\ &= p_k \frac{1 - q^{k+1}}{1 - q^2};\end{aligned}$$

for the numerator, we have calculated the probability of extinction by calculating the probability that each child of the root has only finitely many descendants; while for the denominator, we have calculated the probability of extinction by regarding AGW as [GW •—GW], so that extinction occurs when each of the two GW trees is finite.

The proof of Theorem 3.2 on speed is valid when one conditions on nonextinction in the appropriate places. It gives the following formula for the speed:

$$\lim_{n\to\infty} \frac{|x_n|}{n} = \mathbf{E}\left[\frac{Z_1 - 1}{Z_1 + 1} \;\middle|\; \text{Nonextinction}\right] = \sum_{k \geq 1} \frac{k-1}{k+1} p_k \frac{1 - q^{k+1}}{1 - q^2}.$$

The dynamical system (PathsInTrees, SRW × AGW, S) actually has much stronger mixing properties than simply ergodicity: using **[9]** and some ideas about regeneration points, it may be shown that it is a K-automorphism.

4. *Hölder exponent and dimension*

The *Hausdorff dimension* of a probability measure μ on X is usually defined to be

$$\dim \mu := \min\{\dim E;\ \mu(E) = 1\}.$$

There is another quantity related to the Hausdorff dimension of measures which yields more information when it exists: the *Hölder exponent* of μ at x is defined to be

$$\text{Hö}(\mu)(x) := \lim_{r \downarrow 0} \left(\log \frac{1}{\mu(B_r(x))} \Big/ \log \frac{1}{r}\right) \tag{4.1}$$

when the limit of the above quotient exists.

Example. For a Borel probability measure θ on ∂T, we have

$$\text{Hö}(\theta)(\xi) = \lim_{n\to\infty} \frac{1}{n} \log \frac{1}{\theta(\xi_n)}.$$

The relationship between Hölder exponent to Hausdorff dimension is given in the following result of Billingsley [**4**, §14]; see also [**21**]. (Billingsley proved a more general result for Euclidean space, but the same proof works even more easily on the boundaries of trees.)

LEMMA 4.1. *For any Borel probability measure μ on the boundary of a tree, if the Hölder exponent of μ exists μ-a.e. and is constant, then that constant is the Hausdorff dimension of μ.*

This lemma is actually valid with 'lim inf' in place of 'lim' in (4.1). When the Hölder exponent of μ exists and is constant, however, all reasonable alternative notions of dimension of μ coincide [**21**].

Example. Given a tree T, define simple forward random walk to be the random walk which chooses randomly (uniformly) among the children of the present vertex as the next vertex. The corresponding harmonic measure on ∂T is called *visibility measure*, denoted VIS_T, and corresponds to the *equally-splitting flow*. Suppose now that T is a Galton–Watson tree. Then VIS_T is a flow on the random tree T. Write VIS × GW for the measure

$$(\mathrm{VIS}\times\mathrm{GW})(F) := \int\int \mathbf{1}_F(\xi, T)\, d\mathrm{VIS}_T(\xi)\, d\mathrm{GW}(T).$$

Since

$$\frac{1}{n}\log\frac{1}{\mathrm{VIS}_T(\xi_n)} = \frac{1}{n}\sum_{k=0}^{n-1}\log\frac{\mathrm{VIS}_T(\xi_k)}{\mathrm{VIS}_T(\xi_{k+1})}$$

and the random variables $\mathrm{VIS}_T(\xi_{k-1})/\mathrm{VIS}_T(\xi_k)$ are VIS × GW-i.i.d. with the same distribution as Z_1, the strong law of large numbers gives

$$\mathrm{H\ddot{o}}(\mathrm{VIS}_T)(\xi) = \mathbf{E}[\log Z_1] \quad \mathrm{VIS}\times\mathrm{GW}\text{-a.s. } (\xi, T).$$

Thus $\dim \mathrm{VIS}_T = \mathbf{E}[\log Z_1]$ for GW-a.e. tree T.

The arithmetic mean–geometric mean inequality shows that this dimension is less than $\log m$ except in the deterministic case $Z_1 = m$ a.s.

5. *Markov chains on the space of trees*

Given a flow θ on a tree T and a vertex $x \in T$ with $\theta(x) > 0$, we write θ^x for the (conditional) flow on $T(x)$ given by

$$\theta^x(y) := \theta(y)/\theta(x) \quad (y \in T(x)).$$

We call a Borel function Θ : {trees} → {flows on trees} a *(consistent) flow rule* if $\forall T\ \ \Theta(T)$ is a flow on T such that

$$x \in T,\ |x| = 1,\ \Theta(T)(x) > 0 \quad\Longrightarrow\quad \Theta(T)^x = \Theta(T(x)).$$

A flow rule may also be thought of as a Borel function which assigns to a k-tuple $(T_1, \ldots, T_k)$ of trees a k-tuple of nonnegative numbers adding to one representing the probabilities of choosing the corresponding trees T_i in $\bigvee_{i=1}^k T_i$. A random (forward) walk according to $\Theta(T)$ is Markovian in that once it visits $T(x)$, it walks according to $\Theta(T(x))$. It follows from the definition that for all $x \in T$, not only those at distance 1 from the root, $\Theta(T)(x) > 0 \Rightarrow \Theta(T)^x = \Theta(T(x))$. We shall usually write Θ_T for $\Theta(T)$.

One example of a flow rule has already been encountered, namely, VIS. The principal object of interest in this paper, harmonic measure, also comes from a flow rule, HARM. Another important flow rule, UNIF, is discussed in §6.

PROPOSITION 5.1. *If Θ and Θ' are two flow rules such that for* GW*-a.e. tree T and all vertices $|x| = 1$, $\Theta_T(x) + \Theta'_T(x) > 0$, then* $\mathrm{GW}(\Theta_T = \Theta'_T) \in \{0, 1\}$.

That the hypothesis is needed is seen from examples, say, where two flow rules both follow a 2-ray when it exists (see below) but do different things otherwise.

Proof. Let $s := \mathrm{GW}(\Theta_T = \Theta'_T)$. By the hypothesis,

$$\Theta_T = \Theta'_T, \ |x| = 1 \implies \Theta_{T(x)} = \Theta'_{T(x)}.$$

Therefore, conditioning on Z_1, we see that

$$s \le \sum_k p_k \prod_{i=1}^k \mathrm{GW}(\Theta_{T_i} = \Theta'_{T_i}) = \sum_k p_k s^k$$

and so $s \in \{0, 1\}$. □

Given a flow rule Θ, there is an associated Markov chain on the space of trees given by the transition probabilities

$$\forall T \ \forall x \in T \ |x| = 1 \implies \mathbf{p}_\Theta(T, T(x)) = \Theta_T(x).$$

We say that a (possibly infinite) measure μ on the space of trees is Θ-*stationary* if it is $\mathbf{p}_\Theta$-stationary, i.e., $\mu \mathbf{p}_\Theta = \mu$, or, in other words, for any Borel set A of trees,

$$\begin{aligned}\mu(A) = (\mu \mathbf{p}_\Theta)(A) &= \int \sum_{T' \in A} \mathbf{p}_\Theta(T, T')\, d\mu(T) \\ &= \int \sum_{\substack{|x|=1 \\ T(x) \in A}} \mathbf{p}_\Theta(T, T(x))\, d\mu(T) = \int \sum_{\substack{|x|=1 \\ T(x) \in A}} \Theta_T(x)\, d\mu(T).\end{aligned}$$

The path of such a Markov chain is a sequence $\langle T(\xi_n) \rangle_{n=0}^\infty$ for some tree T and some ray $\xi \in \partial T$. Clearly, we may identify the space of such paths with the ray bundle

$$\mathrm{RaysInTrees} := \{(\xi, T);\ \xi \in \partial T\}.$$

For the corresponding path measure on RaysInTrees, write

$$(\Theta \times \mu)(F) := \int \int \mathbf{1}_F(\xi, T)\, d\Theta_T(\xi)\, d\mu(T),$$

even though this is not a tensor product of measures.

It is well known ([19, pp. 96–97]) that $\Theta \times \mu$ is ergodic iff every Θ-invariant set of trees has μ-measure 0 or 1, where a Borel set A of trees is called *Θ-invariant* if $\mathbf{p}_\Theta(T, A) = \mathbf{1}_A(T)$ μ-a.s. In fact, the shift-invariant σ-field in RaysInTrees corresponds to the Θ-invariant σ-field via the projection π : RaysInTrees $\to$ {trees} onto the second coordinate, which is essentially invertible when restricted to the invariant σ-fields.

PROPOSITION 5.2. *Let Θ be a flow rule such that for* GW*-a.e. tree T and for all $|x| = 1$, $\Theta_T(x) > 0$. Then the Markov chain with transition probabilities $\mathbf{p}_\Theta$ and initial distribution* GW *is ergodic, though not necessarily stationary. Hence, if a (possibly infinite) Θ-stationary measure μ exists which is absolutely continuous with respect to* GW, *then μ is equivalent to* GW *and the associated Markov chain is ergodic.*

Proof. Let A be a Borel set of trees which is Θ-invariant. It follows from our assumption that for GW-a.e. T, we have $T \in A$ iff $T(x) \in A$ whenever $|x| = 1$. Therefore conditioning on the degree of the root of T gives

$$\mathrm{GW}(A) = \sum_k p_k \int_A \cdots \int_A \prod_{i=1}^k d\mathrm{GW}(T_i) = \sum_k p_k \mathrm{GW}(A)^k,$$

so that $\mathrm{GW}(A) \in \{0, 1\}$. □

An example of a flow rule Θ with a Θ-stationary measure which is absolutely continuous with respect to GW but whose associated Markov chain is *not* ergodic is as follows. Call a ray $\xi \in \partial T$ an *n-ray* if every vertex in the ray has exactly n children and write $T \in A_n$ if ∂T contains an n-ray. Note that A_n are pairwise disjoint. Consider the GW process with $p_3 := p_4 := 1/2$. Then $\mathrm{GW}(A_n) > 0$ for $n = 3, 4$. Define Θ_T to choose equally among all children of the root on $(A_3 \cup A_4)^c$ and to choose equally among all children of the root belonging to an n-ray when $T \in A_n$. Then GW_{A_n} is Θ-stationary for both $n = 3, 4$, whence the Θ-stationary measure $(\mathrm{GW}_{A_3} + \mathrm{GW}_{A_4})/2$ gives a non-ergodic Markov chain.

Given a Θ-stationary probability measure μ on the space of trees, we follow Furstenberg [7] and define the *entropy* of the associated stationary Markov chain as

$$\begin{aligned}
\mathrm{Ent}_\Theta(\mu) :&= \int \sum_{|x|=1} \mathbf{p}_\Theta(T, T(x)) \log \frac{1}{\mathbf{p}_\Theta(T, T(x))}\, d\mu(T) \\
&= \int \sum_{|x|=1} \Theta_T(x) \log \frac{1}{\Theta_T(x)}\, d\mu(T) = \int \int \log \frac{1}{\Theta_T(\xi_1)}\, d\Theta_T(\xi)\, d\mu(T) \\
&= \int \log \frac{1}{\Theta_T(\xi_1)}\, d(\Theta \times \mu)(\xi, T).
\end{aligned}$$

[This is not the ergodic-theoretic entropy of the measure-preserving system, only the entropy with respect to a certain (non-generating) partition.] Define $g_\Theta(\xi, T) := \log 1/\Theta_T(\xi_1)$ and let S be the shift on RaysInTrees. The ergodic theorem gives that

$$\begin{aligned}
\mathrm{H\ddot{o}}(\Theta_T)(\xi) &= \lim_{n\to\infty} \frac{1}{n} \log \frac{1}{\Theta_T(\xi_n)} = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \log \frac{\Theta_T(\xi_k)}{\Theta_T(\xi_{k+1})} \\
&= \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \log \frac{1}{\Theta(T)^{\xi_k}(\xi_{k+1})} = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} S^k g_\Theta(\xi, T)
\end{aligned}$$

exists $\Theta \times \mu$-a.s. and satisfies

$$\int \text{Hö}(\Theta_T)(\xi)\, d(\Theta \times \mu)(\xi, T) = \text{Ent}_\Theta(\mu).$$

If the Markov chain is ergodic, then

$$\text{Hö}(\Theta_T)(\xi) = \text{Ent}_\Theta(\mu) \quad \Theta \times \mu\text{-a.s.} \tag{5.1}$$

Note that even if the Markov chain is not ergodic, the Hölder exponent $\text{Hö}(\Theta_T)(\xi)$ is constant Θ_T-a.s. for μ-a.e. T: since $(\xi, T) \mapsto \text{Hö}(\Theta_T)(\xi)$ is a shift-invariant function, it is Θ-invariant (i.e., measurable with respect to the Θ-invariant σ-field) and so depends only on T.

6. *Limit uniform measure*

In this section, we sharpen Hawkes's theorem [**10**] on the Hölder exponent of limit uniform measure. This measure is defined as follows. According to the Seneta–Heyde theorem [**1**, Theorem II.5.1, p. 43], there exist constants c_n such that $c_{n+1}/c_n \to m$ and

$$\tilde{W}(T) := \lim_{n\to\infty} Z_n/c_n$$

exists and is finite non-zero a.s. Note that

$$\tilde{W}(T) = \frac{1}{m} \sum_{|x|=1} \tilde{W}(T(x)). \tag{6.1}$$

Therefore, if we define for every vertex $x \in T$

$$\text{UNIF}_T(x) = \frac{\tilde{W}(T(x))}{m^{|x|}\tilde{W}(T)}, \tag{6.2}$$

then UNIF_T is a unit flow and defines *limit uniform measure* on the boundary of T. The Kesten–Stigum theorem [**1**, Theorem I.2.1, p 23], which says that $\int W(T)\, d\text{GW}(T) > 0$ iff $\int W(T)\, d\text{GW}(T) = 1$ iff $W > 0$ a.s. iff $\mathbf{E}[Z_1 \log Z_1] < \infty$, implies that when $\mathbf{E}[Z_1 \log Z_1] < \infty$, the constants c_n may be taken to be m^n and so W may be used in place of $\tilde{W}$ in (6.2) and (6.1). A theorem of Athreya [**2**] gives that

$$\int \tilde{W}(T)\, d\text{GW}(T) < \infty \iff \mathbf{E}[Z_1 \log Z_1] < \infty. \tag{6.3}$$

We next show that a (possibly infinite) UNIF-stationary measure on trees is $\tilde{W}(T)\, d\text{GW}(T)$. This was also observed by Hawkes [**10**, p. 378]. Related ideas occur in [**11**].

PROPOSITION 6.1. *The Markov chain with transition probabilities* $\mathbf{p}_{\text{UNIF}}$ *and initial distribution* $\tilde{W} \cdot \text{GW}$ *is stationary and ergodic.*

Proof. Apply the definition of stationarity with $\Theta_T(x) = \tilde{W}(T(x))/(m\tilde{W}(T))$: for any

Borel set A of trees, we have

$$
\begin{aligned}
((\tilde{W} \cdot \mathrm{GW})\mathbf{p}_{\mathrm{UNIF}})(A) &= \int \sum_{\substack{|x|=1 \\ T(x)\in A}} \frac{\tilde{W}(T(x))}{m\tilde{W}(T)} \cdot \tilde{W}(T)\, d\mathrm{GW}(T) \\
&= \sum_{k=1}^{\infty} p_k \frac{1}{m} \int_{T_1} \cdots \int_{T_k} \sum_{i=1}^{k} \mathbf{1}_{T_i \in A} \tilde{W}(T_i) \prod_{j=1}^{k} d\mathrm{GW}(T_j) \\
&= \sum_{k=1}^{\infty} p_k \frac{1}{m} \sum_{i=1}^{k} \int_{T_1} \cdots \int_{T_k} \mathbf{1}_{T_i \in A} \tilde{W}(T_i) \prod_{j=1}^{k} d\mathrm{GW}(T_j) \\
&= \sum_{k=1}^{\infty} p_k \frac{1}{m} \sum_{i=1}^{k} \int_A \tilde{W}\, d\mathrm{GW} = \int_A \tilde{W}\, d\mathrm{GW} \\
&= (\tilde{W} \cdot \mathrm{GW})(A),
\end{aligned}
$$

as desired.

The Markov chain is ergodic by our general result on ergodicity (Proposition 5.2) and the fact that $\tilde{W} > 0$ GW-a.s. □

In order to calculate the Hölder exponent of limit uniform measure, we shall use the following well-known lemma of ergodic theory:

LEMMA 6.2. *If S is a measure-preserving transformation on a probability space, g is finite and measurable, and $g - Sg$ is bounded below by an integrable function, then $g - Sg$ is integrable with integral zero.*

Proof. By ergodic decomposition, we may assume that S is ergodic. If $g - Sg$ is not integrable with integral zero, then it has either a finite non-zero integral or $\int (g - Sg) = +\infty$. In either case, the ergodic theorem implies that

$$
0 \neq \int (g - Sg) = \lim_{k\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} S^k(g - Sg)(x) = \lim_{k\to\infty} \frac{1}{n}(g - S^n g)(x)
$$

for a.e. x, whence $S^n g(x) \to \pm\infty$ for a.e. x as $n \to \infty$. But the distribution of $S^n g$ is the same as that of g, a contradiction. Therefore, $g - Sg$ is integrable with integral zero. □

THEOREM 6.3. *If $\mathbf{E}[Z_1 \log Z_1] < \infty$, then the Hölder exponent at ξ of limit uniform measure UNIF_T is equal to $\log m$ for UNIF_T-a.e. ray $\xi \in \partial T$ and GW-a.e. tree T. In particular, $\dim \mathrm{UNIF}_T = \log m$ for GW-a.e. T.*

Proof. The hypothesis and Proposition 6.1 ensure that $W\mathrm{GW}$ is a stationary probability distribution. Let S be the shift on the ray bundle RaysInTrees with the invariant measure $\mathrm{UNIF} \times W\mathrm{GW}$. Define $g(\xi, T) := \log W(T)$ for a Galton–Watson tree T and $\xi \in \partial T$. Then

$$
\begin{aligned}
(g - Sg)(\xi, T) = \log W(T) - \log W(T(\xi_1)) &= \log \frac{mW(T)}{W(T(\xi_1))} - \log m \\
&= \log \frac{1}{\mathrm{UNIF}_T(\xi_1)} - \log m\,.
\end{aligned}
$$

In particular, $g - Sg \geq -\log m$, whence the lemma implies that $g - Sg$ has integral zero.

Now, for UNIF $\times$ WGW-a.e. (ξ, T) (hence for UNIF $\times$ GW-a.e. (ξ, T)), we have that $\text{Hö}(\text{UNIF}_T)(\xi) = \text{Ent}_{\text{UNIF}}(W\text{GW})$ by ergodicity. By definition and the preceding calculation, this in turn is

$$\begin{aligned}\text{Ent}_{\text{UNIF}}(W\text{GW}) &= \int\int \log \frac{1}{\text{UNIF}_T(\xi_1)}\, d\text{UNIF}_T(\xi)\, W(T)\, d\text{GW}(T)\\ &= \log m + \int\int (g - Sg)\, d\text{UNIF}_T(\xi)\, W(T)\, d\text{GW}(T)\\ &= \log m.\end{aligned}$$

□

7. *Dimension drop for other flow rules*

We believe that any flow rule other than limit uniform gives measures of dimension less than $\log m$ GW-a.s. In this section, we prove that this is the case when the flow rule has a finite stationary measure equivalent to GW. (Note that our theorem is valid even when $\mathbf{E}[Z_1 \log Z_1] = \infty$.) To this end, we shall use Shannon's inequality (concavity of the log function):

$$a_i, b_i \in [0, 1], \quad \sum a_i = \sum b_i = 1 \quad \Longrightarrow \quad \sum a_i \log \frac{1}{a_i} \leq \sum a_i \log \frac{1}{b_i},$$

with equality iff $a_i \equiv b_i$.

THEOREM 7.1. *If Θ is a flow rule such that $\Theta_T \neq \text{UNIF}_T$ for GW-a.e. T and there is a finite Θ-stationary measure μ absolutely continuous with respect to GW, then for μ-a.e. T, we have $\text{Hö}(\Theta_T) < \log m$ Θ_T-a.s. and $\dim(\Theta_T) < \log m$.*

Proof. Recall that the Hölder exponent of Θ_T is constant Θ_T-a.s. for μ-a.e. T and equal to the Hausdorff dimension of Θ_T. Thus, it suffices to show that the set of trees

$$A := \{T;\ \dim \Theta_T = \log m\} = \{T;\ \text{Hö}(\Theta_T) = \log m \quad \Theta_T\text{-a.s.}\}$$

has μ-measure 0. Suppose that $\mu(A) > 0$. Now since $\mu \ll \text{GW}$, the limit uniform measure UNIF_T is defined and satisfies (6.2) for μ_A-a.e. T. Since the entropy is the mean Hölder exponent, we have by Shannon's inequality,

$$\begin{aligned}\log m = \text{Ent}_\Theta(\mu_A) &= \int \sum_{|x|=1} \Theta_T(x) \log \frac{1}{\Theta_T(x)}\, d\mu_A(T)\\ &< \int \sum_{|x|=1} \Theta_T(x) \log \frac{1}{\text{UNIF}_T(x)}\, d\mu_A(T) = \int \log \frac{1}{\text{UNIF}_T(\xi_1)}\, d\Theta_T(\xi)\, d\mu_A(T)\\ &= \log m + \int (g - Sg)\, d\Theta_T(\xi)\, d\mu_A(T) = \log m,\end{aligned}$$

where, as in the proof of Theorem 6.3, we have applied Lemma 6.2 to the function $g(\xi, T) := \log \tilde{W}(T)$, which satisfies $g - Sg$ is bounded below by $-\log m$. This contradiction shows that $\mu(A) = 0$, as claimed. □

In order to use this result for harmonic measure, we need to find a stationary measure for the harmonic flow rule with the above properties.

8. *Harmonic-stationary measure*

Consider the set of 'last exit points'

$$\text{Exit} := \left\{(\overleftrightarrow{x}, T) \in \text{PathsInTrees};\ x_{-1} \in x_{-\infty},\ \forall n > 0\ x_n \neq x_{-1}\right\}.$$

This is precisely the event that the path has just exited, for the last time, a horoball centered at $x_{-\infty}$. By almost sure transience of simple random walk, the set Exit has positive measure and for a.e. $(\overleftrightarrow{x}, T)$, there is an $n > 0$ such that $S^n(\overleftrightarrow{x}, T) \in \text{Exit}$. Inducing on this set will yield a key tool:

THEOREM 8.1. *There is a unique ergodic* HARM-*stationary measure* μ_{HARM} *equivalent to* GW.

Proof. The key point is that for $(\overleftrightarrow{x}, T) \in \text{Exit}$, the path of vertices in the tree T given by the first components of the sequence $\langle S^k_{\text{Exit}}(\overleftrightarrow{x}, T)\rangle_{k\geq 0}$ is a sample from $\text{HARM}_{T\setminus T(x_{-1})}$. (Recall that T is rooted at x_0.) Note that the Markov property of the induced system is a consequence of the fact that HARM is a consistent flow rule. Now since $\text{AGW}_{\text{Exit}} \ll \text{AGW}$, we have that the $(\text{SRW} \times \text{AGW})_{\text{Exit}}$-law of $T \setminus T(x_{-1})$ is absolutely continuous with respect to GW. From Proposition 5.2, it follows that the $(\text{SRW} \times \text{AGW})_{\text{Exit}}$-law of $T \setminus T(x_{-1})$ is equivalent to GW. Therefore, the induced measure-preserving system

$$(\text{Exit}, (\text{SRW} \times \text{AGW})_{\text{Exit}}, S_{\text{Exit}})$$

is isomorphic to a HARM-stationary Markov chain on trees with a stationary measure μ_{HARM} equivalent to GW.

The fact that $\text{HARM} \times \mu_{\text{HARM}}$ is ergodic follows from our general result on ergodicity, Proposition 5.2. Ergodicity implies that μ_{HARM} is the unique HARM-stationary measure absolutely continuous with respect to GW. □

Remark. Since (PathsInTrees, SRW × AGW, S) is a tower over Exit, this proves that the former is ergodic, as promised in §3.

Since increases in distance from the root come only at exit points, it is natural that the speed is also the probability of being at an exit point:

PROPOSITION 8.2. *The measure of the exit set is the speed:* $(\text{SRW} \times \text{AGW})(\text{Exit}) = \mathbf{E}[(Z_1 - 1)/(Z_1 + 1)]$.

Proof. See the third proof of the Kac lemma in Petersen [**18**, pp. 47–48]. □

The next proposition is intuitively obvious, but crucial.

PROPOSITION 8.3. *For* GW-*a.e.* T, $\text{HARM}_T \neq \text{UNIF}_T$.

For a proof, define $T_\Delta := [\Delta \bullet\!\!- T]$, where Δ is a single vertex not in T, to be thought of as representing the past. Let $\gamma(T)$ be the probability that simple random walk started at Δ never returns to Δ:

$$\gamma(T) := \text{SRW}_{T_\Delta}(\forall n > 0 \quad x_n \neq \Delta).$$

This is also equal to $\mathrm{SRW}_{[T\bullet\Delta]}(\forall n > 0\ x_n \neq \Delta)$. Let $\mathcal{C}(T)$ denote the effective conductance of T from its root to infinity when each edge has unit conductance. Then [**6**]

$$\gamma(T) = \frac{\mathcal{C}(T)}{1+\mathcal{C}(T)}.$$

It follows that $\gamma(T) = \mathcal{C}(T_\Delta)$.

Proof of Proposition 8.3. In view of the zero-one law, Proposition 5.1, we need merely show that we do not have $\mathrm{HARM}_T = \mathrm{UNIF}_T$ a.s. Now, for any tree T and any $x \in T$ with $|x| = 1$, we have

$$\mathrm{HARM}_T(x) = \frac{\gamma(T(x))}{\sum_{|y|=1} \gamma(T(y))}$$

while

$$\mathrm{UNIF}_T(x) = \frac{\tilde{W}(T(x))}{\sum_{y=1} \tilde{W}(T(y))}.$$

Therefore, if $\mathrm{HARM}_T = \mathrm{UNIF}_T$, the vector

$$\left\langle \frac{\gamma(T(x))}{\tilde{W}(T(x))} \right\rangle_{|x|=1} \tag{8.1}$$

is a multiple of the constant vector **1**. For Galton–Watson trees, each component of this vector has the same law as that of $\gamma(T)/\tilde{W}(T)$. But the independence of $T(x)$ and $T(y)$ for two distinct children x and y of the root implies that the random vector (8.1) is, in fact, constant GW-a.s. Thus, $\gamma(T)/\tilde{W}(T)$ is a constant GW-a.s. But $\gamma < 1$ and, since Z_1 is not constant, $\tilde{W}$ is obviously unbounded, a contradiction. □

Taking stock of our preceding results, we get our main theorem:

THEOREM 8.4. *The dimension of harmonic measure is* GW-*a.s. less than* $\log m$. *The Hölder exponent exists a.s. and is constant.*

Proof. The hypotheses of Theorem 7.1 are verified in Theorem 8.1 and Proposition 8.3. The constancy of the Hölder exponent follows from (5.1). □

Note that no moment assumptions (other than $m < \infty$) were used.

Remark. This theorem holds even if $p_0 > 0$: that is, given nonextinction, the subtree of a Galton–Watson tree consisting of those particles with an infinite line of descent has the law of another Galton–Watson process still with mean m [**3**, p. 49]. Theorem 8.4 applies to this subtree, while harmonic measure on the whole tree is equal to harmonic measure on the subtree.

We now sketch the derivation of the explicit expression (1.1) for the dimension d of harmonic measure. From §5, we have

$$d = \mathrm{Ent}_{\mathrm{HARM}}(\mu_{\mathrm{HARM}}) = \int \log \frac{1}{\mathrm{HARM}_T(\xi_1)}\, d\mathrm{HARM} \times \mu_{\mathrm{HARM}}(\xi, T).$$

Using the relationship between random walks and the conductance $\mathcal{C}(T)$, we may rewrite this as $d = \int \log(1 + \mathcal{C}(T))\, d\mu_{\mathrm{HARM}}(T)$. The formula (1.1) follows from this by substituting the following expression for the Radon–Nikodym derivative of μ_{HARM} with respect to GW:

$$\frac{d\mu_{\mathrm{HARM}}}{d\mathrm{GW}}(T) = \frac{1}{l}\int_{T'} \frac{1}{1+\mathcal{C}(T)^{-1}+\mathcal{C}(T')^{-1}}\, d\mathrm{GW}(T'),$$

where l is the speed of simple random walk. This expression is a consequence of our construction of μ_{HARM} by inducing; we omit the calculation.

9. *The 'hot' part of the tree*

In this section, we demonstrate Corollary 1.2, showing that, with probability arbitrarily close to one, the random walk is confined to an exponentially small part of the whole tree. In the process, we shall need to analyze several other interesting asymptotics of random walk.

We first bound the mean resistance. Note that $1/\gamma(T) = 1 + \mathcal{C}(T)^{-1} = 1 + \mathcal{R}(T)$, one more than the effective resistance $\mathcal{R}(T)$ from the root of T to infinity.

LEMMA 9.1. *We have*

$$\int \frac{d\mathrm{GW}(T)}{\gamma(T)} \le \frac{1}{1 - \mathbf{E}[1/Z_1]}$$

with equality iff Z_1 is constant.

Proof. For a flow θ on T, define

$$\mathcal{E}_n(\theta) := \sum_{0\le|x|\le n} \theta(x)^2$$

and

$$\mathcal{E}(\theta) := \lim_{n\to\infty} \mathcal{E}_n(\theta).$$

Then ([**6**], [**16**])

$$\frac{1}{\gamma(T)} = \min_{\theta(0)=1} \mathcal{E}(\theta) = \mathcal{E}(\mathrm{HARM}_T) \tag{9.1}$$

and HARM_T is the unique minimizer of $\mathcal{E}(\theta)$ among unit flows. In particular, $1/\gamma(T) \le \mathcal{E}(\mathrm{VIS}_T)$ with equality iff $\mathrm{VIS}_T = \mathrm{HARM}_T$. A proof similar to that of Proposition 8.3 shows that $\mathrm{VIS}_T \ne \mathrm{HARM}_T$ for GW-a.e. T unless Z_1 is constant.

Set $a_n := \int \mathcal{E}_n(\mathrm{VIS}_T)\, d\mathrm{GW}(T)$. We have $a_0 = 1$ and

$$a_{n+1} = \int \left\{1 + \sum_{|x|=1} \frac{1}{Z_1^2}\mathcal{E}_n(\mathrm{VIS}_{T(x)})\right\} d\mathrm{GW}(T).$$

Conditioning on Z_1 gives

$$\begin{aligned} a_{n+1} &= 1 + \sum_{k\ge1} p_k \frac{1}{k^2} \sum_{i=1}^{k} \int \mathcal{E}_n(\mathrm{VIS}_{T_i})\, d\mathrm{GW}(T_i) \\ &= 1 + \sum_{k\ge1} p_k \frac{1}{k^2} k a_n = 1 + \mathbf{E}[1/Z_1] a_n. \end{aligned}$$

Therefore, by the monotone convergence theorem,

$$\int \frac{d\mathrm{GW}(T)}{\gamma(T)} \le \int \mathcal{E}(\mathrm{VIS}_T)\, d\mathrm{GW}(T) = \lim_{n\to\infty} a_n = \sum_{k=0}^{\infty} \mathbf{E}[1/Z_1]^k = \frac{1}{1-\mathbf{E}[1/Z_1]}.$$

□

LEMMA 9.2. *For* SRW × AGW-*a.e.* $(\vec{x}, T)$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{|v-x_n|=1} \frac{1}{\gamma\Big(\big(\mathrm{MoveRoot}(T, x_n)\big)(v)\Big)} = 0, \tag{9.2}$$

whence

$$\lim_{n\to\infty} \frac{1}{n\gamma(T(x_n))} = 0, \tag{9.3}$$

and for HARM × μ_{HARM}-*a.e.* (ξ, T),

$$\lim_{k\to\infty} \frac{1}{k\gamma(T(\xi_k))} = 0. \tag{9.4}$$

Proof. By Lemma 9.1, we have

$$\int \sum_{|x|=1} \frac{1}{\gamma(T(x))}\, d\mathrm{AGW}(T) \le \frac{\mathbf{E}[Z_1+1]}{1-\mathbf{E}[1/Z_1]} < \infty. \tag{9.5}$$

Now for any n, the random variable

$$\sum_{|v-x_n|=1} \frac{1}{\gamma\Big(\big(\mathrm{MoveRoot}(T, x_n)\big)(v)\Big)}$$

has the same SRW × AGW-distribution as the AGW-distribution of $\sum_{|v|=1} 1/\gamma(T(v))$. Equation (9.2) is thus a consequence of the Borel–Cantelli lemma and (9.5). This immediately implies (9.3). Let $\tau(0) := \inf\{n \ge 0;\ S^n(\vec{x}, T) \in \mathrm{Exit}\}$ and $\tau(k) := \inf\{n > \tau(k);\ S^n(\vec{x}, T) \in \mathrm{Exit}\}$ be the sequence of exit times of $(\vec{x}, T)$. Set $\xi_k := x_{\tau(k)}$. Then we conclude from (9.3) that

$$\lim_{k\to\infty} \frac{1}{\tau(k)\gamma(T(\xi_k))} = 0$$

for SRW × AGW-a.e. $(\vec{x}, T)$. Since $\lim k/\tau(k)$ is the speed, which is positive a.s., we get (9.4) for SRW × AGW-a.e. $(\vec{x}, T)$, which is the same as for HARM × μ_{HARM}-a.e. (ξ, T), as the latter measure is induced from the former. □

LEMMA 9.3. *For* SRW × AGW-*a.e.* $(\vec{x}, T)$, *we have*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \log \deg(x_k) = \mathbf{E}[\log(Z_1+1)].$$

Proof. Apply the ergodic theorem to the function $(\vec{x}, T) \mapsto \log \deg(x_0)$. □

For the remainder of this section, let $\mathrm{VISIT}_T(x)$ be the probability that simple random walk on T visits x at some time ≥ 0 (starting from the root of T).

THEOREM 9.4. *For* SRW $\times$ AGW*-a.e.* $(\vec{x}, T)$, *we have*

$$\lim_{n\to\infty} \frac{1}{n} \log \frac{1}{\mathrm{VISIT}_T(x_n)} = ld.$$

Proof. Note that for all vertices $v \in T$,

$$\mathrm{HARM}_T(v) \le \mathrm{VISIT}_T(v) \le \mathrm{HARM}_T(v)/\gamma(T(v)).$$

[To see the right-hand inequality, for fixed $v \in T$ and for $\vec{x} \in \vec{T}$, let $\tau := \inf\{n;\ x_n = v\}$. The paths $x_0, x_1, \ldots, x_\tau, y_1, y_2, \ldots$ such that $\tau < \infty$ and $\forall k > 0\ \ y_k \in T(v)$ exit at v and have SRW_T-probability $\mathrm{VISIT}_T(v) \cdot \gamma(T(v))$. On the other hand, the set of *all* paths exiting at v have SRW_T-probability $\mathrm{HARM}_T(v)$.] Thus for all $\xi \in \partial T$,

$$\frac{1}{k} \log \frac{1}{\mathrm{HARM}_T(\xi_k)} \ge \frac{1}{k} \log \frac{1}{\mathrm{VISIT}_T(\xi_k)} \ge \frac{1}{k} \log \frac{1}{\mathrm{HARM}_T(\xi_k)} + \frac{1}{k} \log \gamma(T(\xi_k)).$$

Apply this to HARM $\times$ μ_{HARM}-a.e. (ξ, T) to get

$$\lim_{k\to\infty} \frac{1}{k} \log \frac{1}{\mathrm{VISIT}_T(\xi_k)} = d \tag{9.6}$$

by virtue of Lemma 9.2 and Theorem 8.4 on the Hölder exponent of harmonic measure.

Define $\tau(k)$ as in the proof of Lemma 9.2, so that $\lim_{k\to\infty} k/\tau(k) = l$. Then from (9.6), we have

$$\lim_{k\to\infty} \frac{1}{\tau(k)} \log \frac{1}{\mathrm{VISIT}_T(x_{\tau(k)})} = ld \quad \text{SRW} \times \text{AGW-a.s.}$$

Set $e(n) := \sup\{\tau(k);\ \tau(k) \le n\}$. Then $e(n) - n = o(n)$ a.s. and so

$$\lim_{n\to\infty} \frac{1}{n} \log \frac{1}{\mathrm{VISIT}_T(x_{e(n)})} = ld \quad \text{SRW} \times \text{AGW-a.s.} \tag{9.7}$$

and

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=e(n)}^{n} \log \deg(x_j) = 0$$

by Lemma 9.3. But every path visiting $x_{e(n)}$, i.e., $y_1, \ldots, y_m$ such that $y_m = x_{e(n)}$, can be extended to a path $y_1, \ldots, y_m, x_{e(n)+1}, \ldots, x_n$ visiting x_n, so

$$\mathrm{VISIT}_T(x_n) \ge \mathrm{VISIT}_T(x_{e(n)}) \cdot \prod_{j=e(n)}^{n-1} \frac{1}{\deg(x_j)};$$

similarly,

$$\mathrm{VISIT}_T(x_{e(n)}) \ge \mathrm{VISIT}_T(x_n) \cdot \prod_{j=e(n)+1}^{n} \frac{1}{\deg(x_j)}.$$

Hence

$$\lim_{n\to\infty} \frac{1}{n} \log \frac{1}{\mathrm{VISIT}_T(x_n)} = \lim_{n\to\infty} \frac{1}{n} \log \frac{1}{\mathrm{VISIT}_T(x_{e(n)})} \quad \text{SRW} \times \text{AGW-a.s.}$$

The theorem is now a consequence of (9.7). $\square$

Define the *Green function* of a simple random walk on a tree T as

$$G_T(v) := \int \sum_{n=0}^{\infty} \mathbf{1}_{\{v\}}(x_n)\, d\mathrm{SRW}_T(\vec{x})$$

for a vertex v of T. This is, as usual, the expected number of visits to v.

COROLLARY 9.5. *The Green function G_T satisfies*

$$\lim_{n\to\infty} \frac{1}{n} \log \frac{1}{G_T(x_n)} = ld \quad \mathrm{SRW} \times \mathrm{AGW}\text{-}a.s.$$

Proof. We have the usual formula

$$G_T(x_n) = \mathrm{VISIT}_T(x_n) G_{\mathrm{MoveRoot}(T,x_n)}(x_n).$$

Now $G_{\mathrm{MoveRoot}(T,x_n)}(x_n)$ has the same distribution as $G_T(x_0)$ and

$$G_T(x_0) = \deg(x_0)/\mathcal{C}(T). \tag{9.8}$$

To see this, let $\mathrm{VISIT}'_T(x_0)$ be the probability of returning to x_0 after time 0. Then

$$G_T(x_0) = \frac{1}{1 - \mathrm{VISIT}'_T(x_0)}$$

and

$$\mathrm{VISIT}'_T(x_0) = \sum_{|x|=1} \frac{1 - \gamma(T(x))}{\deg(x_0)} = 1 - \frac{1}{\deg(x_0)} \sum_{|x|=1} \gamma(T(x)) = 1 - \frac{\mathcal{C}(T)}{\deg x_0}.$$

Putting these together gives (9.8). Therefore, Lemma 9.2 and Lemma 9.3 give (recall that $1/\mathcal{C}(T) = 1/\gamma(T) - 1$)

$$\lim_{n\to\infty} \frac{1}{n} \log G_{\mathrm{MoveRoot}(T,x_n)}(x_n) = 0 \quad \mathrm{SRW} \times \mathrm{AGW}\text{-a.s.},$$

whence the result follows from Theorem 9.4 on VISIT. □

Remark. The same result holds for the Green function of simple random walk on GW-a.e. tree. Briefly, this is seen as follows. Since the AGW-law of $T \setminus T(x_{-1})$ is GW, we may examine the rate of decay of $G_{T\setminus T(x_{-1})}(x_n)$ for $\vec{x} \subset T \setminus T(x_{-1})$. On one side, we have $G_{T\setminus T(x_{-1})}(x_n) \geq G_T(x_n)$, to which we apply the above corollary directly. On the other side, write $G'_T(x)$ for the expected number of visits to x *before* returning to the root of T. Then $G_{T\setminus T(x_{-1})}(x_n) \leq G_{T\setminus T(x_{-1})}(x_0) G'_{T\setminus T(x_{-1})}(x_n)$ and $G'_{T\setminus T(x_{-1})}(x_n)/\deg(x_0) \leq G'_T(x_n) \leq G_T(x_n)$, to which we apply the corollary again.

It follows that Theorem 9.4 on VISIT also holds for simple random walk on GW-a.e. tree.

For our next result, we shall need the following lemma.

LEMMA 9.6. *We have*

$$\lim_{n\to\infty} \frac{1}{n} \log \frac{1}{\mathrm{HARM}_T(x_n)} = ld \quad \mathrm{SRW} \times \mathrm{AGW}\text{-}a.s.$$

Proof. As in the proof of Theorem 9.4 on VISIT, we have

$$\gamma(T(x))\mathrm{VISIT}_T(x) \le \mathrm{HARM}_T(x) \le \mathrm{VISIT}_T(x),$$

whence

$$\frac{1}{n}\log\frac{1}{\mathrm{VISIT}_T(x_n)} + \frac{1}{n}\log\frac{1}{\gamma(T(x_n))} \ge \frac{1}{n}\log\frac{1}{\mathrm{HARM}_T(x_n)} \ge \frac{1}{n}\log\frac{1}{\mathrm{VISIT}_T(x_n)}.$$

The result now follows from Lemma 9.1 and Theorem 9.4 on VISIT. □

The following was first proved by Ledrappier **[15]** in the case of random walks on free groups (i.e., nonnecessarily nearest-neighbor (group-invariant) random walks on homogeneous trees). As we mentioned in the introduction, Kaimanovich [12] has proved this for general trees.

Define $\mathrm{SRW}_T^n(x)$ as the probability that simple random walk started at the root of T is at x at time n.

THEOREM 9.7. *The Avez (asymptotic) entropy of simple random walk on Galton–Watson trees is equal to its speed times the dimension of its harmonic measure:*

$$\lim_{n\to\infty}\frac{1}{n}\log\frac{1}{\mathrm{SRW}_T^n(x_n)} = ld \tag{9.9}$$

both SRW × AGW*-a.s. and in* L^1(SRW × AGW) *and*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{x\in T}\mathrm{SRW}_T^n(x)\log\frac{1}{\mathrm{SRW}_T^n(x)} = ld \quad \text{AGW-}a.s. \tag{9.10}$$

Proof. Since $\mathrm{SRW}_T^n(x_n) \le \mathrm{VISIT}_T(x_n)$, we have

$$\liminf_{n\to\infty}\frac{1}{n}\log\frac{1}{\mathrm{SRW}_T^n(x_n)} \ge ld \quad \text{SRW} \times \text{AGW-a.s.}$$

For the other direction, fix $\alpha > ld$ and choose $\epsilon \in (0, (\alpha - ld)/2)$. Define the set of 'bad' points

$$B_n := \left\{x \in T;\ |x| \le n,\ \mathrm{SRW}_T^n(x) < e^{-(\alpha-\epsilon)n},\ \mathrm{HARM}_T(x) > e^{-(ld+\epsilon)n}\right\}.$$

Then

$$n+1 = \sum_{|x|\le n}\mathrm{HARM}_T(x) > \sum_{x\in B_n}\mathrm{SRW}_T^n(x)e^{(\alpha-ld-2\epsilon)n} = \mathrm{SRW}_T^n(B_n)e^{(\alpha-ld-2\epsilon)n},$$

whence

$$\sum_{n\ge 1}\mathrm{SRW}_T^n(B_n) < \infty$$

for every tree T. Therefore, $x_n \in B_n$ only finitely often SRW_T-a.s. In view of Lemma 9.6, it follows that

$$\mathrm{SRW}_T^n(x_n) \ge e^{-(\alpha-\epsilon)n}$$

eventually SRW_T-a.s. for AGW-a.e. tree T. By the choice inherent in α and ϵ, it follows that

$$\limsup_{n\to\infty}\frac{1}{n}\log\frac{1}{\mathrm{SRW}_T^n(x_n)} \le ld \quad \text{SRW} \times \text{AGW-a.s.},$$

which completes the proof of (9.9).

In order to deduce (9.10), set

$$f_n(\vec{x}, T) := \log \frac{1}{\mathrm{SRW}_T^n(x_n)}.$$

We shall establish that for GW-a.e. tree T, the sequence of functions $\frac{1}{n} f_n(\cdot, T)$ is dominated by an SRW_T-integrable function. Note that the left-hand side of (9.10) is simply $\int (1/n) f_n \, d\mathrm{SRW}_T$.

Now since the chance of being at x_{m+n} at time $m+n$ is at least the chance of being at x_m at time m and then going from there to x_{m+n} in n more steps, we have

$$\mathrm{SRW}_T^{m+n}(x_{m+n}) \geq \mathrm{SRW}_T^m(x_m)\mathrm{SRW}^n_{\mathrm{MoveRoot}(T,x_m)}(x_{m+n}).$$

The fact that

$$S^m f_n(\vec{x}, T) = \log \frac{1}{\mathrm{SRW}^n_{\mathrm{MoveRoot}(T,x_m)}(x_{m+n})}$$

allows us to write this inequality as $f_{m+n} \leq f_m + S^m f_n$; i.e., the sequence of functions $\langle f_n \rangle$ is subadditive. Induction then shows that

$$f_n \leq \sum_{k=0}^{n-1} S^k f_1. \tag{9.11}$$

Since $f_1(\vec{x}, T) = \log \deg x_0$, we have trivially that

$$\int f_1 \log^+ f_1 \, d\mathrm{SRW} \times \mathrm{AGW} < \infty. \tag{9.12}$$

Wiener's dominated ergodic theorem [**18**, p. 87] says that (9.12) is equivalent to

$$\int \sup_n \frac{1}{n} \sum_{k=0}^{n-1} S^k f_1 \, d\mathrm{SRW} \times \mathrm{AGW} < \infty,$$

whence from (9.11), we have $\int \sup_n f_n/n \, d\mathrm{SRW} \times \mathrm{AGW} < \infty$. Therefore, for AGW-a.e. T, we have $\int \sup_n f_n/n \, d\mathrm{SRW}_T < \infty$, whence Lebesgue's dominated convergence theorem yields (9.10) from (9.9). □

Theorem 8.4 on the dimension of harmonic measure has the following finitistic version. Recall that T^n denotes the particles of the nth generation of a tree T. Consider the hitting measure HIT_T on T^n of simple random walk, i.e., $\mathrm{HIT}_T(x)$ is the probability that simple random walk started at the root of T first hits the $|x|$th generation $T^{|x|}$ at x. (Note that HIT is not a flow rule.)

THEOREM 9.8. *If η_n denotes the first hitting place in T^n, then*

$$\lim_{n\to\infty} \frac{1}{n} \log \frac{1}{\mathrm{HIT}_T(\eta_n)} = d \tag{9.13}$$

for a.e. walk in GW-*a.e. tree T. Thus, for every $\epsilon > 0$,*

$$\lim_{n\to\infty} \mathrm{HIT}_T\left(\{x \in T^n;\ e^{-(d+\epsilon)n} \leq \mathrm{HIT}_T(x) \leq e^{-(d-\epsilon)n}\}\right) = 1 \quad \text{GW-}a.s. \tag{9.14}$$

Therefore, if $K_T^n(\epsilon)$ denotes the minimum number of points $x \in T^n$ forming a set of hitting measure at least $\epsilon \in (0, 1)$, then

$$\lim_{n\to\infty} \frac{1}{n} \log K_T^n(\epsilon) = d \quad \text{GW-}a.s. \tag{9.15}$$

Proof. Equation (9.14) is merely convergence in probability in (9.13) for GW-a.e. T and equation (9.15) follows immediately from (9.14). To prove (9.13), note that by Theorem 3.2 on speed, $\langle \eta_n \rangle$ is a subsequence of $\langle x_n \rangle$ of density l for a.e. walk in GW-a.e. tree T. Since $\mathrm{HIT}_T(\eta_n) \le \mathrm{VISIT}_T(\eta_n)$, it follows from Theorem 9.4 on VISIT that

$$\liminf_{n\to\infty} \frac{1}{n} \log \frac{1}{\mathrm{HIT}_T(\eta_n)} \ge d$$

for a.e. walk in GW-a.e. tree T. On the other hand, by Lemma 9.6,

$$\lim_{n\to\infty} \frac{1}{n} \log \frac{1}{\mathrm{HARM}_T(\eta_n)} = d \quad \text{SRW} \times \text{GW-a.s.} \tag{9.16}$$

To compare this with HIT and show that

$$\limsup_{n\to\infty} \frac{1}{n} \log \frac{1}{\mathrm{HIT}_T(\eta_n)} \le d \quad \text{SRW} \times \text{GW-a.s.}, \tag{9.17}$$

fix $\alpha > d$ and choose $\epsilon \in (0, (\alpha - d)/2)$. Define the 'bad' points

$$B_n := \left\{x \in T^n;\ \mathrm{HIT}_T(x) < e^{-(\alpha-\epsilon)n},\ \mathrm{HARM}_T(x) > e^{-(d+\epsilon)n}\right\}.$$

Thus,

$$1 = \sum_{|x|=n} \mathrm{HARM}_T(x) > \sum_{x\in B_n} \mathrm{HIT}_T(x) e^{(\alpha-d-2\epsilon)n} > \mathrm{HIT}_T(B_n) e^{(\alpha-d-2\epsilon)n},$$

whence $\sum_{n\ge1} \mathrm{HIT}_T(B_n) < \infty$. (Note that HIT_T is a probability measure on T^n and a measure on T.) Since η_n has law HIT_T on T^n, it follows from the Borel–Cantelli lemma that $\eta_n \in B_n$ only finitely often a.s. In light of (9.16), this means that $\mathrm{HIT}_T(\eta_n) \ge e^{-(\alpha-\epsilon)n}$ eventually a.s. As ϵ and α were essentially arbitrary, we may deduce (9.17). □

We now demonstrate how the walk is essentially restricted to a small subtree of the whole tree. The following is a restatement of Corollary 1.2 from the introduction.

THEOREM 9.9. *For every $\epsilon > 0$, there are subtrees $T^{(\epsilon)} \subset T$ of smaller exponential growth,*

$$\lim_{n\to\infty} \frac{1}{n} \log |T^n \cap T^{(\epsilon)}| = d, \tag{9.18}$$

such that

$$(\text{SRW} \times \text{AGW})\left\{(\vec{x}, T);\ \forall n > 0\ \ x_n \in T^{(\epsilon)}\right\} > 1 - \epsilon. \tag{9.19}$$

Proof. For $\vec{x} \in \vec{T}$, let τ_k be the time of the kth exit, i.e., $\tau_0 := 0$ and

$$\tau_{k+1} := \inf\left\{n > \tau_k;\ S^n(\vec{x}, T) \in \text{Exit}\right\}.$$

SRW × AGW-stationary for $k \geq 1$ with finite mean (the mean being the reciprocal of the measure of the set Exit, according to the Kac lemma), we have

$$\int \sum_{k\geq 1} \frac{\tau_{k+1} - \tau_k}{k^2}\, d\mathrm{SRW} \times \mathrm{AGW} < \infty,$$

whence for AGW-a.e. T,

$$\int \sum_{k\geq 1} \frac{\tau_{k+1} - \tau_k}{k^2}\, d\mathrm{SRW}_T < \infty,$$

whence

$$\int (\tau_{k+1} - \tau_k)\, d\mathrm{SRW}_T = o(k^2) \quad \text{AGW-a.s.},$$

and, finally,

$$\sum_{k=1}^{n} \int (\tau_{k+1} - \tau_k)\, d\mathrm{SRW}_T = o(n^3) \quad \text{AGW-a.s.}$$

From this, we get that the total amount of time $\geq \tau_1$ that the random walk spends in the first n generations of T has SRW_T-expectation $o(n^3)$. *A fortiori*,

$$\sum_{|x|=n} \mathrm{MVISIT}_T(x) = o(n^3) \quad \text{AGW-a.s.}$$

Now the walks which first hit $T^{|x|}$ at x and then stay in $T(x)$ are among those which visit x at time $\geq \tau_1$. Thus,

$$\mathrm{VISIT}_T(x) \geq \mathrm{MVISIT}_T(x) \geq \mathrm{HIT}_T(x)\gamma(T(x)),$$

whence by Theorems 9.4, 9.8, and Lemma 9.2, we have that

$$\lim_{n\to\infty} \frac{1}{n} \log \frac{1}{\mathrm{MVISIT}_T(x_n)} = ld \quad \mathrm{SRW} \times \mathrm{AGW}\text{-a.s.}$$

It follows from Egorov's theorem that for any $\epsilon > 0$, there is a set $A_\epsilon \subseteq \mathrm{PathsInTrees}$ of SRW × AGW-measure greater than $1 - \epsilon$ such that on A_ϵ,

$$\frac{1}{n} \log \frac{1}{\mathrm{MVISIT}_T(x_n)}$$

converges uniformly to ld and $|x_n|/n$ converges uniformly to l. Dividing these limiting relations, we see that

$$\frac{1}{|x_n|} \log \frac{1}{\mathrm{MVISIT}_T(x_n)}$$

converges uniformly to d on A_ϵ. Since $|x_n|$ tends uniformly to infinity on A_ϵ, there is a function $\epsilon_1(k)$ tending to 0 as $k \to \infty$ such that on A_ϵ,

$$\left| \frac{1}{|x_n|} \log \frac{1}{\mathrm{MVISIT}_T(x_n)} - d \right| \leq \epsilon_1(|x_n|).$$

Define

$$\overline{T}^{(\epsilon)} := \left\{ x \in T;\ \left| \frac{1}{|x|} \log \frac{1}{\mathrm{MVISIT}_T(x)} - d \right| \leq \epsilon_1(|x|) \right\} \cup \{\mathrm{root}(T)\},$$

$$T^{(\epsilon)} := \{x \in \overline{T}^{(\epsilon)};\ \forall y \leq x \quad y \in \overline{T}^{(\epsilon)}\}.$$

By definition, on A_ϵ, $x_n \in \overline{T}^{(\epsilon)}$ for all $n \geq 0$, whence $x_n \in T^{(\epsilon)}$ for all $n \geq 0$. Also, the growth of $T^{(\epsilon)}$ is bounded above by

$$\limsup_{n\to\infty} \frac{1}{n} \log |T^n \cap T^{(\epsilon)}| \leq \limsup_{n\to\infty} \frac{1}{n} \log \frac{\sum_{|x|=n} \mathrm{MVISIT}_T(x)}{\inf_{x\in T^n\cap T^{(\epsilon)}} \mathrm{MVISIT}_T(x)}$$
$$\leq \limsup_{n\to\infty} \frac{1}{n} \left(\log o(n^3) + (d + \epsilon_1(n))n\right) = d \quad \text{a.s.}$$

and below by

$$\lim_{n\to\infty} \frac{1}{n} \log K_T^n(1-\epsilon) = d \quad \text{a.s.},$$

where $K_T^n(\epsilon)$ is as in Theorem 9.8 on HIT. (For the same reason, no subtree of growth rate smaller than d exists on which the random walk can stay with positive probability.) □

Remark. The same method shows that the corollary also holds for GW trees.

10. *Open questions*

Several interesting questions remain open. A few follow.

(i) Is it true, as we conjecture, that for every consistent flow rule $\Theta \neq \mathrm{UNIF}$ a.s., the Hausdorff dimension of $\Theta_T < \log m$ a.s.? We have shown in Theorem 7.1 that this is the case provided that there exists a finite Θ-stationary measure equivalent to GW. When do such measures exist?

(ii) In the direction of comparison opposite to that of Theorem 1.1, is $\dim \mathrm{VIS}_T$ a lower bound for $\dim \mathrm{HARM}_T$?

(iii) For the general theory of §5, if a flow rule has a stationary measure equivalent to GW, must the associated Markov chain be ergodic?

(iv) It was shown in Theorem 3.2 that the speed of simple random walk on a Galton–Watson tree with mean m is strictly smaller than the speed of simple random walk on a deterministic tree where each vertex has m children ($m \in \mathbf{Z}$). Since we have also shown that simple random walk is essentially confined to a smaller subtree of growth e^d, it is natural to ask whether its speed is, in fact, smaller than $(e^d - 1)/(e^d + 1)$.

Acknowledgements. We are grateful to David Aldous for drawing our attention to the question of speed for simple random walk on Galton–Watson trees and to Robert Kaufman for conjecturing Theorem 1.1 (private communication to R. L.). We thank Itai Benjamini for numerous helpful discussions. This research was partially supported by an Alfred P. Sloan Foundation Fellowship (Lyons), NSF grant DMS-9103738 (Pemantle), and NSF grant DMS-9213595 (Peres).

REFERENCES

[1] S. Asmussen and H. Hering. *Branching Processes*. Birkhäuser: Boston, 1983.

[2] K. B. Athreya. A note on a functional equation arising in Galton–Watson branching processes. *J. Appl. Prob.* **8** (1971), 589–598.

[3] K. B. Athreya and P. Ney. *Branching Processes*. Springer: New York, 1972.

[4] P. Billingsley. *Ergodic Theory and Information*. Wiley: New York, 1965.

[5] L. Breiman. *Probability*. Addison–Wesley: Reading, MA, 1968.

[6] P. Doyle and J. L. Snell. *Random Walks and Electric Networks*. Mathematical Association of America: Washington, D.C., 1984.

[7] H. Furstenberg. Intersections of Cantor sets and transversality of semigroups. In *Problems in Analysis (Sympos. Salomon Bochner, Princeton University, 1969)*. Princeton University Press: Princeton, NJ, 1970, pp. 41–59.

[8] G. Grimmett and H. Kesten. Random electrical networks on complete graphs. *J. London Math. Soc.* **30** (1984), 171–192.

[9] B. M. Gurevic. Some existence conditions for K-decompositions for special flows. *Trans. Moscow Math. Soc.*, **17** (1967), 99–128.

[10] J. Hawkes. Trees generated by a simple branching process. *J. London Math. Soc.* **24** (1981), 373–384.

[11] A. Joffe and W. A. O'N. Waugh. Exact distributions of kin numbers in a Galton–Watson process. *J. Appl. Prob.* **19** (1982), 767–775.

[12] V. Kaimanovich. Random walks on percolation clusters. Talk at Mathematical Sciences Institute, Cornell University, April 1993.

[13] H. Kesten. Subdiffusive behavior of random walk on a random cluster. *Ann. Inst. Henri Poincaré Probab. Stat.* **22** (1986), 425–487.

[14] Y. Kifer and F. Ledrappier. Hausdorff dimension of harmonic measures on negatively curved manifolds. *Trans. Amer. Math. Soc.* **318** (1990), 685–704.

[15] F. Ledrappier. Some properties of random walks on free groups. Montreal Lectures, in preparation.

[16] R. Lyons. Random walks and percolation on trees. *Ann. Probab.* **18** (1990), 931–958.

[17] N. Makarov. On the distortion of boundary sets under conformal mappings. *Proc. London Math. Soc.* **51**(3) (1985), 369–384.

[18] K. Petersen. *Ergodic Theory*. Cambridge University Press: Cambridge, 1983.

[19] M. Rosenblatt. *Markov Processes: Structure and Asymptotic Behavior*. Springer: New York, 1971.

[20] F. Solomon. Random walks in a random environment. *Ann. Probab.* **3** (1975), 1–31.

[21] L.-S. Young. Dimension, entropy and Lyapunov exponents. *Ergod. Th. & Dynam. Sys.* **2** (1982), 109–124.