Erratum: proof of Theorem 2.3 in Pemantle&Volkov 1999

Unfortunately, there is a mistake the proof of Lemma 3.3 in the displayed formula, and this Lemma is crucial for the proof of Theorem 2.3. It is worth noting that the statement of Lemma 3.3 is however correct as it follows from the alternative proof below.

Throughout the proof we omit mentioning "a.s.", as it applies to nearly all formulas. We also use the fact that both $X_n \to \infty$ and $Y_n \to \infty$, which is very easy to show. Additionally, we make use of the simplifying notation $\mathbb{E}_n[\Delta Z_{n+1}] \equiv \mathbb{E}(Z_{n+1} - Z_n \mid \mathcal{F}_n, Z_n = (x,y))$.

Step 1. Let $Z_n = f(X_n, Y_n)$ where $f(x,y) = \frac{x+cy}{y^2} > 0$ then

$$\mathbb{E}_n[\Delta Z_{n+1}] = -\frac{cy^2 + x(y^2 - y - 1)}{y^2(y+1)^2(x+y)} < 0.$$

Consequently Z_n is a non-negative supermartingale which converges to a (possibly random) limit r, thus implying $X_n/Y_n^2 \to r \geq 0 \implies Y_n > \text{const} \cdot n^{1/2}$ for all large n.

Step 2. let $\tilde{Z}_n = \tilde{f}(X_n, Y_n)$, where $\tilde{f} = \frac{x^2+4cxy+5c^2y^2}{y^3} = \frac{x^2}{y^3} + 4cf(x,y) + \frac{c^2}{y} > 0$ then

$$\mathbb{E}_n\left[\Delta\tilde{Z}_{n+1}\right] = -\frac{c^2y^3\left[y^2+5y+5\right] + \left[2cy(y^3-4y-2) - (y+1)^3\right]x + \left[y^3-3y^2-5y-2\right]x^2}{y^3(y+1)^3(x+y)} < 0$$

for large y. Since $Y_n \to \infty$, $\tilde{Z}_n$ becomes a non-negative supermartingale which converges. Combining this with the result from Step 1, we conclude that $X_n^2/Y_n^3 \to \tilde{r} \geq 0 \implies Y_n > \text{const} \cdot n^{2/3}$ for large enough n.

Step 3. Let $\bar{Z}_n = \bar{f}(X_n, Y_n)$, where $\bar{f} = \left(\frac{x}{y} - c\log(y+1)\right)^2 + \frac{4x^2}{y^3}$ then

$$\mathbb{E}_n\left[\Delta\bar{Z}_{n+1}\right] = -\frac{Ax^2+Bx+C}{y^3(y+1)^3(x+y)} \text{ where}$$
$$A = 2y^3 + O(y^2), \quad B = 2cy^4\log(y) + O(y^4), \quad C = c^2y^5\log(y) + O(y^4\log(y)).$$

Consequently, $\bar{Z}_n$ is eventually a supermartingale which converges. This, together with the result from Step 2, yields that $[X_n/Y_n - c\log(Y_n+1)]^2$ converges. Since one-step increments of $X_n/Y_n - c\log(Y_n+1)$ are decreasing to zero, this means that $X_n/Y_n - c\log(Y_n+1) \cong \pm\sqrt{\tilde{Z}_n - 4\tilde{r} + o(1)}$ cannot change sign infinity often unless it converges to zero. In either case $X_n/Y_n - c\log(Y_n+1)$ must converge. The observation that $\log(Y_n+1) - \log(Y_n) \to 0$ concludes the proof.