RANDOM WALKS IN VARYING DIMENSIONS

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Abstract

We establish recurrence criteria for sums of independent random variables which take values in Euclidean lattices of varying dimension. In particular, we describe transient inhomogenous random walks in the plane which interlace two symmetric step distributions of bounded support.

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1 Introduction

As everyone knows, and Polya proved in 1921, simple random walk is recurrent in one and two dimensions and transient in three or more dimensions. Also widely known is that the transition between recurrence and transience occurs precisely at dimension 2, rather than at some fractional dimension in the interval $(2, 3)$. This is not a precise statement, but one common interpretation is:

> When the dimension parameter d in formulae for quantities such as Green's function is taken to be continuous, then qualitatively different behavior is observed in the two regimes $d \leq 2$ and $d > 2$.

One can interpret the phase transition at dimension 2 probabilistically by considering simple random walk on subgraphs of the three-dimensional lattice; this was done by T. Lyons (1983), who showed that a "slight fattening of a quadrant in $\mathbf{Z}^2$" suffices to obtain transience. Another approach, taken here, is to construct a random walk on a d-dimensional lattice which only occasionally, at some fixed times, moves in directions outside a certain subspace of smaller dimension. The recurrence/transience of such a walk depends (as one would expect) on the frequency of the fully d-dimensional steps, but the location of the phase boundary is not what one would predict from a look at the Green function; in other words, the usual "Borel-Cantelli" criterion for transience fails. A related model, in which the exceptional moves are taken at random times, was analyzed by Scott (1990).

Let F_3 be a truly three dimensional distribution with mean zero and finite variance on the lattice $\mathbf{Z}^3$, and denote by F_2 the projection of this distribution to the x-y plane. Given an increasing sequence of positive integers $\{a_n\}$, we consider the inhomogeneous random walk $\{S_k\}$ whose independent increments $S_k - S_{k-1}$ have distribution F_3 if $k \in \{a_n\}$ and distribution F_2 otherwise. Theorem 2.4 shows that the process $\{S_k\}$ is *recurrent* if

$$a_n \approx \exp(\exp(n^{1/2}))$$

but *transient* if $a_n \approx \exp(\exp(n^\theta))$ for any $\theta \in (0, 1/2)$. Here *recurrence* means that the number of k for which $S_k = \mathbf{0}$ is almost surely infinite, and *transience* means that this number is almost surely finite. (These alternatives are exhaustive, cf. Lemma 2.1.) An easy calculation shows that the *expected* number of visits to the origin by $\{S_k\}$ is infinite when $0 < \theta < 1/2$ as well as when $\theta = 1/2$.

We also consider variants in other dimensions. For instance, there exists a recurrent random walk which interlaces two-dimensional, four-dimensional and six-dimensional steps (but the four-dimensional steps are indispensible here; see Corollary 2.3). Conversely, there is a transient process obtained by alternating blocks of one-dimensional and two-dimensional random walk steps, where both increment distributions are symmetric and have bounded support (Proposition 4.1).

2 Statement of results

To give meaning to the terms "recurrent" and "transient", we first prove a "folklore" lemma which implies a 0-1 law for recurrence of RWVD.

Lemma 2.1 *Let $\{F_j : 1 \le j \le l\}$ be distributions on the abelian group Y and let*

$$(n(1), n(2), \ldots) \in \{1, 2, \ldots, l\}^{\mathbf{Z}^+}$$

be any sequence in which each value $1, \ldots, l$ occurs infinitely often. Let $\{X_k\}$ be independent random variables with corresponding distributions $\{F_{n(k)}\}$. Then any tail event for the sequence of partial sums $S_N = \sum_{k=1}^N X_k$ has probability 0 or 1.

Proof: If $l = 1$, this is a consequence of the Hewitt-Savage 0-1 law. If $l > 1$, assume for induction that the result is true for smaller values of l, and let $\mathcal{F}_{l-1}$ denote the σ-field generated by

$$\{X_k : n(k) \le l - 1\}. \tag{1}$$

Conditional on $\mathcal{F}_{l-1}$, the event B is exchangeable in the remaining variables $\{X_k : n(k) = l\}$; since these variables are identically distributed, the Hewitt-Savage 0-1 law shows that $\mathbf{P}[B \,|\, \mathcal{F}_{l-1}] \in \{0,1\}$ almost surely. The set $\tilde{B} := \{\mathbf{P}[B \,|\, \mathcal{F}_{l-1}] = 1\}$ is $\mathcal{F}_{l-1}$-measurable, and it is a tail event for the partial sums of the variables in (1). By induction, $\mathbf{P}[\tilde{B}] \in \{0,1\}$, which shows that $\mathbf{P}[B] \in \{0,1\}$. □

Definition: Let $d < D$ be positive integers and let $\{a_n\}$ be an increasing sequence of integers. A $\mathbf{Z}^d$*in* $\mathbf{Z}^D$ *random walk in varying dimension* is a process $\{S_k\}$ in $\mathbf{Z}^D$ with independent increments $S_k - S_{k-1}$ distributed according to a truly D-dimensional distribution F_D if $k \in \{a_n : n \geq 1\}$, and according to the projection F_d of F_D to the first d coordinates if $k \notin \{a_n\}$. We assume that:

$$F_D \text{ makes the } D \text{ coordinates independent, and} \qquad (2)$$

$$F_D \text{ has mean zero and finite variance.}$$

We first state an easy qualitative proposition which is sharpened in Theorem 2.4 below.

Proposition 2.2 *Fix distributions F_2 and F_3 satisfying (2). If the sequence $\{a_n\}$ grow sufficiently fast, then the resulting $\mathbf{Z}^2$ in $\mathbf{Z}^3$ random walk in varying dimension is recurrent.*

Proof: Denote by π_z projection to the z-axis and by π_{xy} the projection map to the x-y plane. Since $\{\pi_{xy}(S_k)\}$ is a recurrent planar random walk, we may select a_n inductively to satisfy

$$P[\exists k \in (a_n, a_{n+1}] : \pi_{xy}(S_k) = 0] \geq 1/2. \qquad (3)$$

The process $\{\pi_z(S_{a_n})\}$ is a recurrent one-dimensional random walk, so there is almost surely a random infinite sequence $N(1), N(2), \ldots$ for which $\pi_z(S_{a_{N(j)}}) = 0$ for all $j \geq 1$. Now condition on the sequence $\{\pi_z(S_k)\}$, and use independence of $\{\pi_{xy}(S_k)\}$ and $\{\pi_z(S_k)\}$ to conclude the following:
With probability at least 1/2, there are infinitely many j for which there exists a time $k \in (a_{N(j)}, a_{N(j)+1}]$ such that $S_k = 0$. By the zero-one law, this proves recurrence. □

The argument above is quite general and extends in an obvious way to the product of two recurrent Markov chains. Iterating this argument yields the next corollary.

Corollary 2.3 *If* $d_1 < d_2 < \cdots < d_N$ *and*

$$\max\{d_{j+1} - d_j : 1 \leq j \leq N-1\} \leq 2\,, \tag{4}$$

then there exists a recurrent process $\{S_k\}$ *with independent increments, which interlaces infinitely many* d_j*-dimensional steps for each* j*. More precisely,* $S_{k+1} - S_k$ *has a truly* $D(k)$*-dimensional distribution for each* k*, and the sequence* $\{D(k)\}$ *takes on only the values* $d_1, \ldots, d_N$*, each one infinitely often. If (4) is violated then any such process* $\{S_k\}$ *must be transient.*

(To justify the last assertion, observe that the projection of $\{S_k\}$ to the $d_j + 1, ..., d_{j+1}$ coordinates performs a random walk in $d_{j+1} - d_j$ dimensional space (up to a time change), which is necessarily transient if $d_{j+1} - d_j > 2$.)

Next we state the quantitative version, Theorem 2.4, of Proposition 2.2. This will be proved in detail. We also state similar theorems for RWVD in 2 and 4 dimensions and RWVD in 1 and 3 dimensions and give the necessary modifications to the proof of Theorem 2.4. Define

$$\phi(n) = \frac{\log(a_{n+1}/a_n)}{\log a_{n+1}}; \tag{5}$$

$$\phi_1(n) = \sqrt{\frac{a_{n+1} - a_n}{a_{n+1}}}. \tag{6}$$

Theorem 2.4 *For the* $\mathbf{Z}^2$ *in* $\mathbf{Z}^3$ *random walk in varying dimension* $\{S_k\}$ *considered in Proposition 2.2, we have:*

(*i*) *If* $\sum_n n^{-1/2}\phi(n) < \infty$ *then* $\{S_k\}$ *is transient.*

(*ii*) *If* $\sum_n n^{-1/2}\phi(n) = \infty$ *and the sequence* $\{\phi(n)\}$ *is nonincreasing, then* $\{S_k\}$ *is recurrent.*

Remarks:

1. In particular, S_k is recurrent for $a_n = \exp(e^{n^{1/2}})$ and transient for $a_n = \exp(e^{n^\theta})$ when $\theta < 1/2$.

2. The monotonicity assumption in (ii) is far from necessary, and may be weakened in several ways. If ϕ is bounded below, $\{S_k\}$ is recurrent and the proof is easier. If

$$\sup_{m>n} \phi(m)/\phi(n) < \infty, \tag{7}$$

then $\{S_k\}$ is still recurrent when $\sum_n n^{-1/2}\phi(n) = \infty$. On the other hand, the hypothesis may not be discarded completely. To see this, let $A \subseteq \{1, 2, 3, \ldots\}$ be a set of times such that a simple random walk $\{Y_n\}$ on $\mathbf{Z}^1$ will have $Y_n = 0$ for only finitely many $n \in A$ almost surely, even though $\sum_{n\in A} \mathbf{P}[Y_n = 0] = \infty$ (cf. Example 1 in Section 2 of Benjamini, Pemantle and Peres (1994).) Define the sequence $\{a_n\}$ by $a_{n+1} = 2a_n - 1$ if $n \notin A$ and $a_{n+1} = a_n^2$ if $n \in A$. Each $n \in A$ satisfies $\phi(n) = 1/2$, so the sum in (ii) is infinite by the assumption $\sum_{n\in A} \mathbf{P}[Y_n = 0] = \infty$ and the local CLT. But with probability one, S_{a_n} is in the x-y plane for only finitely many $n \in A$, while by Lemma 3.3 , $\{S_k\}$ visits the origin finitely often in time intervals $[a_n, a_{n+1}]$ for $n \notin A$.

Theorem 2.5 *For any "$\mathbf{Z}^2$ in $\mathbf{Z}^4$" random walk in varying dimension $\{S_k\}$, where the increment distributions satisfy (2) :*
(i) If $\sum_n n^{-1}\phi(n) < \infty$ then $\{S_k\}$ is transient.

(ii) If $\sum_n n^{-1}\phi(n) = \infty$ and the sequence $\{\phi(n)\}$ is nonincreasing, then $\{S_k\}$ is recurrent.

(In particular, this inhomogenous walk is recurrent if $a_n = \exp(e^n)$, and transient if $a_n = \exp(e^{n^\theta})$ with $\theta < 1$.)

Theorem 2.6 *For any "$\mathbf{Z}^1$ in $\mathbf{Z}^3$" random walk in varying dimension, if the increment distributions satisfy (2) then*
(i) If $\sum_n n^{-1}\phi_1(n) < \infty$ then $\{S_k\}$ is transient.

(ii) If $\sum_n n^{-1}\phi_1(n) = \infty$ *and the sequence* $\{\phi_1(n)\}$ *is nonincreasing, then* $\{S_k\}$ *is recurrent.*

(In particular, this inhomogenous walk is recurrent if $a_n = \exp(n/\log^2(n))$, and transient if the exponent 2 in the last formula is replaced by any larger exponent.)

3 Proofs

The proofs begin with some elementary estimates on the probability of returning to the origin in a specified time interval.

Lemma 3.1 *Let* $\{S_k\}$ *be the partial sums of an aperiodic random walk on the one-dimensional integer lattice with mean zero and finite variance. Then there exist constants* c_1 *and* c_2 *depending only on the distribution of the increments, such that for sufficiently large integers* $0 < a < b$*,*

$$c_1\sqrt{\frac{b-a}{b}} \le \mathbf{P}[S_k = 0 \text{ for some } a \le k < b] \le c_2\sqrt{\frac{b-a}{b}}. \tag{8}$$

Lemma 3.2 *Let* $\{S_k\}$ *be the partial sums of an aperiodic random walk on the two-dimensional integer lattice with mean zero and finite variance. Then there exist constants* c_1 *and* c_2 *depending only on the distribution of the increments, such that for sufficiently large integers* $0 < a < b$*,*

$$c_1\frac{\log(b/a)}{\log b} \le \mathbf{P}[S_k = 0 \text{ for some } a \le k < b], \tag{9}$$

and, in the case that $b > 2a$*,*

$$\mathbf{P}[S_k = 0 \text{ for some } a \le k < b] \le c_2\frac{\log(b/a)}{\log b}. \tag{10}$$

Proof of Lemma 3.1: The Local Central Limit Theorem (cf. Spitzer (1964)) gives

$$\mathbf{P}[S_k = 0] = \frac{c}{\sqrt{k}}(1 + o(1)) \tag{11}$$

for some constant c as $k \to \infty$. Write G for the event that $S_k = 0$ for some $k \in [a, b-1]$. Then

$$\mathbf{P}[G] = \frac{\mathbf{E}\#\{k : a \le k < b \text{ and } S_k = 0\}}{\mathbf{E}\left(\#\{k : a \le k < b \text{ and } S_k = 0\} \,|\, G\right)} . \tag{12}$$

Using (11) shows that as $a \to \infty$, the numerator is

$$(c + o(1))(\sqrt{b} - \sqrt{a}) \ge (c + o(1))(b - a)/(2\sqrt{b}).$$

To get an upper bound on the denominator in (12), let $T = \min\{a \le k < b : S_k = 0\}$ be the (possibly infinite) hitting time and condition on T to get

$$\begin{aligned} & \mathbf{E}\left(\#\{k : a \le k < b \text{ and } S_k = 0\} \,|\, G\right) \\ \le\ & \sup_{a \le t < b} \mathbf{E}\left(\#\{k : a \le k < b \text{ and } S_k = 0\} \,|\, T = t\right) \\ =\ & \mathbf{E}\#\{k : 0 \le k < b - a \text{ and } S_k = 0\} \\ \le\ & C\sqrt{b-a}, \end{aligned}$$

for some constant C and all positive integers $a < b$. Thus

$$\mathbf{P}[G] \ge \frac{(c + o(1))(b-a)/(2\sqrt{b})}{C\sqrt{b-a}} \ge c_1 \sqrt{\frac{b-a}{b}} ,$$

for some constant c_1 and all sufficiently large a.

To prove the second inequality, recompute

$$\mathbf{P}[G] = \frac{\mathbf{E}\#\{k : a \le k < 2b - a \text{ and } S_k = 0\}}{\mathbf{E}\left(\#\{k : a \le k < 2b - a \text{ and } S_k = 0\} \,|\, G\right)} .$$

The numerator is now $(c + o(1))(\sqrt{2b - a} - \sqrt{a})$ which is at most $(2c + o(1))(b - a)/\sqrt{a}$. The denominator is at least

$$\inf_{a \le t < b} \mathbf{E}\left(\#\{k : a \le k < 2b - a \text{ and } S_k = 0\} \,|\, T = t\right) \ge (c + o(1))\sqrt{b-a}.$$

Taking the quotient proves the second inequality in the case $b \le 2a$; the case $b > 2a$ is trivial.
$\square$

PROOF OF LEMMA 3.2: The Local CLT now gives

$$\mathbf{P}[S_k = 0] = \frac{c}{k}(1 + o(1)).$$

Defining G and T as in the preceding proof, it again follows that

$$\mathbf{P}[G] = \frac{\mathbf{E}\#\{k : a \leq k < b \text{ and } S_k = 0\}}{\mathbf{E}\left(\#\{k : a \leq k < b \text{ and } S_k = 0\} \,|\, G\right)} .$$

Using the Local CLT and conditioning on T as before, shows this to be at least

$$(1 + o(1))\frac{\log b - \log a}{\log(b - a)},$$

which proves (9). On the other hand, using the alternate expression

$$\mathbf{P}[G] = \frac{\mathbf{E}\#\{k : a \leq k < 2b - a \text{ and } S_k = 0\}}{\mathbf{E}\left(\#\{k : a \leq k < 2b - a \text{ and } S_k = 0\} \,|\, G\right)}$$

gives

$$\mathbf{P}[G] \leq (1 + o(1))\frac{\log(2b - a) - \log a}{\log(b - a)}$$

which is at most $c_2 \frac{\log(b/a)}{\log b}$ as long as $b > 2a$, proving (10). □

PROOF OF THEOREM 2.4: The second moment method will be used. It is possible to get a good second moment estimate on the number of intervals $[a_n, a_{n+1} - 1]$ that contain a return to zero, but only after pruning the short intervals. We must first prove:

Lemma 3.3 *The number of k for which $S_k = 0$ and $a_n \leq k < a_{n+1}$ for some n satisfying $a_{n+1} < 2a_n$ is almost surely finite.*

PROOF: Let $m(1), m(2), \ldots$ enumerate the integers m for which $[2^{m-1}, 2^{m+1} - 1]$ contains some a_n. It suffices to show that finitely many intervals of the form $[2^{m(j)-1}, 2^{m(j)+1} - 1]$ contain values of k for which $S_k = 0$, since these cover all intervals of the form $[a_n, a_{n+1} - 1]$ satisfying $a_{n+1} < 2a_n$.

Fix j and let $n(j)$ denote the least n such that $a_n \in [2^{m(j)-1}, 2^{m(j)+1} - 1]$. By the independence of the coordinates of $\{S_k\}$, and by the Local CLT in one and two dimensions, one sees that for each $k \in [2^{m(j)-1}, 2^{m(j)+1}-1]$, the probability of $S_k = 0$ is at most $c/(k\sqrt{n(j)})$. Summing this over all k in the interval gives

$$\mathbf{P}[S_k = 0 \text{ for some } 2^{m(j)-1} \leq k < 2^{m(j)+1}] \leq \frac{c}{\sqrt{n(j)}} .$$

Another way to get an upper bound on this is to see that the probability of this event is at most the product of the probability that the walk returns to the x-y plane during the interval with the probability that it returns to the z-axis during the interval. Lemmas 3.1 and 3.2 applied to the intervals $[n(j), n(j+1)-1]$ and $[2^{m(j)-1}, 2^{m(j)+1}-1]$ respectively show this product to be at most

$$c\sqrt{\frac{n(j+1)-n(j)}{n(j+1)}}\,\frac{1}{m(j)}.$$

Since $m(j) \geq j$, these two upper bounds may be written as

$$\mathbf{P}[S_k = 0 \text{ for some } 2^{m(j)-1} \leq k < 2^{m(j)+1}] \leq c \min\left(\frac{1}{\sqrt{n(j)}}\,,\,\sqrt{\frac{n(j+1)-n(j)}{n(j+1)}}\,\frac{1}{m(j)}\right).$$

Lemma 3.4 with $b_j = n(j+1) - n(j)$ now shows that these probabilities are summable in j, and Borel-Cantelli finishes the proof. For continuity's sake, the lemma (which is a fact about deterministic integer sequences) is given at the end of the section. □

Proof of Theorem 2.4 (continued): Let $I_n = 1$ if $a_{n+1} \geq 2a_n$ and $S_k = 0$ for some $k \in [a_n, a_{n+1}-1]$, and let $I_n = 0$ otherwise. Part (i) of the theorem is just Borel-Cantelli: the hypothesis in (i) and the estimate (10) in the case $b > 2a$ together imply that $\mathbf{E}I_n \leq n^{-1/2}\phi(n)$ is summable. Thus the random walk visits zero finitely often in intervals $[a_n, a_{n+1}-1]$ for which $a_{n+1} \geq 2a_n$; this, together with Lemma 3.3, proves (i).

To prove (ii), it suffices, by the 0-1 law (Lemma 2.1), to show that the probability of S_k returning to the origin infinitely often is positive. This follows from the two assertions: $\sum_{n=1}^\infty \mathbf{E}I_n = \infty$, and $\mathbf{E}(\sum_{n=1}^M I_n)^2 \leq c(\sum_{n=1}^M \mathbf{E}I_n)^2$ (cf. Kochen and Stone (1964)).

Seeing that $\sum_{n=1}^\infty \mathbf{E}I_n = \infty$ is easy, since $\sum_{n=1}^\infty n^{-1/2}\phi(n)$ is assumed to be infinite; the difference between the two sums is

$$\sum_{n=1}^{\infty} \frac{\phi(n)}{n^{1/2}} \mathbf{1}_{\{a_{n+1} < 2a_n\}}.$$

Letting $m(j)$ enumerate those integers m such that $2^{m-1} \leq a_n < 2^{m+1}$ for some n, the difference comes out to at most

$$\sum_{j=1}^{\infty} \frac{\log 4}{j^{1/2} \log(2^{m(j)-1})}$$

which is summable since $m(j) \geq j$. For use below, let C_0 denote this finite sum.

For the second moment computation, take M large enough so that $\sum_{n=1}^{M} \mathbf{E} I_n \geq 1$. The expected square of $\sum_{n=1}^{M} I_n$ may be expanded into terms $\mathbf{E} I_n I_r$, which we now bound. Of course, $\mathbf{E} I_n I_r = 0$ if $a_{n+1} < 2a_n$ or $a_{r+1} < 2a_r$. Assume now that $\mathbf{E} I_n I_r > 0$ and that $n < r$ are not consecutive among numbers k with $\mathbf{E} I_k > 0$. Then

$$\begin{aligned} \mathbf{E} I_n I_r &= (\mathbf{E} I_n) \mathbf{E}(I_r \,|\, I_n = 1) \\ &\leq c \frac{\phi(n)}{n^{1/2}} \max_{t < a_{n+1}} \mathbf{E}(I_r \,|\, S_t = 0). \end{aligned}$$

The inequality (10) may be used with $b = a_{r+1} - t$ and $a = a_r - t$ to see that

$$\mathbf{E} I_n I_r \leq c \max_{t < a_{n+1}} \frac{\phi(n)}{n^{1/2}} \frac{\log(a_{r+1} - t) - \log(a_r - t)}{\sqrt{r - n} \log(a_{r+1} - t)}.$$

As t increases, the numerator of the last term increases and the denominator decreases, and since $t < a_{n+1} < a_r/2$, this yields

$$\begin{aligned} \mathbf{E} I_n I_r &\leq c \frac{\phi(n)}{n^{1/2}} \frac{\log(a_{r+1} - a_r/2) - \log(a_r - a_r/2)}{\sqrt{r - n} \log(a_{r+1} - a_r/2)} \\ &\leq c' \frac{\phi(n)}{n^{1/2}} \frac{\phi(r)}{(r-n)^{1/2}} \leq c' \frac{\phi(n)}{n^{1/2}} \frac{\phi(r-n)}{(r-n)^{1/2}}, \end{aligned}$$

since $\phi(n)$ is assumed to be monotone decreasing.

Using the bound $\mathbf{E} I_n I_r \leq \mathbf{E} I_n$ for consecutive or identical nonzero terms, and changing variables $l = r - n$, yields

$$\mathbf{E}(\sum_{n=1}^{M} I_n)^2 \leq 5c'(C_0 + \sum_{n=1}^{M} \mathbf{E} I_n)^2.$$

By our choice of M, the second moment is thus bounded by a constant multiple of the square of the first moment, completing the proof of the theorem. □

Proof of Theorem 2.5 : This is completely analogous to the previous proof. It suffices to change $n^{-1/2}\phi(n)$ to $n^{-1}\phi(n)$ everywhere. The analogue of Lemma 3.3 goes through without alteration, since $n^{-1} < n^{-1/2}$, and the second moment estimate is easily completed.

Proof of Theorem 2.6 : Here we cannot use a version of Lemma 3.3 - indeed the critical growth rate of a_n is subexponential.

(i) Write I_n for the indicator function of the existence of a $k \in [a_n, a_{n+1} - 1]$ for which $S_k = 0$. Then

$$c_1 \frac{\phi_1(n)}{n} \le \mathbf{E} I_n \le c_2 \frac{\phi_1(n)}{n}$$

for all large n by Lemma 3.1. Invoking the Borel-Cantelli lemma proves part (i).

(ii) The assumption of this part ensures that $\sum_{n=1}^{M} \mathbf{E} I_n \to \infty$ as $M \to \infty$. To estimate the second moment, we consider seperately the contributions of "long" and "short" intervals:

$$\mathbf{E}(\sum_{n=1}^{M} I_n)^2 = \sum_{n=1}^{M} \left(\mathbf{E} I_n + 2 \sum_{r=n+1}^{M} \mathbf{E} I_n I_r \mathbf{1}_{\{a_{r+1} \ge 2a_{n+1}\}} + 2 \sum_{r=n+1}^{M} \mathbf{E} I_n I_r \mathbf{1}_{\{a_{r+1} < 2a_{n+1}\}} \right) . \quad (13)$$

Each summand in the middle term of (13) may be estimated using Lemma 3.1 :

$$\mathbf{E} I_n I_r \le c \frac{\phi_1(n)}{n} \frac{\sqrt{(a_{r+1} - a_r)/(a_{r+1} - a_{n+1})}}{r - n} \le 2c \frac{\phi_1(n)}{n} \frac{\phi_1(r)}{r - n} , \quad (14)$$

provided that $a_{r+1} \ge 2a_{n+1}$. Monotonicity of ϕ_1 allows us to bound $\phi_1(r)$ from above by $\phi_1(r - n)$. This implies that the middle term of (13) is at most

$$4c \sum_{n=1}^{M} \sum_{r=n+1}^{M} \frac{\phi_1(n)}{n} \frac{\phi_1(r-n)}{r-n} \le 4c (\sum_{n=1}^{M} \mathbf{E} I_n)^2.$$

To bound the last term of (13), fix n and write

$$\sum_{r=n+1}^{M} \mathbf{E} I_n I_r \mathbf{1}_{\{a_{r+1} < 2a_{n+1}\}} \le c \frac{\phi_1(n)}{n} \left(\sum_{r=n+1}^{M} \frac{c}{r-n} \sqrt{\frac{a_{r+1} - a_r}{a_{r+1} - a_{n+1}}} \mathbf{1}_{\{a_{r+1} < 2a_{n+1}\}} \right) . \quad (15)$$

Since $\phi_1(r)^2 = \frac{a_{r+1} - a_r}{a_{r+1}}$ is nonincreasing, and since $a_{r+1} < 2a_n$ for all nonzero summands in (15), it follows that each difference $a_{r+1} - a_r$ in these summands is at most twice greater than any preceding difference $a_{r'+1} - a_{r'}$ where $r' < r$. Thus

$$\frac{a_{r+1} - a_r}{a_{r+1} - a_{n+1}} \leq \frac{2}{r-n}$$

for each r under consideration, and hence (denoting $l = r - n$ in the last step) :

$$\sum_{r=n+1}^{M} \frac{1}{r-n} \sqrt{\frac{a_{r+1} - a_r}{a_{r+1} - a_{n+1}}} \mathbf{1}_{\{a_{r+1} < 2a_{n+1}\}} \leq c_0 + \sum_{l=1}^{\infty} \frac{c}{l^{3/2}} < c'.$$

It now follows from (15) and Lemma 3.1 that the last term of (13) is at most a constant multiple of the sum $\sum_{n=1}^{M} \mathbf{E} I_n$. Taking M large enough so that this sum is greater than 1, the second moment bound is established. □

Now that the proofs of Theorems 2.4, 2.5 and 2.6 are complete, it remains to prove the lemma that was used in the first proof.

Lemma 3.4 *Let $b_1, b_2, \ldots$ be any sequence of positive integers and let $B_n = \sum_{k=1}^{n} b_k$ be the partial sums. Then*

$$\sum_{n=1}^{\infty} \min\left(\frac{1}{\sqrt{B_{n-1}}}, \frac{1}{n}\sqrt{\frac{b_n}{B_n}}\right) < \infty.$$

Proof: Breaking down the terms according to whether $B_{n-1} \geq n^3$ gives

$$\begin{aligned} & \sum_{n=1}^{\infty} \min\left(\frac{1}{\sqrt{B_{n-1}}}, \frac{1}{n}\sqrt{\frac{b_n}{B_n}}\right) \\ \leq \quad & \sum_{n=1}^{\infty} \left(\mathbf{1}_{\{B_{n-1} \geq n^3\}} \frac{1}{\sqrt{B_n}} + \mathbf{1}_{\{B_{n-1} < n^3\}} \frac{1}{n}\sqrt{\frac{b_n}{B_n}}\right) \\ \leq \quad & C + \sum_{n=1}^{\infty} \mathbf{1}_{\{B_{n-1} < n^3\}} \frac{1}{n}\sqrt{\frac{b_n}{B_n}}. \end{aligned}$$

To show that the second term is finite, we estimate the sum over intervals $[M, 2M]$. Let $\delta_n = \sqrt{b_n / B_n}$, and, assuming the sum to be nonzero, let $T = \max\{j \leq 2M : B_j \leq (2M)^3\}$.

Then

$$\sum_{n=M}^{2M} \frac{1}{n}\sqrt{\frac{b_n}{B_n}}\mathbf{1}_{\{B_{n-1}<n^3\}}$$

$$\leq \frac{1}{M}\sum_{n=M}^{2M} \delta_n \mathbf{1}_{\{B_{n-1}<(2M)^3\}}$$

$$= \frac{1}{M}\sum_{n=M}^{T+1} \delta_n .$$

Since $1+\delta_n^2 = B_{n+1}/B_n$ and each $\delta_n < 1$, we have

$$\prod_{n=M}^{T}(1+\delta_n^2) \leq 2B_T/B_M \leq 16M^3.$$

Taking logs gives

$$\sum_{n=M}^{T}\delta_n^2 \leq \sum_{n=M}^{T}(\log 2)^{-1}\log(1+\delta_n^2) \leq c\log M$$

for $c > 3/\log 2$ and large M. By Cauchy-Schwarz,

$$\sum_{n=M}^{T+1}\delta_n \leq \sqrt{M+2}\sqrt{\sum_{n=M}^{T}\delta_n^2} \leq c\sqrt{M\log M}.$$

Thus

$$\sum_{n=M}^{2M}\mathbf{1}_{\{B_{n-1}<n^3\}}\frac{1}{n}\sqrt{\frac{b_n}{B_n}} \leq c\sqrt{\frac{\log M}{M}} .$$

This is summable as M varies over powers of 2, which proves the lemma. □

4 A transient inhomogenous random walk with fair bounded steps in one and two dimensions

Let $\{a_n\}$ and $\{b_n\}$ be sequences of positive integers, and let $\{S_k\}$ be an inhomogenous random walk in $\mathbf{Z}^2$ which does a_1 steps of a random walk uniform over all four diagonal neighbors, then

b_1 steps of a horizontal simple random walk, then a_2 diagonal steps, then b_2 horizontal steps, and so on.

Proposition 4.1 *If the sequences* $\{a_n\}$ *and* $\{b_n\}$ *satisfy conditions (16) and (17) below then* $\{S_k\}$ *is transient. (In particular these conditions hold for* $a_n = n^2$ *and* $b_n = 2^n$ *.)*

PROOF: Denote $A_n = \sum_{j=1}^n a_j$ and $B_n = \sum_{j=1}^n b_j$. Let (X_k, Y_k) denote the coordinates of S_k, and observe that the two sequences $\{X_k\}$ and $\{Y_k\}$ are independent of each other. The first sequence is a simple random walk on the integers, while the increments $Y_{k+1} - Y_k$ are simple RW steps if $k \in [A_n + B_n, A_{n+1} + B_n)$ for some n; for k in the complementary time intervals, $Y_{k+1} = Y_k$.

If $A_n + B_{n-1} \le k < A_n + B_n$ then $Y_k = Y_{A_n+B_{n-1}}$ is the position of a simple random walk after A_n steps, so Y_k vanishes with probability $1/\sqrt{A_n}$, up to a bounded factor. The probability that $X_k = 0$ for some $k \in [A_n + B_{n-1}, A_n + B_n)$ may be estimated by $C[\sqrt{A_n + B_n} - \sqrt{A_n + B_{n-1}}]/\sqrt{b_n}$ (using Lemma 3.1). Thus the summability condition

$$\sum_n \frac{1}{\sqrt{A_n}} \frac{\sqrt{A_n + B_n} - \sqrt{A_n + B_{n-1}}}{\sqrt{b_n}} < \infty \tag{16}$$

rules out infinitely many returns of S_k to the origin in intervals $k \in [A_n + B_{n-1}, A_n + B_n)$.

The only other way $\{S_k\}$ can return to the origin is during intervals of the type $[A_n + B_n, A_{n+1} + B_n)$. The analogous calculation yields the condition

$$\sum_{n=1}^{\infty} \frac{1}{\log a_n} \sum_{j=1}^{a_n} (B_n + A_n + j)^{-1/2} (A_n + j)^{-1/2} < \infty \tag{17}$$

which is sufficient to ensure finitely many returns of S_k to the origin during such intervals. □

To see a concrete example where Proposition 4.1 applies, with $\{a_n\}$ and $\{b_n\}$ growing almost as slowly as is allowed, let $a_n = (\log n)^{2+\epsilon}$ and $b_n = (\log n)^{4+\epsilon}$ for some $\epsilon > 0$.

Remark: Durrett, Kesten and Lawler (1991) analyze a random walk in one dimension that interlaces several increment distributions all having mean zero. In that setting, distributions without second moments are necessary in order to obtain transience.

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