I: INTRO

Where problem comes from: [Handout = 4-5 pix of random tiling objects]. All have simple GF's = P/Q.
E.g. $\mathbb{P}((i,j)$ tile in k-◇ goes $N)$, $F = \sum \mathbb{P}(i,j,k)\, x^i y^j z^k$
$= 1/(1-(x+x^{-1}+y+y^{-1})\, z/2 + z^2)$.

Problem: given $F(z) = \dfrac{P(z)}{Q(z)} = \sum_r a_r z^r$ in d variables
Find asym. formulae for a_r as $r \to \infty$, $r/|r| \to \hat{r}$.

First 2/3 of talk, focus on case where $\mathcal{V} := \mathcal{V}_Q := \{z : Q(z) = 0\}$ is smooth hypersurface in $\mathbb{C}^d$. Some accessible open problems I'd like to see solved, putting out for collaboration.
Last part, if time, a phenomenon occurring due to topological quirk.

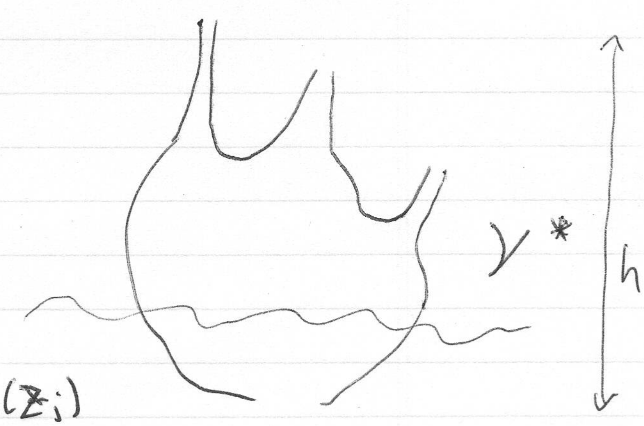
II: Jump ahead to preview of topological problem.

Work on perma-board

Let $\mathcal{V}^* = \mathcal{V} \cap (\mathbb{C}^*)^d = \{z \in \mathbb{C}^d : Q(z) = 0,\ z_1 \cdots z_d \neq 0\}$.
Def $h : \mathcal{V}^* \to \mathbb{R}$ by $h(z) = \log |z|^{-r} = \mathrm{Re}\{-r \cdot \log z\}$.

1. FACT (will go back & justify): There is a homology class $\mathcal{C} \in H_{d-1}(\mathcal{V}^*)$ and a $(d-1)$-form $\eta \in H^{d-1}(\mathcal{V}^*)$ s.t.

$$a_r = \int_{\mathcal{C}} z^{-r} \eta$$



2. FACT: (Work rel. to low height, global top. irrelevant as $h \to -\infty$)

$H_{d-1}(\mathcal{V}^*)$ generated by cycles $\sigma(z_i)$ "local downward $(d-1)$-handles" as $\{z_i\}$ varies over saddles of h on $\mathcal{V}^*$.

3. FACT: $\int_{\sigma(z_i)} z^{-r} \eta$ can be "read off" in terms of Q, derivatives.

From These we understand that if we can resolve e in The $\{\sigma(z_j)\}$ basis,

$$e = \sum_j n_j \, \sigma(z_j) \,, \qquad (\star)$$

then we have solved the asymptotic extraction problem.

Today's talk:
1. Go back and explain FACTS 1-3
2. Show solution to $(\star)$ when $d=2$
3. Show conjectured solution when $d=3$
4. Outline gaps in conjecture
5. IF time, show a topological oddity in a related problem.

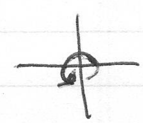
1. Cauchy Formula 2. Residue 3. Saddle pt. method 4. Morse theory

---

III: FACTS

Step I: Cauchy Formula.

$$a_r = \left(\frac{1}{2\pi i}\right)^d \int_{T^d} z^{-r} \, \frac{P(z)}{Q(z)} \, \frac{dz}{z} \, .$$

Here, T^d is a small torus, $\prod_{j=1}^{d} \gamma_j$  circle OF radius ε in j^{th} coord. winding ctrclk.

Here, $\frac{dz}{z}$ is the (logarithmic) holomorphic volume form on $(\mathbb{C}^*)^d$, a (middle-dim.) d-Form.

Let $\mathcal{M} := \mathbb{C}^d \setminus (\mathcal{V} \cup \{z_1 \cdots z_d = 0\})$.
Then integrand $z^{-r}\omega := z^{-r}\left(\left(\frac{1}{2\pi i}\right)^d \frac{P(z)}{Q(z)} \, \frac{dz}{z}\right)$ is holomorphic on $\mathcal{M}$.

Consequently, $\int_T z^{-r}\omega$ depends only on class $[T] \in H_d(\mathcal{M})$

(3)

Step 2: Residue. In this story, T and $z^{-r}\omega$ can be replaced by any cycle and co-cycle. Recall $\int_T z^{-r}\omega$ depends only on $[T] \in H_d(\mathcal{M})$ and $[z^{-r}\omega]$ in $H^d(\mathcal{M})$.

Claim:

Fixing Q, hence $\mathcal{V}, \mathcal{V}^*, \mathcal{M}$, there are functors

$$\begin{aligned} \vartheta &: H_d(\mathcal{M}) \to H_{d-1}(\mathcal{V}^*) \quad \text{"intersection cycle"} \\ \mathrm{Res} &: H^d(\mathcal{M}) \to H^{d-1}(\mathcal{V}^*) \quad \text{"residue"} \end{aligned}$$

such that $\int_T z^{-r}\omega = \int_{\vartheta(T)} \mathrm{Res}(z^{-r}\omega) = \int_{\vartheta(T)} z^{-r}\,\mathrm{Res}(\omega)$

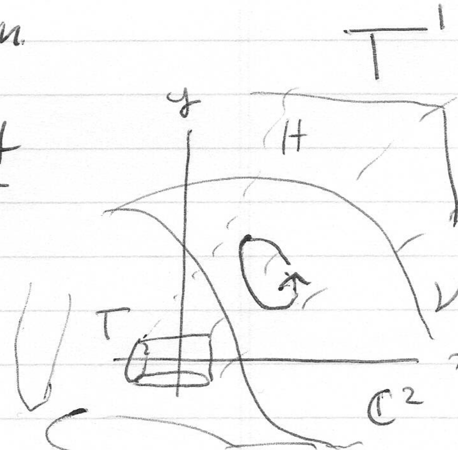
$\rightarrow$ This is FACT I with $\mathcal{C} := \vartheta(T)$ and $\eta = \mathrm{Res}(\omega)$
$= \vartheta(T^d)$ $\quad = \mathrm{Res}\left(\left(\frac{1}{2\pi i}\right)^d \frac{P}{Q}\frac{dz}{z}\right)$

Proof by picture:

This is really the Thom isomorphism.

Let $H : \mathbb{C}^d \times [0,1]$ be a homotopy from T to $\emptyset$ or to something far away we don't care about, T'.

ϑ is the intersection with $\mathcal{V}$.



Count dimensions: $\left.\begin{aligned} \dim(T) &= d \\ \dim(H) &= d+1 \\ \mathrm{codim}(\mathcal{V}) &= 2 \end{aligned}\right\} \Rightarrow \dim \vartheta(T) = d-1.$

Thom iso $\Rightarrow \vartheta(T) \times S^1 = T - T'$ in $H_d(\mathcal{M})$. Therefore

$\int_T z^{-r}\omega = \int_{\vartheta(T)\times S^1} z^{-r}\omega + \cancel{\int_{T'} z^{-r}\omega} \approx 0$. Take residue in S^1 coord. to prove claim.

(4)

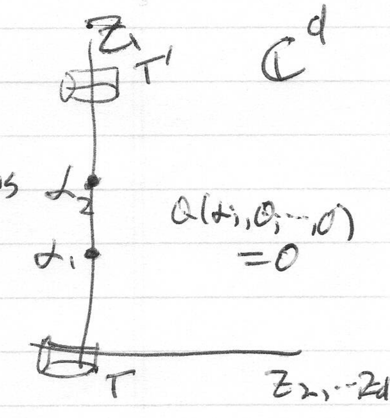
Remarks: (1) Really $\partial(T)$ should be $\partial(T, T')$ but we mod out by all cycles supported on some low set $\{h \leq -L\}$. Thus $\partial(T)$ is a well defined cycle relative ~~to~~ to $\{h \leq -L\}$.

(2) Various explicit formulae exist for $\eta := \text{Res}\ \omega$. The nicest description is the equation
$$\eta \wedge dQ = P\, dZ$$

Step 3: Saddle Point Method

We want to evaluate $\int_{\mathcal{C}} z^{-r} \eta$ where $\mathcal{C} = \partial(T)$.

FYI: we can envision $\partial(T)$ as follows, though saddle pt. method will work for any $\mathcal{C}$. To envision $\partial(T)$ let H move T far up the z_1-axis, leaving the other coordinates alone. Intersections will occur near solutions α_j to $Q(\alpha_j, 0, \ldots, 0) = 0$. These will be graphs over small tori $\{|z_2| = \cdots = |z_d| = \varepsilon\}$ of $z_1(z_2, \ldots, z_d)$ implicit coord. of V.



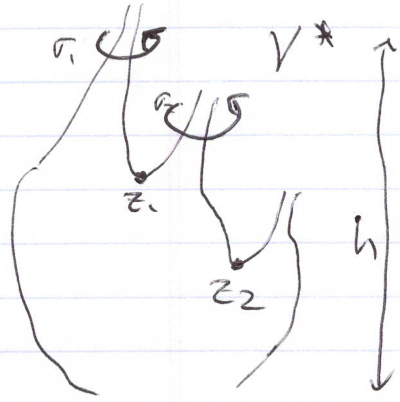
Originally, $\left|\int_{\mathcal{C}} z^{-r} \eta\right| \ll |z^{-r} \eta|$ because of oscillation.

Deform $\mathcal{C}$ so that $h(\mathcal{C}) := \max\{h(z) : z \in \mathcal{C}\}$ is minimized. This implies gradient of $-r \cdot \log z$ vanishes on V at such pt. In other words oscillation has agogic pause at argmax h.

At such a point, $\left|\int_{\mathcal{C}} z^{-r} \eta\right| \approx |z^{-r} \eta(z)|$ and can be read off from higher derivatives of Q by saddle point formulae.

Example $d=2$

$C = \sigma_1 + \sigma_2$



Step 4: Morse Theory

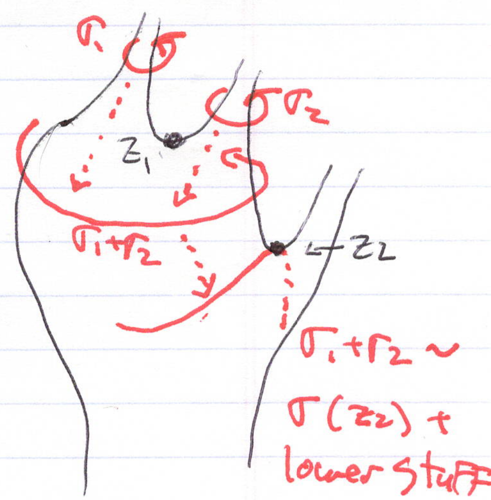
How to deform C to minimize $\max_C h$?

Draw V^* so that h is vertical axis.
In our example, $C = \sigma_1 + \sigma_2$, small circles around points $(\alpha_1, 0)$, $(\alpha_2, 0)$ in V^*.

Push C down, e.g. via gradient flow.
Obstructions *may* occur at saddles, shown as z_1, z_2.
When obstruction actually does occur, C is "draped over a saddle" and is homologous to $\sigma(z_j)$ + possible lower stuff.

In example in picture, σ_1 & σ_2 have opposite orientations at z_1 so $\sigma_1 + \sigma_2$ is not obstructed there and can be pushed to z_2 where obstruction does occur.

Morse Theory says:



Homotopy type of V^* is built by successive addition of "handles" = topological k-ball / boundary at saddles of index k.

h = real part of analytic fn = harmonic so all crit pts. have index $d-1$

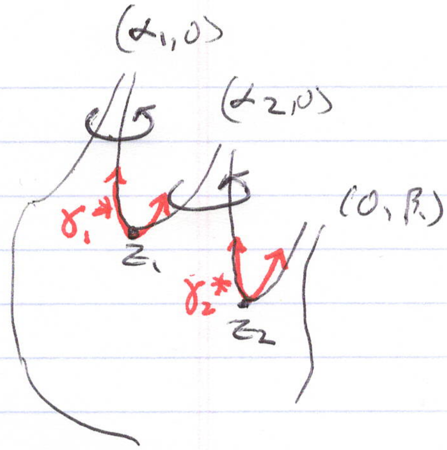
All handles are $\simeq B^{d-1}/\partial B^{d-1} = S^{d-1}$

hence $H_{d-1}(V^*) = \bigoplus_j \sigma(z_j)$ where $\sigma(z_j) =$ 

downward local $\mathbb{R}^{d-1}$ patch.

IV Solutions

Solution when $d=2$:



Recall $\mathcal{C} = \sum \gamma_{\alpha_j}$
where α_j are, say, all x-axis zeros.

(Note: Could have ~~also~~ instead summed $\sum \gamma_{\beta_j}$.
Same homology class.)

Idea: $\{\sigma(z_i)\}$ is nearly dual to chains $\{\gamma_i^*\}$
where γ_i^* is the upward flowing $(d-1)$-manifold from z_i.

Arrange in order of decreasing height: $h(z_1) \geq h(z_2) \geq \cdots$:
Signed intersection number is a pairing s.t.

$I(\gamma_j^*, \sigma(z_i)) = 1$ when $i=j$ and 0 when $j<i$.

Upper triangular pairing
means Leading-Term-Invertible

$$\begin{array}{cc} & \gamma_j^* \\ \sigma(z_i) & \begin{pmatrix} 1 & \cdot & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \end{array}$$

$\mathrm{Min}\{j : n_j \neq 0 \text{ in } \mathcal{C} = \sum n_j \sigma(z_j)\}$

$= \mathrm{Min}\{j : \gamma_j^* \cap \mathcal{C} \neq 0\}$.

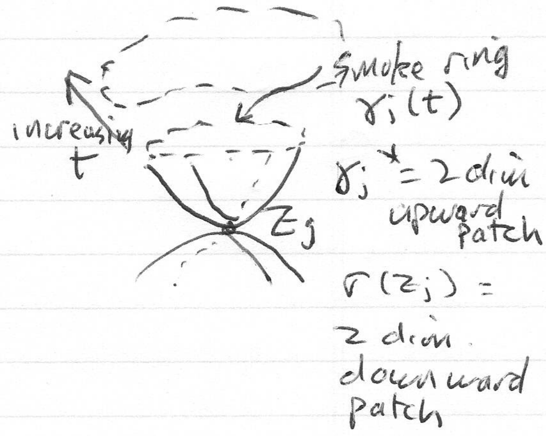
To compute $\gamma_j^* \cap \mathcal{C}$, check whether ascends to 2 of same or to one x, one y. If two of same then intersection # $=0$ (will be opposite signs). If different, then intersection # $\neq 0$ and we can stop if only want leading term.

In example, γ_1^* goes up to x, x so get zero and move on.
Next, γ_2^* goes up to x, y so can stop and output

$a_r \sim \int_{\sigma(z_2)} z^{-r} \eta$ = known formula.

V conjecture

Conjecture when $d=3$:

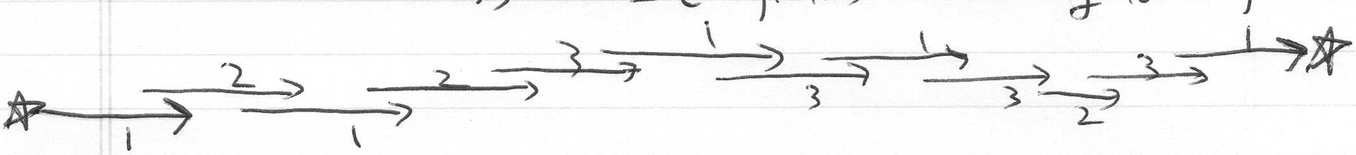


Again order saddles high to low.
$h(z_1) \geq h(z_2) \geq \cdots h(z_m)$.

Starting at highest, flow a smoke ring up the ascending manifold.
Note: index$(z_j) = 2$, all j, so ascending manifold has dim $= 2$. Hence boundaries of increasing balls form 1-d smoke rings, homeomorphic to S^1, carried by upward flow.

For t large, $h(\gamma_j(t))$ uniformly large $\Rightarrow$ min. coordinate modulus $\leq \varepsilon$ for every pt. of $\gamma_j(t)$.

Also clearly not all coords $\leq \varepsilon$ because V avoids nbd. of 0.
Therefore, for sufficiently large t and small ε, reading around the smoke ring $\gamma_j(t)$ and marking which coordinate(s) are $\leq \varepsilon$ yields something looking like this.



Let $N(j)$ denote winding number $= \# 1 \to 2 - \# 2 \to 1$, say.

Conjecture: First j s.t. $N(j) \neq 0$ is first j s.t. $n_j \neq 0$ and $n_j = N(j)$ if sign/orientation properly defined.

Problem 1: Prove or disprove this.

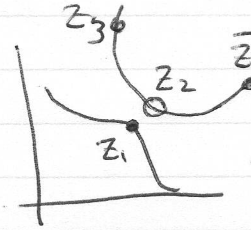
Problem 2: $h : V^* \to \mathbb{R}$ not in general proper when $d > 2$.
Therefore, can't conclude $\{\Gamma(z_j)\}$ basis for $H_{d-1}(V^*)$.
Nightmare is that gradient flows out to ∞ at finite ht.
Problem is to give suf. conditions to rule this out, e.g., no crit pts of h on $P(V^*)$ at ∞.

VI topological fun

Lacuna analysis: Remind me of kids trick (arm gesture)

Case study, take $P(z)=1$, $Q(z) = 1-x-y-z-w + c\,xyzw$
Family of rational series in four variables.

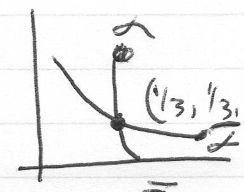
V is smooth except when $c=27$.
What happens there?
Look at pictures.



$C=26$

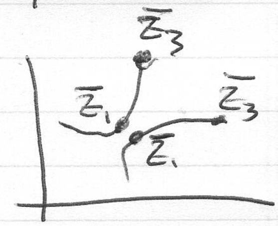
z_1, z_2 real
z_3, z_4 complex

For $c<27$ contributing saddle is at z_1, integrate over $T(z_1)$ get $a_{n,n,n,n} \sim b^n$, $b \to 81$ as $c \to 27^-$.
(asym. positive)



$C=27$

For $c>27$ contributing saddles are two complex conjugate points at $z_1, \bar{z}_1$. Get $a_{n,n,n,n} \sim 2\,\mathrm{Re}\{b^n\}$, $|b| \to 81$ as $c \to 27^-$ but have factor of $\cos(n \operatorname{Arg}(b))$.
(asym oscillating)



$C=28$

When $c=27$, have exponential drop:

$$a_{n,n,n,n} \sim c\,\mathrm{Re}\{\alpha^n\}, \quad |\alpha| = 9$$

So get oscillating behavior and modulus growing at exp. rate 9^n rather than 81^n, because $\alpha = (\beta, \beta, \beta, \beta)$ with $|\beta| = 1/\sqrt{3}$ and $\operatorname{Arg}(\beta) \neq 0$.

What happens to contribution from $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})$?

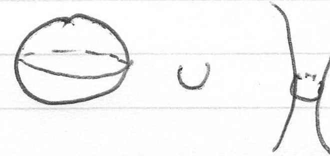
ANS: Topologically, it's null.
We haven't discussed integral in non-smooth case but can understand via perturbation.

Set $c=27$ but perturb Q to $Q+\varepsilon$ for small pos. real ε.

Compute $\theta(T)$ locally using explicit homotopy, obtaining

$$\mathcal{C} = S^{d-1} \cup \mathcal{H}^{d-1} / S^{d-2}, \text{ That is,}$$

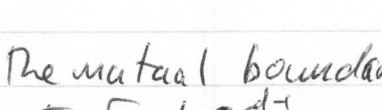
a $(d-1)$-sphere together with a 1-sheeted $(d-1)$-hyperboloid sewn together on a common boundary where they intersect, This being the equator of the sphere and the neck of the hyperb.

Picture:  sewn along .

This chain is not a manifold, having a local structure of an X at the common intersection.

Moreover, orientation of $\mathcal{C}$ changes across the mutual boundary! Therefore, the sphere on its own is not a cycle. In fact $\partial S^{d-1} = 2S^{d-2}$. Likewise for $\mathcal{H}$, though $S+H$ <u>is</u> a cycle.

Can deform $\mathcal{C}$ by flowing down so that S^{d-2} goes to south pole and $\mathcal{C}$ become S^{d-1} with unif. orient. plus $\mathcal{H}'$.

Sending $\varepsilon \to 0$, small sphere S shrinks to a point while $\mathcal{H}$ folds in on itself and double covers (if you flow it) the bottom sheet of a two-sheeted hyperboloid in <u>opposite</u> direx. This is local, so after some distance, the double cover separates. Thus, $\mathcal{H} \to$ 

$$a_r = \int_{S+H} z^{-r}\eta = 0 + \int_{\mathcal{H}'} z^{-r}\eta \text{ where } \mathcal{H}' \text{ is supported at } \{h \le h(z_\cdot) - \underbrace{\Delta h}_{\text{const.}}\}.$$