Due: May 3, 2024

Math 6030 / Problem Set 12 (two pages)

Prolongations of valuations. Let $R = R_v$ be a valuation ring of a field K with canonical valuation v, valuation ideal $\mathfrak{m}_v$, valuation group $vK = K^\times/R_v^\times$ and residue field $\kappa_v = R_v/\mathfrak{m}_v$. Let $L|K$ be a field extension and R_w be a valuation ring of L with canonical valuation w, valuation ideal $\mathfrak{m}_w$, value group $wL = L^\times/R_w^\times$ and residue field κ_w.

Recall: We say that: w prolongs v if $vK \subset wL$ and $v = w|_K$, *denoted* $w|v$, and that R_w prolongs R_v if $R_w, \mathfrak{m}_w$ dominates $R_v, \mathfrak{m}_v$, i.e., $\mathfrak{m}_v = \mathfrak{m}_w \cap R$, *denoted* $R_w|R_v$. If so, define: $e(w|v) := (wL : vK)$ the ramification index, and $f(w|v) := [\kappa_w : \kappa_v]$ the residue degree of $w|v$.

1) In the above notation, prove/disprove/answer:

a) One has: $w|v$ iff $R_w|R_v$ iff $R^\times = K \cap R_w^\times$.

b) Suppose that $w|$ or equivalently, $R_w|R_v$.
- If $u_i \in L$ and $w(u_i) \neq w(u_{i'})$ for all $i \neq i'$, then $w(\sum_i u_i) = \min_i w(u_i)$.
- If $x_j \in R$ and $(\overline{x}_j)_j$ are κ_v lin. indep. in κ_w, then $(x_j)_j$ are K-lin. indep. in L.

2) Suppose that $R_w|R_v$, $R_{w_l}|R_v$, $l \leqslant n$ be distinct. In the above notation/context, TFH:

I) **The (weak) Fundamental Inequality**.
Let $(y_i)_i$ in L and $(x_j)_j$ in R_w be s.t. $\big(w(y_i)\big)_i$ are distinct in $wL \twoheadrightarrow wL/vK$ and $(\overline{x}_j)_j$ are κ_v-linearly independent in κ_w. Then $(x_i y_j)_{i,j}$ is K-linearly independent in L.

II) **The Fundamental Inequality**. One has $\sum_l e(w_l|v) f(w_l|v) \leqslant [L : K]$.

3) In the above context, let $L|K$ be algebraic, $S|R_v$ be the integral closure of R_v in L, and $X_v = \{w \in \mathrm{Val}(L) \mid w|v\}$ be the set of prolongations of v to L. Prove/disprove/answer:

a) The map $X_v \to \mathrm{Max}(S)$, $\mathfrak{m}_w \mapsto \mathfrak{n} := \mathfrak{m}_w \cap S$ is a well defined bijection and $R_w = S_\mathfrak{n}$.

b) For every $w \in X_v$ one has: wL/vK is a torsion group, and $\kappa_w|\kappa_v$ is algebraic. Moreover, if $L = \overline{K}$, then wL is divisible and $\kappa_w = \overline{\kappa}_w$. Does the converse hold?

[**Hint** to a): For $x \in L^\times$, let $\mathrm{Mipo}_K(x) = t^n + \cdots + a_0 \in K[t]$. If i_0 is maximal s.t. $v(a_{i_0}) = \min_i v(a_i)$, set $b_i = a_{i+i_0}/a_{i_0}$ for $i \leqslant n - i_0$, $c_j = a_j/a_{i_0}$ for $j < m$. Then $p(t) = \sum_i b_i t^i \in 1 + t\,\mathfrak{m}_v[t]$ (WHY) and $q(t) = \sum_j c_j t^j \in R_v[t]$ (WHY). Further, $p(x) = \frac{1}{x} q(\frac{1}{x})$ (WHY), and $p(x) \in 1 + \mathfrak{m}_v[x]$ (WHY). Next, $\forall\, R_w|R_v$ have: $x \in R_w \Rightarrow p(x) \in R_w$, thus $q(\frac{1}{x}) \in R_w$; $\frac{1}{x} \in R_w \Rightarrow q(\frac{1}{x}) \in R_w$, thus $p(x) \in R_w$ (WHY). Conclude: $\forall\, x \in K$ have $p(x), q(\frac{1}{x}) \in S$ (WHY). Now suppose that $x \in \mathfrak{m}_w$. Then $p(x) \in 1 + \mathfrak{n}$ (WHY) and $x\,p(x) = q(\frac{1}{x})$ implies: $q(\frac{1}{x}) \in \mathfrak{m}_w \cap R = \mathfrak{n}$ (WHY). Thus $x = p(x)^{-1} q(\frac{1}{x}) \in \mathfrak{n}_\mathfrak{n}$ (WHY), etc.

4) In the above context/notation, let R_v be a DVR and $[L\!:\!K]$ be finite. Prove/disprove:

a) S^+ is a finite R_v-module iff the *fundamental equality* $\sum_l e(w_l|v) f(w_l|v) = [L : K]$ holds.

b) If $L|K$ is finite separable, the fundamental equality $\sum_l e(w_l|v) f(w_l|v) = [L : K]$ holds.

More about the integral closure. Let $\widehat{K} := k((t))$ endowed with $\hat{v}$. Then $\mathrm{td}(\widehat{K}|k) = \infty$ (WHY), and for $t, u \in \widehat{K}$ alg. indep. over k, set $K = k(t, u^{p^e})$, $L = k(t, u)$, and $w = \hat{v}|_L$, $v = \hat{v}|_K$.

5) In the above notation, let $\mathrm{char}(k) = p > 0$. Prove/disprove:

a) $L|K$ is a purely inseparable of degree p^e.

b) w is the unique extension of v to L and $e(w|v) = 1 = f(w|v)$.

c) $R_w|R_v$ is the integral closure of R_v in L, but R_w is not a finite R_v-module.

6) Complete the proof of the assertion from the class:
Let $R = k[x_1, \dots, x_n]$ is a finitely generated domain over a field k, $K = \mathrm{Quot}(R)$, $L|K$ is a finite field extension, and $S|R$ be the integral closure of R in L. Then S is a finite R-module.

[**Hint**: Use the explanations from the class to reduce to the case $R_0 = k[t_1, \dots, t_d]$, hence $K_0 = k(t_1, \dots, t_d)$ and $L_0|K_0$ purely inseparable. Conclude by induction on $[L_0 : K_0]$ as follows: If $\alpha^p = p(t_1,\dots, t_d)$, then $\alpha \in K_1 = k_1(u_1, \dots, u_d)$, where $k_1|k$ is the finite field extension obtained by adjoining the p-th roots of all the coefficients of $p(t_1,\dots, t_d)$ and $u_i^p = t_i$ for $1 \leqslant i \leqslant d$, etc. . . .]

7) Describe the decomposition groups of the prime ideals $\mathfrak{p} \in \mathrm{Spec}(R)$ in $S|R$ below:

a) $R = \mathbb{Z}$, S the ring of algebraic integers in $K = \mathbb{Q}[\sqrt{d}\,]$ with $1 < |d| < 7$.

b) $R = \mathbb{Z}$, S the ring of algebraic integers in $K = \mathbb{Q}[\zeta_3, \sqrt[3]{2}\,]$, $\mathfrak{p} = 2\mathbb{Z}, 3\mathbb{Z}, 5\mathbb{Z}, 7\mathbb{Z}$.

More about fractional ideals.

8) Let R be a Noetherian domain, $\mathrm{Min}(R)$ minimal prime ideals $\mathfrak{p} \neq (0)$. Prove/disprove/answer:

a) An ideal $\mathfrak{a} \subset R$ is invertible iff $\mathrm{Spec}(\mathfrak{a}) \subset \mathrm{Min}(R)$, and all $\mathfrak{p} \in \mathrm{Min}(\mathfrak{a})$ are invertible.

b) If $\mathfrak{p} \in \mathrm{Spec}(R)$ is invertible, then $\mathfrak{p} \in \mathrm{Min}(R)$ and $R_\mathfrak{p}$ is integrally closed. Conversely ?

c) R is integrally closed iff all $\mathfrak{p} \in \mathrm{Min}(R)$ are invertible.

Conclude: $\mathrm{Div}(R) = \oplus_\mathfrak{p} \mathbb{Z}\,\mathfrak{p}$ with $\mathfrak{p} \in \mathrm{Min}(R)$ invertible.

[**Hint** to a), b): Use Krull Principal Ideal Thm combined with the Lemma from the proof of the Characterization Thm, etc. . . .]

9) Prove the following basic facts about Dedekind domains R:

- **Gauss Lemma for Dedekind domains** R: *For $f = a_n t^n + \dots + a_0 \in R[t]$ let $c(f) := (a_0,\dots, a_n) \in \mathcal{I}d(R)$ be the content of f. Then one has $c(fg) = c(f)c(g)$.*
- *For $\mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in \mathcal{I}d(R)$ one has: $\mathfrak{a} \cap (\mathfrak{b}+\mathfrak{c}) = (\mathfrak{a}\cap\mathfrak{b})+(\mathfrak{a}\cap\mathfrak{c})$, $\mathfrak{a}+(\mathfrak{b}\cap\mathfrak{c}) = (\mathfrak{a}+\mathfrak{b})\cap(\mathfrak{a}+\mathfrak{c})$.*
- *A finite R-module M is flat iff M is torsion free.*
- *Let M be a finite torsion R-module. There are $(\mathfrak{p}_i)_i$, $(e_i)_i$, $i \in I$ finite s.t. $M \cong_R \oplus_i R/\mathfrak{p}_i^{e_i}$.*
- *For $N \subset R^N$ an R-submodule, $\exists\, \mathfrak{a} \in \mathcal{I}d(R)$, $N_0 \subset N$ R-free such that $N \cong_R N_0 \oplus \mathfrak{a}$.*

[**Hint**: Localize at each $\mathfrak{p} \in \mathrm{Max}(R)$, and use the fact that two modules are equal, morphisms are injective/surjective, etc. iff the corresponding assertions hold everywhere locally, etc.... For the last assertion, show that $\mathfrak{a} \oplus \mathfrak{b} \cong_R R \oplus \mathfrak{c}$, etc....]