Due: Feb 9, 2024

Math 6030 / Problem Set 2 (two pages)

Pontrjagin Duality for abelian m-torsion groups. Let $\mathcal{A}$ be the category of abelian torsion groups, considered as discrete topological groups, and $\widehat{\mathcal{A}}$ be the category of profinite abelian groups. Further, let $\mathcal{A}_m$ and $\widehat{\mathcal{A}}_m$ be the corresponding full subcategories of m-torsion groups. Finally, let C_m be a typical cyclic group of order m, e.g. $C_m = \mathbb{Z}/m, \frac{1}{m}\mathbb{Z}/\mathbb{Z}, \mu_m$, which we consider as a discrete topological group.

For Γ in $\mathcal{A}_m$ or in $\widehat{\mathcal{A}}_m$, let $\mathcal{C}(\Gamma, C_m)$ be the space of continuous functions endowed with the open-compact topology. Not that $\mathcal{C}(\Gamma, C_m)$ is a toplogical group (WHY). Finally, denote $G^* := \mathrm{Hom}(G, C_m) \subset \mathcal{C}(G, C_m)$ the group of continuous homomorphism.

1) Prove that $G^* \subset \mathcal{C}(G, C_m)$ is topologically closed in $\mathcal{C}(G, C_m)$, and further one has:

a) If $G \in \mathcal{A}_m$, then G^* is a profinite m-torsion group, hence $G^* \in \widehat{\mathcal{A}}_m$.
Further, $G \rightsquigarrow G^*$ defines a contravariant functor $\mathcal{A}_m \rightsquigarrow \widehat{\mathcal{A}}_m$.

b) If $G \in \widehat{\mathcal{A}}_m$, then G^* is a discrete m-torsion group, hence $G^* \in \mathcal{A}_m$.
Further, $G \rightsquigarrow G^*$ defines a contravariant functor $\widehat{\mathcal{A}}_m \rightsquigarrow \mathcal{A}_m$.

c) Further, G and G^* are isomorphic as topological groups iff G id a finite.

2) Duality. Let $\imath_G : G \to (G^*)^* = \mathrm{Hom}(G^*, C_m)$, $g \mapsto \Phi_g, \Phi_g(\varphi) = \varphi(g) \,\forall\, \varphi \in G^*$. Prove:

a) The canonical map $\imath_G : G \to (G^*)^*$ is an isomorphism of topological groups, **hence**:
$\widehat{\mathcal{A}}_m \rightsquigarrow \mathcal{A}_m$ and $\mathcal{A}_m \rightsquigarrow \widehat{\mathcal{A}}_m$ are inverse to each other, thus $\mathcal{A}_m$, $\widehat{\mathcal{A}}_m$ are (anti)equivalent.

b) The subgroups of G correspond functorially to the factor groups of G^*.

Artin–Schreier Thm and (formally) real fields. Recall that a (formally) real field is any field K which admits a total ordering $\leqslant$ which is compatible with the field operations (HOW). Obviously, in a real field K one has $-1_K < 0_K < 1_K$ (WHY), hence $\mathrm{char}(K) = 0$ (WHY). A real field is called real closed if there is no proper algebraic extension $K'|K$ of real fields, i.e., K' carrying a total field ordering $\leqslant'$ which prolongs $\leqslant$ to K'. It turns out the total orderings of fields relate in a subtle with the (sums of) squares in K. In the sequel, $K^{\bullet,2} := \{x^2 \,|\, x \in K\}$ is the set of squares and $\sum K^{\bullet,2} := \{\sum_i x_i^2 \,|\, x_i \in K\}$ is the set of finite sums of squares in K.

3) Without invoking the Artin–Schreier Thm, setting $\boldsymbol{i} = \sqrt{-1}$, prove directly:

a) TFAE: (i) $1 < [\overline{K} : K] < \infty$; (ii) $1 < [K^s : K] < \infty$; (iii) $K \neq K^s = K(\boldsymbol{i})$, $\mathrm{char}(K) = 0$.

b) If (i), (ii), (iii) are satisfied, then $K^\times = -K^{\bullet,2} \cup K^{\bullet,2}$, $-K^{\bullet,2} \cap K^{\bullet,2} = \{0\}$, $\sum K^{\bullet,2} = K^{\bullet,2}$.

Conclude: $x \leqslant y \overset{\text{def}}{\Longleftrightarrow} y - x \in K^{\bullet,2}$ defines a total field ordering on K.

4) Prove that for an arbitrary field K one has: $\sum K^{\bullet,2}$ is a semifield, i.e., it is closed w.r.t. $+, \cdot$ and inverses x^{-1} for nonzero $x \in \sum K^{\bullet,2}$. Invoking this fact, prove:

Artin's Theorem. *K is a real field if and only if $-1 \notin \sum K^{\bullet,2}$.*

[**Hint** to Artin's Thm: First, if K is real, then $-1 \notin \sum K^{\bullet,2}$ (WHY) and $\mathrm{char}(K) = 0$ (WHY). For the converse prove:

- Let $K_1|K$ be an algebraic extension with $[K_1 : K]$ odd. Then $\sum K_1^{\bullet,2} \subset K_1^\times$ is a semifield and $-1 \notin \sum K^{\bullet,2}$.
- Let $K_2 = K[\sqrt{\sum K^{\bullet,2}}]$. Then $\sum K_2^{\bullet,2} \subset K_2^\times$ is a semifield and $-1 \notin \sum K^{\bullet,2}$.

Conclude: If $\tilde{K} \subset \overline{K}$ is a maximal real subfield, then $\overline{K} = \tilde{K}[\sqrt{-1}]$, hence K is a real closed by Problem 3) above.]

Basics about $\mathrm{Max}_\bullet(R), \mathrm{Spec}(R) \subset \mathfrak{Id}(R)$.

Recall that for a ring R we defined $\mathrm{Max}_\bullet(R) \subset \mathfrak{Id}_\bullet(R)$, $\mathrm{Spec}(R)$, where $\bullet$ can be l (left), r (right), or empty, and the latter case, $\mathrm{Max}(R) \subset \mathfrak{Id}(D)$, means two-sided. Further, $\mathcal{J}(R) = \cap_{\mathfrak{m}\in\mathrm{Max}_\bullet(R)}\mathfrak{m}$ is the Jacobson radical of R, and $\mathcal{N}(R) = \cap_{\mathfrak{p}\in\mathrm{Spec}(R)}\mathfrak{p}$ is the nil-radical of R. Similarly, given $\mathfrak{a} \in \mathfrak{Id}(R)$, $\mathcal{J}(\mathfrak{a}) = \cap_{\mathfrak{a}\subset\mathfrak{m}\in\mathrm{Max}_\bullet(R)}\mathfrak{m}$ is the Jacobson radical of $\mathfrak{a}$, and $\mathcal{N}(\mathfrak{a}) = \cap_{\mathfrak{a}\subset\mathfrak{p}\in\mathrm{Spec}(R)}\mathfrak{p}$ is the nil-radical of $\mathfrak{a}$.

5) For an arbitrary ring R with 1_R prove/disprove/answer:

a) $\mathrm{Max}(R) \subset \mathrm{Spec}(R)$ and $\mathcal{J}(\mathfrak{m}) = \mathfrak{m}$ for all $\mathfrak{m} \in \mathrm{Max}(R)$.

b) Every $\mathfrak{m} \in \mathrm{Max}_l(R)$ contains maximal ideals $\mathfrak{p} \subset \mathfrak{m}$, and all such $\mathfrak{p}$ are prime ideals.

c) How do $\mathcal{N}(R)$ and $\mathcal{J}(R)$ compare?

6) For an arbitrary ring R with 1_R prove/disprove/answer:

a) Let $\mathfrak{a}_1, \dots, \mathfrak{a}_n \in \mathfrak{Id}(R)$, $\mathfrak{p} \in \mathrm{Spec}(R)$. If $\cap_i\mathfrak{a}_i \subset \mathfrak{p}$, then $\exists\, i_0$ such that $\mathfrak{a}_{i_0} \subset \mathfrak{p}$.
The same question with "$\subset$" replaced by "$=$".

b) Let $\mathfrak{p}_1, \dots, \mathfrak{p}_n \in \mathrm{Spec}(R)$, $\mathfrak{a} \in \mathfrak{Id}(R)$. If $\mathfrak{a} \subset \cup_i\,\mathfrak{p}_i$, then $\exists\, i_0$ such that $\mathfrak{a} \subset \mathfrak{p}_{i_0}$.
What is the corresponding assertion with "$\subset$" replaced by "$=$" ?

7) For an arbitrary ring R with 1_R and $\mathfrak{a}, \mathfrak{a}_i \in \mathfrak{Id}(R)$, prove/disprove/answer:

a) $\mathcal{N}\big(\mathcal{N}(\mathfrak{a})\big) = \mathcal{N}(\mathfrak{a})$, $\mathcal{N}(\sum_i \mathfrak{a}_i) = \mathcal{N}\big(\sum_i \mathcal{N}(\mathfrak{a}_i)\big)$.

b) $\mathcal{N}(\prod_i \mathfrak{a}_i) = \cap_i \mathcal{N}(\mathfrak{a}_i) = \mathcal{N}(\cap_i \mathfrak{a}_i)$.

($*$) What are the corresponding assertions for the Jacobson radical $\mathcal{J}(\bullet)$?

Extension/Contraction of Ideals.

Let $f : R \to S$ be a ring morphism with $f(1_R) = 1_S$, and recall the $\bullet$-ideal extension map $f_* : \mathfrak{Id}_\bullet(R) \to \mathfrak{Id}_\bullet(S)$, $\mathfrak{a} \mapsto \mathfrak{a}^e := \big(f(\mathfrak{a})\big)_\bullet$, respectively the $\bullet$-ideal contraction map $f^* : \mathfrak{Id}_\bullet(S) \to \mathfrak{Id}_\bullet(R)$, $\mathfrak{b} \mapsto \mathfrak{b}^c := f^{-1}(\mathfrak{b})$. These maps a well defined (WHY). Further, let $\mathfrak{Id}^c_\bullet(R) \subset \mathfrak{Id}_\bullet(R)$ be the subset of ideals which are contracted, and $\mathfrak{Id}^e_\bullet(S) \subset \mathfrak{Id}_\bullet(S)$ be the subset of ideals which are extended.

8) For $\mathfrak{a}, \mathfrak{a}_i \in \mathfrak{Id}_\bullet(R)$ and $\mathfrak{b}, \mathfrak{b}_i \in \mathfrak{Id}_\bullet(S)$, prove/disprove the assertions (same made in class):

a) $\mathfrak{a}^{ec} := (\mathfrak{a}^e)^c \supset \mathfrak{a}$ and $\mathfrak{b}^{ce} := (\mathfrak{b}^c)^e \subset \mathfrak{b}$.

b) $f_*\colon \mathfrak{Id}^c_\bullet(R) \to \mathfrak{Id}^e_\bullet(S)$ and $f^*\colon \mathfrak{Id}^e_\bullet(S) \to \mathfrak{Id}^c_\bullet(R)$ are well defined bijections and $f^* = f_*^{-1}$.

c) $(\sum_i \mathfrak{a}_i)^e = \sum_i \mathfrak{a}_i^e$ and $(\prod_i \mathfrak{a}_i)^e = \prod_i \mathfrak{a}_i^e$. Further, $(\sum_i \mathfrak{b}_i)^c = \sum_i \mathfrak{a}_i^c$ and $(\prod_i \mathfrak{b}_i)^c = \prod_i \mathfrak{a}_i^c$.
The same questions for $\mathfrak{a}_i \in \mathfrak{Id}^c_\bullet(R)$, respectively $\mathfrak{b}_i \in \mathfrak{Id}^e_\bullet(S)$.

9) In the above notation, prove/disprove/answer the following:

a) If $\mathfrak{q} \in \mathrm{Spec}(S)$, then $\mathfrak{q}^c \in \mathrm{Spec}(R)$, i.e., contractions of prime ideals are prime ideals.
Hence $f^* : \mathrm{Spec}(S) \to \mathrm{Spec}(R)$, $\mathfrak{q} \mapsto \mathfrak{p} := \mathfrak{q}^c = f^{-1}(\mathfrak{q})$ is well defined.

b) For $\mathfrak{b} \in \mathfrak{Id}(S)$ one has $\mathcal{N}(\mathfrak{b})^c = \mathcal{N}(\mathfrak{b}^c)$. The same question for $\mathfrak{b} \in \mathfrak{Id}^e(S)$.

10) Give examples to show that, in general, maximal ideals do not behave well under extension and/or contraction, and that prime ideals do not behave well under extension.