Due: April 5, 2024

Math 6030 / Problem Set 8 (two pages)

Special classes of commutative rings. In the sequel, R is a commutative ring with 1_R.

1) Which of the following are Noetherian rings:

a) The ring R of rational functions $f(t) \in \mathbb{C}(t)$ which have no poles on the unit circle.
b) The ring of analytic functions on the whole complex plane $\mathbb{C}$.
c) the ring of germs of analytic functions around the origin $0 \in \mathbb{C}$.
d) The ring R of all the polynomials $f(t) \in \mathbb{C}[t]$ with $f'(0) = 0$.

2) Prove/disprove/answer:

a) R is Noetherian/Artinian iff all its localizations $R_\mathfrak{p}$ are so.
b) R is Noetherian iff every ascending sequences of ideals is uniformly locally stationary, i.e., $\forall\, (\mathfrak{a}_i)_i$ ascending sequence of ideals $\exists\, i_0$ s.t. $\mathfrak{a}_{i,\mathfrak{p}} \subset \mathfrak{a}_{i_0,\mathfrak{p}}$ for all $\mathfrak{p} \in \mathrm{Spec}(R)$.
c) Formulate and prove/disprove the corresponding assertion of Artin rings and descending sequences of ideals.

3) For $\mathfrak{a} \in \mathcal{I}d(R)$, set $\mathcal{I}d(\mathfrak{a}^n) := \{\, \mathfrak{b} \in \mathcal{I}d(R) \,|\, \mathfrak{a}^{n+1} \subset \mathfrak{b} \subset \mathfrak{a}^n\}$, $\mathcal{I}d(\overline{\mathfrak{a}^n}) = \{\, \mathfrak{b}/\mathfrak{a}^{n+1} \,|\, \mathfrak{b} \in \mathcal{I}d(\mathfrak{a}^n)$. Define an outer multiplication of $\overline{R} := R/\mathfrak{a}$ on $\overline{\mathfrak{a}^n} = \mathfrak{a}^n/\mathfrak{a}^{n+1}$ by $\overline{r}\cdot\overline{x} := \overline{rx}$. Prove/disprove:

a) $\overline{\mathfrak{a}^n}$ is an $\overline{R}$-module via $\overline{r}\cdot\overline{x} = \overline{rx}$, and $\mathcal{I}d(\overline{\mathfrak{a}^n})$ is the set of $\overline{R}$-submodules of $\overline{\mathfrak{a}^n}$.
b) If $\mathfrak{a} = \mathfrak{m} \in \mathrm{Max}(R)$ and $\kappa := R/\mathfrak{m}$ is the residue field, the following are equivalent:

(i) $\mathcal{I}d(\mathfrak{m}^n)$ satisfies ACC; (ii) $\mathcal{I}d(\mathfrak{m}^n)$ satisfies DCC; (iii) $\dim_\kappa(\overline{\mathfrak{m}^n}) < \infty$.

4) Let R be a commutative ring, k a field. Prove/disprove:

a) R is Artinian iff R is Noetherian and $\mathrm{Spec}(R)$ is a Hausdorff topological space.
b) Suppose that R is a k-algebra of finite type. Then R is Artinian iff $\dim_k(R^+) < \infty$.

5) Let R be a Noether ring. Prove in all detail the assertion form the class:

a) If $\mathfrak{p} \in \mathrm{Spec}(R)$ has $\mathrm{ht}(\mathfrak{p}) = m$, there is a regular sequence $\underline{\boldsymbol{r}} = (r_1, \dots, r_m)$ with $r_i \in \mathfrak{p}$.
b) Every descending sequence $(\mathfrak{p}_i)_i$ in $\mathrm{Spec}(R)$ is stationary.

More about valuation rings.

Recall that for a valuation ring R of $K = \mathrm{Quot}(R)$, we denote by $\mathfrak{m}_R$ its valuation ideal, $\kappa_R = R/\mathfrak{m}$ its residue field, $v_R : K \to vK = K^\times/R^\times \cup \infty$ its canonical value group. Further, $a \in R^\times$ iff $v_R(a) = 0$ (WHY), and $a \in \mathfrak{m}_R$ iff $v_R(a) > 0$ (WHY). Recall that an absolute value $|\ | : K \to \mathbb{R}_{\leqslant 0}$ is called non-archimedean, if $|x + y| \leqslant \max(|x|, |y|)$.

6) Let K be a field, v a valuation of K. Prove the assertions form the class:

a) Let $x, y \in K$ be given. If $v(x) \neq v(y)$, then $v(x + y) = \min\big(v(x), v(y)\big)$.
b) A valuation ring R of K is Noetherian iff $R = K$ or R is discrete.
c) If $|\ |$ is a non-archimedean absolute value of K, then $R_{|\,|} := \{x \in K \mid |x| \leqslant 1\}$ is a valuation ring of K with valuation ideal $\mathfrak{m}_{|\,|} = \{x \in K \mid |x| < 1\}$.
- Conversely, if $R \subset K$ is a valuation ideal with $vK \subset \mathbb{R}, +$ then for every $0 < \rho < 1$ one has: $|\ |_R : K \to \mathbb{R}$, $x \mapsto \rho^{v_R(x)}$ is a non-archimedean absolute value with $R = R_{|\,|_R}$.

In the above notation, let R_v be a valuation ring with valaution v, and $vK^\times \subset \Gamma$ be an inclusion of totally ordered abelian groups. Let $F = K(t)$ be the rational function field in the variable t. For $\gamma \in \Gamma_{\geqslant 0}$ define $w_{v,t,\gamma} : F \to \Gamma \cup \infty$ as follows. For $f = f(t) = \sum_i a_i t_i \in K[t]$, set $w_{v,t,\gamma}(f) = \min_i \big(v(a_i) + i\gamma\big)$, and for $f/g \in F(t)$ set $w_{v,t,\gamma}(f/g) = w_{v,t,\gamma}(f) - w_{v,t,\gamma}(g)$.

7) In the above notation, let $w_{v,t} := w_{v,t,0_\Gamma}$. Prove/answer:

a) $w_{v,t} : F \to \Gamma \cup \infty$ is a valuation with value group vK and whose restriction to K is v.

Terminology. The valuation $w_{v,t}$ is called the *Gauss valuation* defined by v and $t \in F$. Hence if $R_{w_{v,t}}$ is the valuation ring of $w_{v,t}$, then $\mathfrak{m}_v = \mathfrak{m}_{w_{v,t}} \cap R_v$ (WHY).

b) Every $f \in K[t]$ is of the form $f = a_f f_0$ with $a_f \in K$, $w_{v,t}(f) = v(a_f)$, $f_0 \in R^\times_{w_{v,t}}$.

c) $t \in R^\times_{w_{v,t}}$ is a unit in $R_{w_{v,t}}$, and let $\bar{t} \in \kappa_{w_{v,t}}$ be the image of t in the residue field of $\kappa_{w_{v,t}} := R_{w_{v,t}}/\mathfrak{m}_{w_{v,t}}$. Then $\kappa_{w_{v,t}}$ is nothing but the rational function field $Kv(\bar{t})$.

8) In the above notation, suppose that $n\gamma \notin vK\ \forall\, n \in \mathbb{N}$. Setting $w_{v,\gamma} := w_{v,t,\gamma}$, prove/answer:

a) $w_{v,\gamma} : F \to \Gamma$ is a valuation with value group $vK + \mathbb{Z}\gamma$ and whose restriction to K is v.

b) $f \in K[t]$ has $w_{v,\gamma}(f) = 0$ iff $f \in R^\times$, hence constant. Describe the $w_{v,\gamma}$-units in $F(t)$.

c) The residue field $\kappa_{w_{v,\gamma}} = R_{w_{v,\gamma}}/\mathfrak{m}_{w_{v,\gamma}}$ equals the residue field Kv of v.

Miscellaneous. Let M be an R-module, and $(\mathcal{M}) :\ \cdots \to M_1 \to M_0 \to M \to 0$ be a projective resolution. If $M_n = (0)$ for $n \gg 0$, we say that $(\mathcal{M})$ is finite, and if n_0 is minimal with $M_{n_0} = (0)$, we say that $(\mathcal{M})$ has length n_0. Further, recall $\mathrm{Tor}_i^R(M,N)$ and $\mathrm{Ext}_R^i(M,N)$ and the related facts.

9) In the above notation/context, prove/disprove/answer:

a) Let R be a PID. Find projective resolutions $(\mathcal{M})$ of minimal length.
What can you say about $\mathrm{Tor}_i^R(M,N)$ and $\mathrm{Ext}_R^i(M,N)$?

b) Let R be a valuation ring. Then:
- every finite torsion-free R-module is free.
- every R-submodule N of a free R-module M is R-free iff R is a DVR.
- every R-module M has a finite resolution iff R is a DVR.

What can you say about $\mathrm{Tor}_i^R(M,N)$ and $\mathrm{Ext}_R^i(M,N)$ in case b) above?