5.5 Substitution

THE SUBSTITUTION RULE If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int f(g(x))\, g'(x)\, dx = \int f(u)\, du$$

inside function

outside function

$$\text{outside function} = \frac{\text{derivative of the}}{\text{inside function}} \left(\begin{array}{c}\text{or some}\\ \text{multiple of it}\end{array}\right)$$

Integration by Substitution: Evaluating $\int f(g(x))g'(x)\,dx$

Step 1 Let $u = g(x)$, where $g(x)$ is part of the integrand, usually, the "inside function" of the composite function $f(g(x))$.

Step 2 Compute $du = g'(x)\,dx$.

Step 3 Use the substitution $u = g(x)$ and $du = g'(x)\,dx$ to transform the integral into one that involves *only* u: $\int f(u)\,du$.

Step 4 Find the resulting integral.

example 1

$$\int_0^2 x^2\sqrt{1+x^3}\,dx$$

$$u = 1 + x^3$$

$$du = 3x^2 dx$$

$$\frac{1}{3}du = x^2 dx$$

Option 1 : Limit Switch

$$x = 2 \Rightarrow u = 9$$

$$x = 0 \Rightarrow u = 1$$

$$= \int_1^9 \sqrt{u}\cdot\frac{1}{3}du = \frac{1}{3}\int_1^9 u^{1/2}\cdot du$$

$$= \frac{1}{3}\cdot\frac{2}{3}\left[u^{3/2}\right]_1^9 = \frac{2}{9}\left(9^{3/2} - 1^{3/2}\right)$$

$$= \frac{2}{9}\left(9\sqrt{9} - 1\sqrt{1}\right) = \frac{2}{9}(27-1) = \mathbf{\frac{52}{9}}$$

example 2

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$$\int_{\pi/6}^{\pi/4} \csc^2 x \cot x \, dx$$

$f = \cot x$ $\quad u = \cot x$

$f' = -\csc^2 x$ $\quad du = -\csc^2 x dx$

$-du = \csc^2 x dx$

$$\int u(-du)$$

Option 2 : Don't Limit Switch

$$-\int u du = -\frac{u^2}{2}$$

now substitute back into x

$u = \cot x$

$$= \left[-\frac{1}{2}\cot^2 x \right]_{\pi/6}^{\pi/4} = -\frac{1}{2}\left[\left(\frac{\cos x}{\sin x} \right)^2 \right]_{\pi/6}^{\pi/4}$$

$$= -\frac{1}{2}\left[\underbrace{\left(\frac{\cos\left(\frac{\pi}{4}\right)}{\sin\left(\frac{\pi}{4}\right)} \right)^2}_{1} - \underbrace{\left(\frac{\cos\left(\frac{\pi}{6}\right)}{\sin\left(\frac{\pi}{6}\right)} \right)^2}_{3} \right] = -\frac{1}{2}[1-3] = \mathbf{1}$$

$\tan x = \frac{\sin x}{\cos x}$ so $\cot x = \frac{\cos x}{\sin x}$

$$\frac{\cos\left(\frac{\pi}{4}\right)}{\sin\left(\frac{\pi}{4}\right)} = \frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = 1$$

$$\frac{\cos\left(\frac{\pi}{6}\right)}{\sin\left(\frac{\pi}{6}\right)} = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3}$$

example 3

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$$\int_0^1 \sin(3\pi x) dx$$

$u = 3\pi x$

$du = 3\pi dx \qquad \frac{1}{3\pi} du = dx$

$$= \int \sin u \cdot \frac{1}{3\pi} du$$

$$= -\frac{1}{3\pi}\cos u$$

$$= \left[-\frac{1}{3\pi}\cos 3\pi x \right]_0^1$$

$$= -\frac{1}{3\pi}(\cos 3\pi - \cos 0)$$

$$= -\frac{1}{3\pi}(-1-1) = \mathbf{\frac{2}{3\pi}}$$

example 4

$$\int \frac{1}{\sqrt{x}\left(\sqrt{x}+1\right)^2}\,dx$$

$$u=\sqrt{x}+1 \qquad du=\frac{1}{2\sqrt{x}}\,dx$$

$$2du=\frac{1}{\sqrt{x}}\,dx$$

$$=2\int \frac{1}{u^2}\,du = 2\int u^{-2}\,du$$

$$=-2u^{-1}+C$$

Change back into x

$$=-\frac{2}{\sqrt{x}+1}+C$$

example 5

$$\int_2^5 \frac{x}{\sqrt{x-1}}\,dx$$

$$u = x-1 \quad \Rightarrow \quad x = u+1$$
$$du = dx$$

$$u = x-1$$
$$x=5 \Rightarrow u=4$$
$$x=2 \Rightarrow u=1$$

$$\int_2^5 \frac{x}{\sqrt{x-1}}\,dx = \int_1^4 \frac{u+1}{\sqrt{u}}\,du = \int_1^4\left(\frac{u}{\sqrt{u}}+\frac{1}{\sqrt{u}}\right)du = \int_1^4\left(\sqrt{u}+\frac{1}{\sqrt{u}}\right)du$$

$$=\int_1^4\left(u^{1/2}+u^{-1/2}\right)du = \left[\frac{u^{3/2}}{\frac{3}{2}}+\frac{u^{1/2}}{\frac{1}{2}}\right]_1^4 = \left[\frac{2}{3}u^{3/2}+2\sqrt{u}\right]_1^4$$

$$=\left(\frac{2}{3}4^{3/2}+2\sqrt{4}\right)-\left(\frac{2}{3}1^{3/2}+2\sqrt{1}\right)=\left(\frac{2}{3}\cdot 8+2\cdot 2\right)-\left(\frac{2}{3}+2\right)$$

$$=\frac{16}{3}-\frac{2}{3}+4-2 = \frac{14}{3}+2 = \frac{20}{3}$$

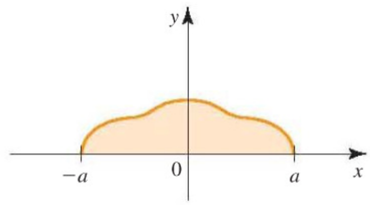
Integrals of Odd and Even Functions

Suppose that f is continuous on $[-a, a]$.

a. If f is even, then $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$.

$$\int_{-\pi}^{\pi}\left(1+x^2-\cos x\right)dx = 2\int_{0}^{\pi}\left(1+x^2-\cos x\right)dx$$

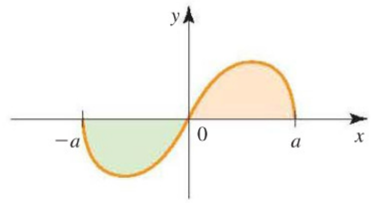
$$= 2\left(x+\frac{x^3}{3}+\sin x\right)\Bigg|_0^{\pi} = \mathbf{2\left(\pi+\frac{\pi^3}{3}\right)}$$



(a) $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$

b. If f is odd, then $\int_{-a}^{a} f(x)\,dx = 0$.

$$\int_{-2}^{2}\left(x^3-\sin x\right)dx = \mathbf{0}$$



(b) $\int_{-a}^{a} f(x)\,dx = 0$