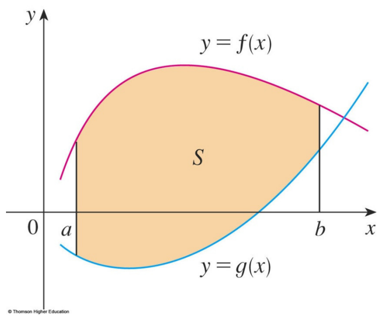
5.6 Area Between Curves



Consider the region S that lies between two curves $y = f(x)$ and $y = g(x)$ and between the vertical lines $x = a$ and $x = b$.

Here, f and g are continuous functions and $f(x) \geq g(x)$ for all x in $[a,b]$.

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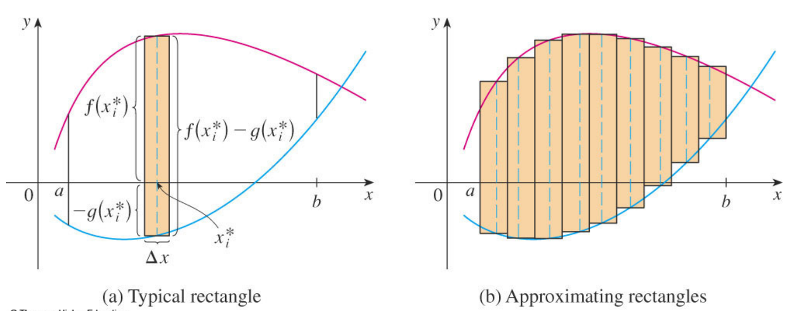

(a) Typical rectangle (b) Approximating rectangles

We divide S into n strips of equal width and approximate the i th strip by a rectangle with base Δx and height $f(x_i{*}) - g(x_i{*})$

The Riemann sum $\sum_{i=1}^{n}[f(x_i^*)-g(x_i^*)]\,\Delta x$

is therefore an approximation to what we intuitively think of as the area of S.

This approximation appears to become better and better as $n \to \infty$.

Thus, we define the area A of the region S as the limiting value of the sum of the areas of these approximating rectangles.

$$A=\lim_{n\to\infty}\sum_{i=1}^{n}[f(x_i^*)-g(x_i^*)]\Delta x$$

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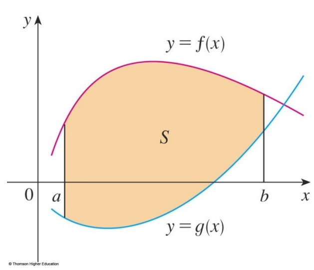
Thus, we have the following formula for area :

$$A=\int_a^b\left[f(x)-g(x)\right]dx$$



Remember S is described as the region bounded by the curves $y=f(x)$ and $y=g(x)$ and the lines $x=a$ and $x=b$, where, f and g are continuous and $f(x)\geq g(x)$ for all x in $[a,b]$.

$$A = \text{the area of } S$$

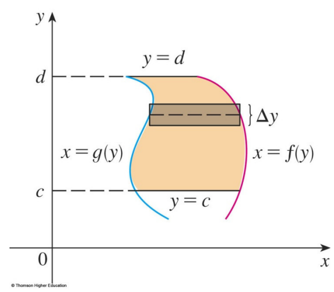
$$A=\int_a^b\left[\begin{pmatrix}\text{upper}\\\text{function}\end{pmatrix}-\begin{pmatrix}\text{lower}\\\text{function}\end{pmatrix}\right]dx$$

Some regions are best treated by regarding x as a function of y.

If a region is bouned by the curves $x = f(y)$ and $x = g(y)$ and the lines $y = c$ and $y = d$, where f and g are continuous and $f(y) \geq g(y)$ for all y in $[c, d]$, then its area is:

$$A = \int_c^d [f(y) - g(y)]\, dy$$

$$A = \int_c^d \left[\begin{pmatrix} \text{right} \\ \text{function} \end{pmatrix} - \begin{pmatrix} \text{left} \\ \text{function} \end{pmatrix} \right] dy$$

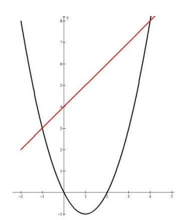


Find the area of the region bounded by the curves.

$y = x^2 - 2x, \ y = x + 4$

Select the correct answer.

a. $\frac{125}{3}$ b. $\frac{25}{3}$ c. 5 d. 20 e. $\frac{125}{6}$



Find where the curves intersect:

$$x^2 - 2x = x + 4$$
$$x^2 - 3x - 4 = 0$$
$$(x-4)(x+1) = 0$$
$$\Rightarrow x = 4, x = -1$$

$$\Rightarrow A = \int_{-1}^{4} \left[(x+4) - (x^2 - 2x) \right] dx = \int_{-1}^{4} \left[-x^2 + 3x + 4 \right] dx = \left[\frac{-x^3}{3} + \frac{3x^2}{2} + 4x \right]_{-1}^{4}$$

$$= \left[\frac{-4^3}{3} + \frac{3 \cdot 4^2}{2} + 4 \cdot 4 \right] - \left[\frac{-(-1)^3}{3} + \frac{3 \cdot (-1)^2}{2} + 4 \cdot (-1) \right]$$

$$= \left[\frac{-64}{3} + 24 + 16 \right] - \left[\frac{1}{3} + \frac{3}{2} - 4 \right]$$

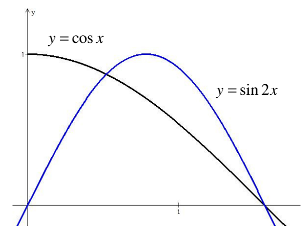
$$= \frac{-65}{3} + 44 - \frac{3}{2} = \frac{-130 + 264 - 9}{6} = \boxed{\frac{125}{6}}$$

Find the area of the region bounded by the curves.

$y = \cos x,\ y = \sin 2x,\ x = 0,\ x = \pi/2$

Select the correct answer.

a. $\frac{1}{4}$ b. $\frac{1}{2}$ c. $\frac{1}{3}$ d. 2 e. 4



Find where the curves intersect:

$\cos x = \sin 2x$

$\sin 2x - \cos x = 0$

Use the trig. ident. :

$\sin 2x = 2\sin x \cos x$

$2\sin x\cos x - \cos x = 0$

$\cos x(2\sin x - 1) = 0$

$\cos x = 0$ or $2\sin x - 1 = 0$

$\Rightarrow x = \frac{\pi}{2}$ or $\sin x = \frac{1}{2} \Rightarrow x = \frac{\pi}{6}$

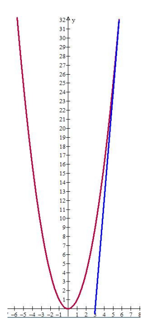
$$\Rightarrow A = \int_0^{\pi/6} [\cos x - \sin 2x]\,dx + \int_{\pi/6}^{\pi/2} [\sin 2x - \cos x]\,dx$$

$$= [\sin x + \tfrac{1}{2}\cos 2x]_0^{\pi/6} + [-\tfrac{1}{2}\cos 2x - \sin x]_{\pi/6}^{\pi/2}$$

$$= \left[\sin(\pi/6) + \tfrac{1}{2}\cos(\pi/3)\right] - \left[\sin(0) + \tfrac{1}{2}\cos(0)\right] + \left[-\tfrac{1}{2}\cos(\pi) - \sin(\pi/2)\right] - \left[-\tfrac{1}{2}\cos(\pi/3) - \sin(\pi/6)\right]$$

$$= [\tfrac{1}{2} + \tfrac{1}{4}] - [0 + \tfrac{1}{2}] + [\tfrac{1}{2} - 1] - [-\tfrac{1}{4} - \tfrac{1}{2}] = \tfrac{1}{4} + \tfrac{1}{4} = \boxed{\tfrac{1}{2}}$$

Find the area of the region bounded by the parabola $y = x^2$, the tangent line to this parabola at (6, 36), and the x-axis.



We need the equation of the tangent line

$y = x^2 \Rightarrow y' = 2x$ evaluated at $x = 6 \Rightarrow y' = 12$

this is the slope of the tangent line. The tangency pt. is $(6, 36)$

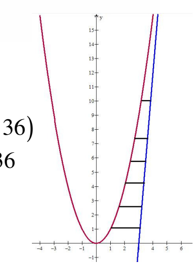
using $y = mx + b \Rightarrow 36 = 12(6) + b \Rightarrow 36 = 72 + b \Rightarrow b = -36$

$\Rightarrow$ the equation of the line is $y = 12x - 36$

$\Rightarrow$ the integral should be done in terms of dy

since the lower limit will change when you reach 3

$\Rightarrow$ the curves need to be expressed as functions of y



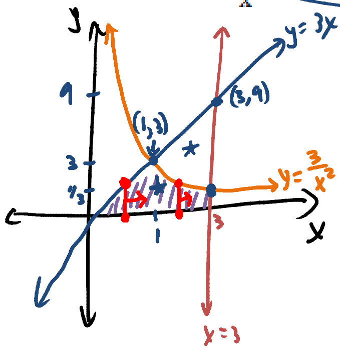
$y = x^2 \Rightarrow x = \sqrt{y}$

$y = 12x - 36 \Rightarrow x = \frac{y + 36}{12} \Rightarrow x = \frac{y}{12} + 3$

$$\Rightarrow A = \int_0^{36} \left[\left(\frac{y}{12} + 3\right) - \left(\sqrt{y}\right)\right] dy = \left(\frac{y^2}{24} + 3y - \frac{2y^{3/2}}{3}\right)_0^{36}$$

$$= \left(\frac{36^2}{24} + 3\cdot 36 - \frac{2\cdot 36^{3/2}}{3}\right) = 36\left(\frac{36}{24} + 3 - \frac{2\cdot 36^{1/2}}{3}\right) = 36\left(\frac{3}{2} + 3 - 4\right) = 36\left(\frac{1}{2}\right) = \boxed{18}$$

Find the area of the region in the first quadrant bounded by the line $y = 3x$, the line $x = 3$, the curve $y = \frac{3}{x^2}$, and the x-axis.



intersection b/w $y = 3x$ and $x = 3$

$y = 3(3) = 9$ $(3,9)$

intersection b/w $y = 3x$ and $y = \frac{3}{x^2}$

$x^2 \cdot 3x = \frac{3}{x^2} \cdot x^2$

$3x^3 = 3$

$x^3 = 1 \Rightarrow x = 1$

$y = 3$ $(1,3)$

int. b/w $y = \frac{3}{x^2}$ & $x = 3$ $(3, \frac{1}{3})$

$y = \frac{3}{(3)^2} = \frac{3}{9} = \frac{1}{3}$

$$A = \int_0^1 (3x - 0)\,dx + \int_1^3 \left(\frac{3}{x^2} - 0\right) dx$$

$$A = \int_0^1 3x\,dx + \int_1^3 3x^{-2}\,dx = \frac{3x^2}{2}\Big|_0^1 + \frac{3x^{-1}}{-1}\Big|_1^3 = \frac{3}{2}(1)^2 - 0 + \left[\frac{-3}{x}\right]_1^3 = \frac{3}{2} + (-1 - -3) = \boxed{7/2}$$