12.10 Taylor and Maclaurin Series

Suppose f is a function which has a power series representation

$$f(x)=\sum_{n=0}^{\infty}c_n(x-a)^n=c_0+c_1(x-a)+c_2(x-a)^2+c_3(x-a)^3+\cdots$$

for $|x-a|<R$

We can find the coefficients c_n in the following manner:

$$f(a)=c_0+c_1\underbrace{(a-a)}_{0}+c_2\underbrace{(a-a)^2}_{0}+c_3\underbrace{(a-a)^3}_{0}+\cdots \Rightarrow f(a)=c_0$$

Now let's take the derivative:

$$f'(x)=c_1+2c_2(x-a)+3c_3(x-a)^2+4c_4(x-a)^3+\cdots$$

$$f'(a)=c_1+2c_2\underbrace{(a-a)}_{0}+3c_3\underbrace{(a-a)^2}_{0}+4c_4\underbrace{(a-a)^3}_{0}+\cdots \Rightarrow f'(a)=c_1$$

Now let's take the second derivative:

$$f''(x)=2c_2+2\cdot 3c_3(x-a)+3\cdot 4c_4(x-a)^2+\cdots$$

$$f''(a)=2c_2+2\cdot 3c_3\underbrace{(a-a)}_{0}+3\cdot 4c_4\underbrace{(a-a)^2}_{0}+4c_5\underbrace{(a-a)^3}_{0}+\cdots$$

$$\Rightarrow f''(a)=2c_2$$

$$c_2=\frac{f''(a)}{2}$$

Finally let's take the third derivative:

$$f'''(x)=2\cdot 3c_3+2\cdot 3\cdot 4c_4(x-a)+3\cdot 4\cdot 5c_5(x-a)^2+\cdots$$

$$f'''(a)=2\cdot 3c_3+2\cdot 3\cdot 4c_4(a-a)+3\cdot 4\cdot 5c_5(a-a)^2+\cdots \Rightarrow f'''(a)=2\cdot 3c_3$$

$$c_3=\frac{f'''(a)}{2\cdot 3}$$

Continuing in this manner, we'll obtain:

$$c_4=\frac{f^{(4)}(a)}{2\cdot 3\cdot 4} \qquad c_5=\frac{f^{(5)}(a)}{2\cdot 3\cdot 4\cdot 5} \qquad \cdots \qquad c_n=\frac{f^{(n)}(a)}{n!}$$

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n = f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\frac{f'''(a)}{3!}(x-a)^3+\cdots$$

Taylor series of the function f centered at a.

If $a=0$, then we call the series the **Maclaurin series** of the function f.

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n = f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3+\cdots$$

Find the Maclaurin series for $f(x)=e^x$

$f(x)=e^x$	$f(0)=1 \div 0!$	$=1$
$f'(x)=e^x$	$f'(0)=1 \div 1!$	$=1\,x$
$f''(x)=e^x$	$f''(0)=1 \div 2!$	$=\frac{1}{2!}x^2$
$f'''(x)=e^x$	$f'''(0)=1 \div 3!$	$=\frac{1}{3!}x^3$
$f^{(4)}(x)=e^x$	$f^{(4)}(0)=1 \div 4!$	$=\frac{1}{4!}x^4$
$\vdots$	$\vdots$	$\vdots$

$$\frac{f^{(n)}(0)}{n!}=\frac{1}{n!} \qquad \frac{f^{(n)}(0)}{n!}x^n=\frac{x^n}{n!}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{n!}{(n+1)!}\cdot\frac{x^{n+1}}{x^n}\right|$$

$$=\lim_{n\to\infty}\left|\frac{\cancel{n!}}{(n+1)\cdot\cancel{n!}}\cdot\frac{\cancel{x^n}\cdot x}{\cancel{x^n}}\right|$$

$$=\lim_{n\to\infty}\left|\frac{x}{(n+1)}\right| = 0<1$$

for all x

$\Rightarrow R=\infty$

$$e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots \quad \text{, with } R=\infty$$

Find the Maclaurin series for $f(x) = \sin x$

$$
\begin{array}{lll}
f(x) = \sin x & f(0) = 0 \ \div 0! & = 0 \\
f'(x) = \cos x & f'(0) = 1 \ \div 1! & = 1\,x \\
f''(x) = -\sin x & f''(0) = 0 \ \div 2! & = 0\,x^2 \\
f'''(x) = -\cos x & f'''(0) = -1 \div 3! & = -\frac{1}{3!}x^3 \\
f^{(4)}(x) = \sin x & f^{(4)}(0) = 0 \ \div 4! & = 0\,x^4 \\
f^{(5)}(x) = \cos x & f^{(5)}(0) = 1 \ \div 5! & = \frac{1}{5!}\,x^5
\end{array}
$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(2n+1)!}{(2(n+1)+1)!}\cdot\frac{x^{\overset{2n+3}{2(n+1)+1}}}{x^{2n+1}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{\cancel{(2n+1)!}}{(2n+3)(2n+2)\cdot\cancel{(2n+1)!}}\cdot\frac{\cancel{x^{2n+1}}\cdot x^2}{\cancel{x^{2n+1}}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{x^2}{(2n+3)(2n+2)}\right| = 0 < 1 \text{ for all } x$$

$$\Rightarrow R = \infty$$

only odd powers so we should use $2n-1$ or $2n+1$
the first term (when $n=0$) is x^1 so we choose $2n+1$

$$\frac{f^{(n)}(0)}{n!}x^n = \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\sin x = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \quad \text{, with } R = \infty$$

Find the Maclaurin series for $f(x) = (1+x)^k$

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$$
\begin{array}{ll}
f(x) = (1+x)^k & f(0) = 1 \\
f'(x) = k(1+x)^{k-1} & f'(0) = k \\
f''(x) = k(k-1)(1+x)^{k-2} & f''(0) = k(k-1) \\
f'''(x) = k(k-1)(k-2)(1+x)^{k-3} & f'''(0) = k(k-1)(k-2) \\
f^{(4)}(x) = k(k-1)(k-2)(k-3)(1+x)^{k-4} & f^{(4)}(0) = k(k-1)(k-2)(k-3) \\
 & \vdots
\end{array}
$$

$$f^{(n)}(0) = k(k-1)(k-2)\cdots\underbrace{(k-(n-1))}_{k-n+1}$$

stop when you get to this term

$$\underbrace{\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n}_{\text{Maclaurin Series for } f(x)} = \sum_{n=0}^{\infty}\frac{k(k-1)(k-2)\cdots(k-n+1)}{n!}x^n$$

$$(1+x)^k = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots$$

$$(1+x)^k=\sum_{n=0}^{\infty}\frac{k(k-1)(k-2)\cdots(k-n+1)}{n!}x^n$$

Find its radius of convergence.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{k(k-1)(k-2)\cdots(k-(n+1)+1)}{k(k-1)(k-2)\cdots(k-n+1)}\cdot\frac{n!}{(n+1)!}\cdot\frac{x^{n+1}}{x^n}\right|$$

$$=\lim_{n\to\infty}\left|\frac{\cancel{k(k-1)(k-2)\cdots(k-n+1)}(k-n)}{\cancel{k(k-1)(k-2)\cdots(k-n+1)}}\cdot\frac{\cancel{n!}}{(n+1)\cdot\cancel{n!}}\cdot\frac{\cancel{x^n}\cdot x}{\cancel{x^n}}\right|$$

$$=\lim_{n\to\infty}\left|\frac{k-n}{n+1}\cdot x\right|=|-x|=|x|<1 \qquad \Rightarrow R=1$$

for convergence

$$\lim_{n\to\infty}\frac{k-n}{n+1}=-1$$

In a probablility course you learn that $\dfrac{k(k-1)(k-2)\cdots(k-n+1)}{n!}$ is the number of ways to select n objects out of a total of k objects

The symbol often used for this is $\binom{k}{n}$

List of important Maclaurin series :

$$\frac{1}{1-x}=\sum_{n=0}^{\infty}x^n=1+x+x^2+x^3+\cdots \quad R=1$$

$$\frac{1}{(1-x)^2}=\sum_{n=1}^{\infty}nx^{n-1}=1+2x+3x^2+4x^3+\cdots \quad \text{with } R=1$$

$$\ln(1-x)=-\sum_{n=0}^{\infty}\frac{x^{n+1}}{n+1}=-x-\frac{x^2}{2}-\frac{x^3}{3}-\frac{x^4}{4}-\cdots \quad \text{with } R=1$$

$$\arctan x=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}=x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\cdots \quad \text{,with } R=1$$

$$e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots \quad \text{,with } R=\infty$$

$$\sin x=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots \quad \text{,with } R=\infty$$

$$\cos x=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots \quad \text{,with } R=\infty$$

$$(1+x)^k=\sum_{n=0}^{\infty}\binom{k}{n}x^n=1+kx+\frac{k(k-1)}{2!}x^2+\frac{k(k-1)(k-2)}{3!}x^3+\cdots \quad \text{,with } R=1$$

Find the Taylor series for $f(x)=\sqrt{x}$ centered at $x=1$.

$$f(x)=x^{1/2} \qquad f(1)=1 \div 0! \quad =1$$
$$f'(x)=\tfrac{1}{2}x^{-1/2} \qquad f'(1)=\tfrac{1}{2} \div 1! \quad =\tfrac{1}{2}\,(x-1)$$
$$f''(x)=\tfrac{-1}{4}x^{-3/2} \qquad f''(1)=\tfrac{-1}{4} \div 2! \quad =\tfrac{-1}{4\cdot 2!}(x-1)^2$$
$$f'''(x)=\tfrac{3}{8}x^{-5/2} \qquad f'''(1)=\tfrac{3}{8} \div 3! \quad =\tfrac{3}{8\cdot 3!}(x-1)^3$$
$$f^{(4)}(x)=\tfrac{-15}{16}x^{-7/2} \quad f^{(4)}(1)=\tfrac{-15}{16}\div 4! = \tfrac{-5\cdot 3}{16\cdot 4!}(x-1)^4$$
$$f^{(5)}(x)=\tfrac{7\cdot 15}{32}x^{-9/2} \quad f^{(5)}(1)=\tfrac{7\cdot 15}{32}\div 5! = \tfrac{7\cdot 5\cdot 3}{32\cdot 5!}(x-1)^5$$
$$f^{(6)}(x)=\tfrac{-9\cdot 7\cdot 15}{64}x^{-11/2} \quad f^{(6)}(1)=\tfrac{-9\cdot 7\cdot 15}{64}\div 6! = \tfrac{-9\cdot 7\cdot 5\cdot 3}{64\cdot 6!}(x-1)^6$$

$$f(x)=1+\tfrac{1}{2}(x-1)+\sum_{n=2}^{\infty}\frac{(-1)^{n-1}\left(1\cdot 3\cdot 5\cdot \cdots \cdot (2n-3)\right)}{2^n n!}(x-1)^n$$

Find the Taylor series for $f(x)=\cos x$ centered at $x=\dfrac{\pi}{4}$.

$$f(x)=\cos x \qquad f\left(\tfrac{\pi}{4}\right)=\tfrac{\sqrt{2}}{2} \div 0! = \tfrac{\sqrt{2}}{2}$$
$$f'(x)=-\sin x \qquad f'\left(\tfrac{\pi}{4}\right)=-\tfrac{\sqrt{2}}{2} \div 1! = -\tfrac{\sqrt{2}}{2}\left(x-\tfrac{\pi}{4}\right)$$
$$f''(x)=-\cos x \qquad f''\left(\tfrac{\pi}{4}\right)=-\tfrac{\sqrt{2}}{2} \div 2! = -\tfrac{\sqrt{2}}{2\cdot 2!}\left(x-\tfrac{\pi}{4}\right)^2$$

these are both negative

$$f'''(x)=\sin x \qquad f'''\left(\tfrac{\pi}{4}\right)=\tfrac{\sqrt{2}}{2} \div 3! = \tfrac{\sqrt{2}}{2\cdot 3!}\left(x-\tfrac{\pi}{4}\right)^3$$
$$f^{(4)}(x)=\cos x \qquad f^{(4)}\left(\tfrac{\pi}{4}\right)=\tfrac{\sqrt{2}}{2} \div 4! = \tfrac{\sqrt{2}}{2\cdot 4!}\left(x-\tfrac{\pi}{4}\right)^4$$
$$f^{(5)}(x)=-\sin x \quad f^{(5)}\left(\tfrac{\pi}{4}\right)=-\tfrac{\sqrt{2}}{2} \div 5! = -\tfrac{\sqrt{2}}{2\cdot 5!}\left(x-\tfrac{\pi}{4}\right)^5$$

we have to split into odd and even powers and have each alternate

$$\underset{\text{centered at } x=\frac{\pi}{4}}{\cos x}=\frac{\sqrt{2}}{2}\left[\sum_{n=0}^{\infty}\frac{(-1)^{2n}\left(x-\frac{\pi}{4}\right)^{2n}}{(2n)!}+\sum_{n=0}^{\infty}\frac{(-1)^{n+1}\left(x-\frac{\pi}{4}\right)^{2n+1}}{(2n+1)!}\right]$$

Use a power series to integrate a function when there is no integration technique you could use.

$$\int x^3 e^{-x^3}\,dx$$

$$e^{-x} = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} \quad \Rightarrow e^{-x^3} = \sum_{n=0}^{\infty} \frac{(-x^3)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n}}{n!}$$

$$x^3 e^{-x^3} = x^3 \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n}}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n} x^3}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n+3}}{n!}$$

$$\int x^3 e^{-x^3}\,dx = \int \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n+3}}{n!}\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int x^{3n+3}\,dx$$

$$= \boxed{C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n+4}}{n!(3n+4)}}$$

Use a power series to evaluate a limit.

$$\lim_{x\to 0} \frac{\sin x - x + \frac{x^3}{6}}{x^5}$$

$$\lim_{x\to 0} \frac{\left(\cancel{x} - \cancel{\frac{x^3}{3!}} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right) \cancel{-x} + \cancel{\frac{x^3}{6}}}{x^5} \qquad = \lim_{x\to 0} \frac{\frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \cdots}{x^5}$$

$$= \lim_{x\to 0} \left(\frac{x^5}{5!\,x^5} - \frac{x^7}{7!\,x^5} + \frac{x^9}{9!\,x^5} - \frac{x^{11}}{11!\,x^5} + \cdots \right)$$

$$= \lim_{x\to 0} \left(\frac{1}{5!} - \underbrace{\frac{x^2}{7!}}_{0} + \underbrace{\frac{x^4}{9!}}_{0} - \underbrace{\frac{x^6}{11!}}_{0} + \cdots \right) = \boxed{\frac{1}{120}}$$

Use a power series to find the sum of a series.

$$1+\pi+\frac{\pi^2}{2!}+\frac{\pi^3}{3!}+\frac{\pi^4}{4!}+\cdots=\sum_{n=0}^{\infty}\frac{\pi^n}{n!}=\boxed{e^{\pi}}$$

$$\frac{\pi}{2}-\frac{\pi^3}{3!\cdot 8}+\frac{\pi^5}{5!\cdot 32}-\frac{\pi^7}{7!\cdot 128}+\cdots=\sum_{n=0}^{\infty}\frac{(-1)^n\pi^{2n+1}}{2^{2n+1}(2n+1)!}=\sin\frac{\pi}{2}=\boxed{1}$$

$$1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots=\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}=\arctan 1=\boxed{\frac{\pi}{4}}$$

Find the first three non-zero terms of the Maclaurin series for $f(x)=e^x\ln(1-x)$ by **multiplying** two series.

$$e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots\text{ , with } R=\infty \text{ and } I=(-\infty,\infty)$$

$$\ln(1-x)=-\sum_{n=0}^{\infty}\frac{x^{n+1}}{n+1}=-x-\frac{x^2}{2}-\frac{x^3}{3}-\frac{x^4}{4}-\cdots\text{ , with } R=1 \text{ and } I=|x|<1$$

$$e^x\ln(1-x)=\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots\right)\left(-x-\frac{x^2}{2}-\frac{x^3}{3}-\frac{x^4}{4}-\cdots\right)$$

$$-x\ -\frac{x^2}{2}-\frac{x^3}{3}\ \cdots$$

$$-x^2-\frac{x^3}{2}\ \cdots$$

$$-\frac{x^3}{2}\ \cdots$$

$$e^x\ln(1-x)=-x-\frac{3}{2}x^2-\frac{4}{3}x^3-\cdots\text{ , with } R=1 \text{ and } I=|x|<1$$

the overlap b/w
the two intervals

Find the first three non-zero terms of the Maclaurin series for $f(x) = \tan x$ by **dividing** two series.

$$\tan x = \frac{\sin x}{\cos x} = \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots}{1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots}$$

$$x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots$$

$$1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \cdots \Big) \, x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \cdots$$

$$-\left(x - \frac{x^3}{2} + \frac{x^5}{24} - \frac{x^7}{720} + \cdots \right)$$

$$\frac{x^3}{3} - \frac{x^5}{30} + \frac{x^7}{840} + \cdots$$

$$-\left(\frac{x^3}{3} - \frac{x^5}{6} + \frac{x^7}{72} - \cdots \right)$$

$$\frac{2x^5}{15} - \frac{4x^7}{315} + \cdots$$

$$-\frac{x^3}{6} + \frac{x^3}{2} = \frac{-1+3}{6} x^3 = \frac{x^3}{3}$$

$$\frac{x^5}{120} - \frac{x^5}{24} = \frac{1-5}{120} x^5 = -\frac{x^5}{30}$$

$$-\frac{x^7}{5040} + \frac{x^7}{720} = \frac{-1+7}{5040} x^7 = \frac{x^5}{840}$$

$$-\frac{x^5}{30} + \frac{x^5}{6} = \frac{-1+5}{30} x^5 = \frac{2x^5}{15}$$

$$\frac{x^7}{840} - \frac{x^7}{72} = \frac{3-35}{2520} x^7 = \frac{-4x^7}{315}$$

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots$$