12.10 Taylor and Maclaurin Series

Suppose f is a function which has a power series representation

$$f(x)=\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$$

for $|x-a| < R$

We can find the coefficients c_n in the following manner:

$$f(a) = c_0 + c_1\underbrace{(a-a)}_{0} + c_2\underbrace{(a-a)^2}_{0} + c_3\underbrace{(a-a)^3}_{0} + \cdots \Rightarrow f(a) = c_0$$

Now let's take the derivative:

$$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + \cdots$$

$$f'(a) = c_1 + 2c_2\underbrace{(a-a)}_{0} + 3c_3\underbrace{(a-a)^2}_{0} + 4c_4\underbrace{(a-a)^3}_{0} + \cdots \Rightarrow f'(a) = c_1$$

Now let's take the second derivative:

$$f''(x) = 2c_2 + 2\cdot 3c_3(x-a) + 3\cdot 4c_4(x-a)^2 + \cdots$$

$$f''(a) = 2c_2 + 2\cdot 3c_3\underbrace{(a-a)}_{0} + 3\cdot 4c_4\underbrace{(a-a)^2}_{0} + 4c_5\underbrace{(a-a)^3}_{0} + \cdots$$

$$\Rightarrow f''(a) = 2c_2$$

$$c_2 = \frac{f''(a)}{2}$$

Finally let's take the third derivative:

$$f'''(x) = 2\cdot 3c_3 + 2\cdot 3\cdot 4c_4(x-a) + 3\cdot 4\cdot 5c_5(x-a)^2 + \cdots$$

$$f'''(a) = 2\cdot 3c_3 + 2\cdot 3\cdot 4c_4(a-a) + 3\cdot 4\cdot 5c_5(a-a)^2 + \cdots \Rightarrow f'''(a) = 2\cdot 3c_3$$

$$c_3 = \frac{f'''(a)}{2\cdot 3}$$

Continuing in this manner, we'll obtain:

$$c_4 = \frac{f^{(4)}(a)}{2\cdot 3\cdot 4} \qquad c_5 = \frac{f^{(5)}(a)}{2\cdot 3\cdot 4\cdot 5} \qquad \cdots \qquad c_n = \frac{f^{(n)}(a)}{n!}$$

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n = f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\frac{f'''(a)}{3!}(x-a)^3+\cdots$$

__________ **series** of the function f centered at a.

If $a=0$, then we call the series the ____________ **series** of the function f.

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n = f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3+\cdots$$

Find the Maclaurin series for $f(x)=e^x$

$f(x)=e^x$	$f(0)=1$
$f'(x)=e^x$	$f'(0)=1$
$f''(x)=e^x$	$f''(0)=1$
$f'''(x)=e^x$	$f'''(0)=1$
$f^{(4)}(x)=e^x$	$f^{(4)}(0)=1$
$\vdots$	$\vdots$

$\vdots$

$$\frac{f^{(n)}(0)}{n!}x^n =$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{n!}{(n+1)!}\cdot\frac{x^{n+1}}{x^n}\right|$$

$$=\lim_{n\to\infty}\left|\frac{\cancel{n!}}{(n+1)\cdot\cancel{n!}}\cdot\frac{\cancel{x^n}\cdot x}{\cancel{x^n}}\right|$$

$$=\lim_{n\to\infty}\left|\frac{x}{(n+1)}\right| = 0<1$$

for all x

$\Rightarrow R=\infty$

$e^x =$, with $R=\infty$

Find the Maclaurin series for $f(x) = \sin x$

$$f(x) = \sin x \qquad f(0) =$$
$$f'(x) = \cos x \qquad f'(0) =$$
$$f''(x) = -\sin x \qquad f''(0) =$$
$$f'''(x) = -\cos x \qquad f'''(0) =$$
$$f^{(4)}(x) = \sin x \qquad f^{(4)}(0) =$$
$$f^{(5)}(x) = \cos x \qquad f^{(5)}(0) =$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(2n+1)!}{(2(n+1)+1)!}\cdot\frac{x^{2(n+1)+1}}{x^{2n+1}}\right| \quad (2(n+1)+1 = 2n+3)$$

$$= \lim_{n\to\infty}\left|\frac{(2n+1)!}{(2n+3)(2n+2)\cdot(2n+1)!}\cdot\frac{x^{2n+1}\cdot x^2}{x^{2n+1}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{x^2}{(2n+3)(2n+2)}\right| = 0 < 1$$ for all x

$\Rightarrow R = \infty$

only odd powers so we should use $2n-1$ or $2n+1$
the first term (when $n=0$) is x^1 so we choose $2n+1$

$$\frac{f^{(n)}(0)}{n!}x^n =$$

$\sin x =$, with $R = \infty$

Find the Maclaurin series for $f(x) = (1+x)^k$

$$f(x) = (1+x)^k \qquad f(0) = 1$$
$$f'(x) = k(1+x)^{k-1} \qquad f'(0) = k$$
$$f''(x) = k(k-1)(1+x)^{k-2} \qquad f''(0) = k(k-1)$$
$$f'''(x) = k(k-1)(k-2)(1+x)^{k-3} \qquad f'''(0) = k(k-1)(k-2)$$
$$f^{(4)}(x) = k(k-1)(k-2)(k-3)(1+x)^{k-4} \qquad f^{(4)}(0) = k(k-1)(k-2)(k-3)$$
$$\vdots$$
$$f^{(n)}(0) = k(k-1)(k-2)\cdots(k-(n-1))$$

$k-n+1$
stop when you get to this term

$$\underbrace{\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n}_{\text{Maclaurin Series for } f(x)} = \sum_{n=0}^{\infty}\frac{k(k-1)(k-2)\cdots(k-n+1)}{n!}x^n$$

$$(1+x)^k = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots$$

$$(1+x)^k = \sum_{n=0}^{\infty} \frac{k(k-1)(k-2)\cdots(k-n+1)}{n!} x^n$$

Find its radius of convergence.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{k(k-1)(k-2)\cdots(k-(n+1)+1)}{k(k-1)(k-2)\cdots(k-n+1)}\cdot\frac{n!}{(n+1)!}\cdot\frac{x^{n+1}}{x^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{\cancel{k(k-1)(k-2)\cdots(k-n+1)}(k-n)}{\cancel{k(k-1)(k-2)\cdots(k-n+1)}}\cdot\frac{\cancel{n!}}{(n+1)\cdot\cancel{n!}}\cdot\frac{\cancel{x^n}\cdot x}{\cancel{x^n}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{k-n}{n+1}\cdot x\right| = |-x| = |x| < 1 \quad \Rightarrow R = 1$$

for convergence

$$\lim_{n\to\infty}\frac{k-n}{n+1} = -1$$

In a probablility course you learn that $\frac{k(k-1)(k-2)\cdots(k-n+1)}{n!}$ is the number of ways to select n objects out of a total of k objects

The symbol often used for this is $\binom{k}{n}$

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List of important Maclaurin series :

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots \quad R = 1$$

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} nx^{n-1} = 1 + 2x + 3x^2 + 4x^3 + \cdots \quad \text{with } R = 1$$

$$\ln(1-x) = -\sum_{n=0}^{\infty}\frac{x^{n+1}}{n+1} = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots \quad \text{with } R = 1$$

$$\arctan x = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \quad \text{,with } R = 1$$

$$e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \quad \text{,with } R = \infty$$

$$\sin x = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \quad \text{,with } R = \infty$$

$$\cos x = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \quad \text{,with } R = \infty$$

$$(1+x)^k = \sum_{n=0}^{\infty}\binom{k}{n}x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots \quad \text{,with } R = 1$$

Find the Taylor series for $f(x)=\sqrt{x}$ centered at $x=1$.

$f(x)=x^{1/2}$ $\qquad f(1)=1$

$f'(x)=\frac{1}{2}x^{-1/2}$ $\qquad f'(1)=\frac{1}{2}$

$f''(x)=\frac{-1}{4}x^{-3/2}$ $\qquad f''(1)=\frac{-1}{4}$

$f'''(x)=\frac{3}{8}x^{-5/2}$ $\qquad f'''(1)=\frac{3}{8}$

$f^{(4)}(x)=\frac{-15}{16}x^{-7/2}$ $\qquad f^{(4)}(1)=\frac{-15}{16}$

$f^{(5)}(x)=\frac{7\cdot 15}{32}x^{-9/2}$ $\qquad f^{(5)}(1)=\frac{7\cdot 15}{32}$

$f^{(6)}(x)=\frac{-9\cdot 7\cdot 15}{64}x^{-11/2}$ $\qquad f^{(6)}(1)=\frac{-9\cdot 7\cdot 15}{64}$

$f(x)=$

Find the Taylor series for $f(x)=\cos x$ centered at $x=\dfrac{\pi}{4}$.

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$f(x)=\cos x$ $\qquad f\left(\frac{\pi}{4}\right)=\frac{\sqrt{2}}{2}\ \div 0! = \frac{\sqrt{2}}{2}$

$f'(x)=-\sin x$ $\qquad f'\left(\frac{\pi}{4}\right)=-\frac{\sqrt{2}}{2}\ \div 1! = -\frac{\sqrt{2}}{2}\left(x-\frac{\pi}{4}\right)$

$f''(x)=-\cos x$ $\qquad f''\left(\frac{\pi}{4}\right)=-\frac{\sqrt{2}}{2}\div 2! = -\frac{\sqrt{2}}{2\cdot 2!}\left(x-\frac{\pi}{4}\right)^2$

these are both negative

$f'''(x)=\sin x$ $\qquad f'''\left(\frac{\pi}{4}\right)=\frac{\sqrt{2}}{2}\ \div 3! = \frac{\sqrt{2}}{2\cdot 3!}\left(x-\frac{\pi}{4}\right)^3$

$f^{(4)}(x)=\cos x$ $\qquad f^{(4)}\left(\frac{\pi}{4}\right)=\frac{\sqrt{2}}{2}\ \div 4! = \frac{\sqrt{2}}{2\cdot 4!}\left(x-\frac{\pi}{4}\right)^4$

$f^{(5)}(x)=-\sin x$ $\qquad f^{(5)}\left(\frac{\pi}{4}\right)=-\frac{\sqrt{2}}{2}\div 5! = -\frac{\sqrt{2}}{2\cdot 5!}\left(x-\frac{\pi}{4}\right)^5$

we have to split into odd and even powers and have each alternate

$$\underset{\text{centered at } x=\frac{\pi}{4}}{\cos x}=\frac{\sqrt{2}}{2}\left[\sum_{n=0}^{\infty}\frac{(-1)^{2n}\left(x-\frac{\pi}{4}\right)^{2n}}{(2n)!}+\sum_{n=0}^{\infty}\frac{(-1)^{n+1}\left(x-\frac{\pi}{4}\right)^{2n+1}}{(2n+1)!}\right]$$

Use a power series to integrate a function when there is no integration technique you could use.

$$\int x^3 e^{-x^3}\,dx$$

$$e^{-x} = \sum_{n=0}^{\infty} \qquad \Rightarrow e^{-x^3} = \sum_{n=0}^{\infty} \qquad = \sum_{n=0}^{\infty}$$

$$x^3 e^{-x^3} =$$

Use a power series to evaluate a limit.

$$\lim_{x\to 0} \frac{\sin x - x + \frac{x^3}{6}}{x^5}$$

Use a power series to find the sum of a series.

$$1+\pi+\frac{\pi^2}{2!}+\frac{\pi^3}{3!}+\frac{\pi^4}{4!}+\cdots$$

$$\frac{\pi}{2}-\frac{\pi^3}{3!\cdot 8}+\frac{\pi^5}{5!\cdot 32}-\frac{\pi^7}{7!\cdot 128}+\cdots$$

$$1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots$$

Find the first three non-zero terms of the Maclaurin series for $f(x)=e^x\ln(1-x)$ by __________ two series.

$$e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots$$, with $R=\infty$ and $I=(-\infty,\infty)$

$$\ln(1-x)=-\sum_{n=0}^{\infty}\frac{x^{n+1}}{n+1}=-x-\frac{x^2}{2}-\frac{x^3}{3}-\frac{x^4}{4}-\cdots$$, with $R=1$ and $I=|x|<1$

$$e^x\ln(1-x)=$$

Find the first three non-zero terms of the Maclaurin series for $f(x) = \tan x$ by ________ two series.

$$\tan x = \frac{\sin x}{\cos x} = \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots}{1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots}$$

$$1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \cdots \overline{\big)\, x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \cdots}$$

$\tan x =$

$$-\frac{x^3}{6} + \frac{x^3}{2} = \frac{-1+3}{6} x^3 = \frac{x^3}{3}$$

$$\frac{x^5}{120} - \frac{x^5}{24} = \frac{1-5}{120} x^5 = -\frac{x^5}{30}$$

$$-\frac{x^7}{5040} + \frac{x^7}{720} = \frac{-1+7}{5040} x^7 = \frac{x^5}{840}$$

$$-\frac{x^5}{30} + \frac{x^5}{6} = \frac{-1+5}{30} x^5 = \frac{2x^5}{15}$$

$$\frac{x^7}{840} - \frac{x^7}{72} = \frac{3-35}{2520} x^7 = \frac{-4x^7}{315}$$