12.2 Series

We will now add the terms of an infinite sequence $\{a_n\}_{n=1}^{\infty}$

to get $\underbrace{a_1 + a_2 + a_3 + \cdots + a_n + a_{n+1} + \cdots}$

this is called an infinite **series**

Notation: $\sum_{n=1}^{\infty} a_n$

Example:

$$2 + \frac{2}{3} + \frac{2}{9} + \frac{2}{27} + \frac{2}{81} + \cdots + \frac{2}{3^{n-1}} + \cdots$$

S_n = the sum of the first n terms

it is called the nth partial sum

$$S_n = a_1 + a_2 + a_3 + \cdots + a_n = \sum_{k=1}^{n} a_k$$

$S_1 = 2$

$S_2 = 2 + \frac{2}{3} = \frac{8}{3}$

$S_3 = 2 + \frac{2}{3} + \frac{2}{9} = \frac{26}{9}$

$S_4 = 2 + \frac{2}{3} + \frac{2}{9} + \frac{2}{27} = \frac{80}{27}$

The partial sums form a sequence $\{S_n\}_{n=1}^{\infty}$

$$\{S_n\}_{n=1}^{\infty} = \left\{2, \frac{8}{3}, \frac{26}{9}, \frac{80}{27}, \cdots\right\}$$

n	S_n
1	2
2	2.66666
3	2.88888
4	2.96296
5	2.98765
10	2.99995
15	2.99999
20	2.99999
25	**2.99999**

$\lim_{n\to\infty} S_n = s$ (the limit of the sequence of partial sums exists and is finite) $\Rightarrow$ We call s the **sum** of the infinite series

$$\sum_{n=1}^{\infty} a_n = s$$

and the series is called **convergent**

(by adding sufficiently many terms of the series, we can get as close as we like to the number s.)

otherwise the series is called **divergent**

The harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.
We will show this in 12.3

$\{S_n\}_{n=1}^{\infty} = \left\{2, \frac{8}{3}, \frac{26}{9}, \frac{80}{27}, \cdots\right\}$ It seems like $\lim_{n\to\infty} S_n = 3$

$$\Rightarrow 2 + \frac{2}{3} + \frac{2}{9} + \frac{2}{27} + \cdots + \frac{2}{3^{n-1}} + \cdots = \sum_{n=1}^{\infty} \frac{2}{3^{n-1}} = 3$$

We can show that the sum is 3 since this series is an example of a special type of series called a geometric series.

A **geometric series** is one in which each term is obtained from the preceding one by multiplying it by the common ratio r.

$$a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1}$$

this only converges for certain values of r.

$r = 1$

$S_n = a + a + a + a + \cdots = na \Rightarrow \lim_{n\to\infty} na = \infty$ so it diverges for $r = 1$

$r = -1$

$S_n = a - a + a - a + \cdots = (-1)^{n-1} a \Rightarrow \underset{\left(\text{it could be } a\text{, it could be } 0 \text{ depending on the value of } n\right)}{\lim_{n\to\infty} (-1)^{n-1} a}$ does not exist, so it diverges for $r = -1$

$r \neq \pm 1$

$$S_n = a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} = a\left(1 + r + r^2 + r^3 + \cdots + r^{n-1}\right)$$

$$-rS_n = -ar - ar^2 - ar^3 - \cdots - ar^n = -a\left(r + r^2 + r^3 + \cdots + r^n\right)$$

$$S_n - rS_n = a\left(1 - r^n\right)$$

$$\Rightarrow S_n(1-r) = a\left(1 - r^n\right)$$

$$\Rightarrow S_n = \frac{a\left(1 - r^n\right)}{1 - r}$$

so, $\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{a\left(1-r^n\right)}{1-r} = \frac{a\left(1 - \lim_{n\to\infty} r^n\right)}{1-r} = \frac{a\left(1 - \overset{=0}{\lim_{n\to\infty} r^n}\right)}{1-r} = \frac{a}{1-r}$

if $-1 < r < 1$

We saw in section 12.1 :

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } -1 < r < 1 \\ 1 & \text{if } r = 1 \end{cases}$$

$$\lim_{n\to\infty} r^n = \begin{cases} \text{undefined} & \text{if } r \in (-\infty, -1] \\ \infty & \text{if } r \in (1, \infty) \end{cases}$$

so, $\lim_{n\to\infty} S_n = \frac{a}{1-r}$ provided that $-1 < r < 1$ or $|r| < 1$.

The geometric series $\sum_{n=1}^{\infty} ar^{n-1}$ converges to the sum of $\frac{a}{1-r}$ if $|r| < 1$

The geometric series diverges for all other values of r

Back to our example:

$$2+\frac{2}{3}+\frac{2}{9}+\frac{2}{27}+\cdots+\frac{2}{3^{n-1}}+\cdots = 2\left(1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\cdots+\frac{1}{3^{n-1}}+\cdots\right)$$

$a =$ the first term

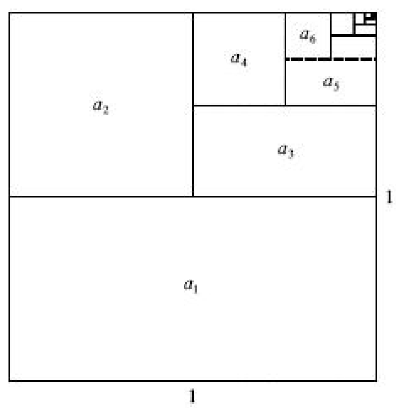
$r =$ the ratio b/w the terms

$a = 2$

$r = \frac{1}{3}$

$|r| < 1$

$$s = \frac{a}{1-r} = \frac{2}{1-\frac{1}{3}} = \frac{2}{\frac{2}{3}} = 2\cdot\frac{3}{2} = \boxed{3}$$



Area of square $= 1$

sum of the series should also be 1

Find $\sum_{n=1}^{\infty} a_n$.

$a = \frac{1}{2}, r = \frac{1}{2}$ $\quad s = \frac{a}{1-r} = \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1$

$$\sum_{n=1}^{\infty} a_n = \frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots+\frac{1}{2^n}+\cdots = 1$$

Represent $2.\overline{15}$ as an improper fraction by using a geometric series.

$$2.\overline{15} = 2.151515\ldots$$

$$= 2 + \underbrace{\frac{15}{100} + \frac{15}{10,000} + \frac{15}{1,000,000} + \cdots}$$

a geometric series with $a = \frac{15}{100}$ and $r = \frac{1}{100}$

$$s = \frac{a}{1-r} = \frac{\frac{15}{100}}{1-\frac{1}{100}} = \frac{\frac{15}{100}}{\frac{99}{100}} = \frac{15}{99} = \frac{5}{33}$$

$$2.\overline{15} = 2\frac{5}{33} = \frac{66+5}{33}$$

$$2.\overline{15} = \boxed{\frac{71}{33}}$$

A **telescoping series** is one in which the middle terms cancel and the sum collapses into just a few terms.

Example:

$$\sum_{n=1}^{\infty} \frac{3}{n(n+3)} = \sum_{n=1}^{\infty} \left[\frac{1}{n} - \frac{1}{n+3}\right]$$

$$\frac{3}{n(n+3)} = \frac{A}{n} + \frac{B}{n+3}$$

$$3 = A(n+3) + Bn$$

$$n = 0 \Rightarrow A = 1$$

$$n = -3 \Rightarrow B = -1$$

$$S_n = \underbrace{\left(1-\frac{1}{4}\right)}_{n=1} + \underbrace{\left(\frac{1}{2}-\frac{1}{5}\right)}_{n=2} + \underbrace{\left(\frac{1}{3}-\frac{1}{6}\right)}_{n=3} + \underbrace{\left(\frac{1}{4}-\frac{1}{7}\right)}_{n=4} + \underbrace{\left(\frac{1}{5}-\frac{1}{8}\right)}_{n=5} + \underbrace{\left(\frac{1}{6}-\frac{1}{9}\right)}_{n=6} + \cdots$$

$$+ \underbrace{\left(\frac{1}{n-2}-\frac{1}{n+1}\right)}_{(n-2)\text{ term}} + \underbrace{\left(\frac{1}{n-1}-\frac{1}{n+2}\right)}_{(n-1)\text{ term}} + \underbrace{\left(\frac{1}{n}-\frac{1}{n+3}\right)}_{\text{nth term}}$$

$$S_n = 1 + \frac{1}{2} + \frac{1}{3} - \frac{1}{n+1} - \frac{1}{n+2} - \frac{1}{n+3}$$

$$\text{so, } \lim_{n\to\infty} S_n = \lim_{n\to\infty}\left[1 + \frac{1}{2} + \frac{1}{3} - \underbrace{\frac{1}{n+1}}_{0} - \underbrace{\frac{1}{n+2}}_{0} - \underbrace{\frac{1}{n+3}}_{0}\right] = \frac{6+3+2}{6} = \boxed{\frac{11}{6}}$$

If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$.

Converse :

If $\lim_{n\to\infty} a_n = 0$, then the series $\sum_{n=1}^{\infty} a_n$ is convergent. This is **False**!

(Just because $\lim_{n\to\infty} a_n = 0$, you **cannot** conclude that the series $\sum_{n=1}^{\infty} a_n$ is convergent.)

$\lim_{n\to\infty} \frac{1}{n} = 0$, but $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

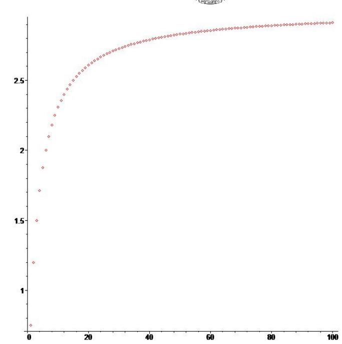
Contrapositive :

Test for Divergence :

If $\lim_{n\to\infty} a_n \neq 0$ or $\lim_{n\to\infty} a_n$ does not exist, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

This is **True**!

$$\sum_{n=1}^{\infty} \frac{3n^2}{n(n+3)} = \frac{3}{4} + \frac{6}{5} + \cdots + \underbrace{\frac{150}{53}}_{n=50} + \cdots + \underbrace{\frac{300}{103}}_{n=100} + \cdots$$



There is no way that this sum could converge to some finite number since the terms approach 3 as $n \to \infty$.

$\text{sum} = 0.75 + 1.2 + \cdots + 3 + 3 + 3 + \cdots + 3 + 3 + 3 + \cdots$

$$\sum_{n=1}^{\infty} \frac{3n^2}{n(n+3)} \qquad \lim_{n\to\infty} \frac{3n^2}{n(n+3)} = 3 \neq 0$$

so this series diverges by the **Test for Divergence**.

Just remember that if you get 0 for the limit, you **can't** conclude that the series converges. This just means that it has a chance to converge.

If $\sum a_n$ and $\sum b_n$ are convergent series,

then so are the series $\sum ca_n$ (where c is a constant),

$\sum(a_n+b_n)$, and $\sum(a_n-b_n)$,

i) $\sum ca_n = c\sum a_n$

example:

$$\sum_{n=1}^{\infty} 7\cdot\frac{2}{3^{n-1}} = 7\cdot\sum_{n=1}^{\infty}\frac{2}{3^{n-1}}$$

$$=7\cdot 3$$

$$=\boxed{21}$$

ii) $\sum(a_n+b_n)=\sum a_n+\sum b_n$

iii) $\sum(a_n-b_n)=\sum a_n-\sum b_n$

example:

$$\sum_{n=1}^{\infty}\left[\frac{2}{3^{n-1}}-\frac{3}{n(n+3)}\right]=\underbrace{\sum_{n=1}^{\infty}\frac{2}{3^{n-1}}}_{3}-\underbrace{\sum_{n=1}^{\infty}\frac{3}{n(n+3)}}_{\frac{11}{6}}$$

$$=3-\frac{11}{6}$$

$$=\frac{18-11}{6}=\boxed{\frac{7}{6}}$$