12.2 Series

Math 104 – Rimmer
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We will now add the terms of an infinite sequence $\{a_n\}_{n=1}^{\infty}$

to get $\underbrace{a_1 + a_2 + a_3 + \cdots + a_n + a_{n+1} + \cdots}$

this is called an infinite ________

Notation:

Example:

$$2 + \frac{2}{3} + \frac{2}{9} + \frac{2}{27} + \frac{2}{81} + \cdots + \frac{2}{3^{n-1}} + \cdots$$

S_n = the sum of the first n terms

it is called the ______________

$$S_n = a_1 + a_2 + a_3 + \cdots + a_n = \sum_{k=1}^{n} a_k$$

$S_1 =$

$S_2 = 2 + \frac{2}{3} =$

$S_3 = 2 + \frac{2}{3} + \frac{2}{9} =$

$S_4 = 2 + \frac{2}{3} + \frac{2}{9} + \frac{2}{27} =$

The partial sums form a sequence $\{S_n\}_{n=1}^{\infty}$

$\{S_n\}_{n=1}^{\infty} =$

n	S_n
1	2
2	2.66666
3	2.88888
4	2.96296
5	2.98765
10	2.99995
15	2.99999
20	2.99999
25	**2.99999**

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$\lim_{n\to\infty} S_n = s \quad \Rightarrow$ We call s the _____ of the infinite series

(the limit of the sequence of partial sums exists and is finite)

$$\sum_{n=1}^{\infty} a_n = s$$

and the series is called ____________

(by adding sufficiently many terms of the series, we can get as close as we like to the number s.)

otherwise the series is called ___________

The harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ ________.
We will show this in 12.3

$\{S_n\}_{n=1}^{\infty} = \left\{2, \frac{8}{3}, \frac{26}{9}, \frac{80}{27}, \cdots\right\}$ It seems like $\lim_{n\to\infty} S_n = 3$

$$\Rightarrow 2 + \frac{2}{3} + \frac{2}{9} + \frac{2}{27} + \cdots + \frac{2}{3^{n-1}} + \cdots = \sum_{n=1}^{\infty} \frac{2}{3^{n-1}} = 3$$

We can show that the sum is 3 since this series is an example of a special type of series called a ___________ series.

A ________________ is one in which each term is obtained from the preceding one by multiplying it by the common ratio r.

$$a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1}$$

this only converges _________________.

$\underline{r = 1}$

$S_n = a + a + a + a + \cdots = na \Rightarrow \lim_{n\to\infty} na =$

$\underline{r = -1}$

$S_n = a - a + a - a + \cdots = (-1)^{n-1} a \Rightarrow \lim_{n\to\infty} (-1)^{n-1} a$

(it could be a, it could be 0 depending on the value of n)

$\underline{r \neq \pm 1}$

$$S_n = a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} = a\left(1 + r + r^2 + r^3 + \cdots + r^{n-1}\right)$$

$$rS_n = ar + ar^2 + ar^3 + \cdots + ar^n = a\left(r + r^2 + r^3 + \cdots + r^n\right)$$

$$S_n - rS_n = \qquad a\left(1 - r^n\right)$$

$$\Rightarrow S_n(1-r) = a\left(1-r^n\right)$$

$$\Rightarrow S_n = \frac{a\left(1-r^n\right)}{1-r}$$

$$\text{so, } \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{a\left(1-r^n\right)}{1-r} = \frac{a\left(1-\lim_{n\to\infty} r^n\right)}{1-r} = \frac{a\left(1-\overbrace{\lim_{n\to\infty} r^n}^{=0}\right)}{1-r} = \frac{a}{1-r}$$

if $-1 < r < 1$

We saw in section 12.1 :

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } -1 < r < 1 \\ 1 & \text{if } r = 1 \end{cases}$$

$$\lim_{n\to\infty} r^n = \begin{cases} \text{undefined} & \text{if } r \in (-\infty, -1] \\ \infty & \text{if } r \in (1, \infty) \end{cases}$$

so, $\lim_{n\to\infty} S_n = \frac{a}{1-r}$ provided that $-1 < r < 1$ or $|r| < 1$.

The geometric series $\sum_{n=1}^{\infty} ar^{n-1}$ converges to the sum of $\frac{a}{1-r}$ if $|r| < 1$

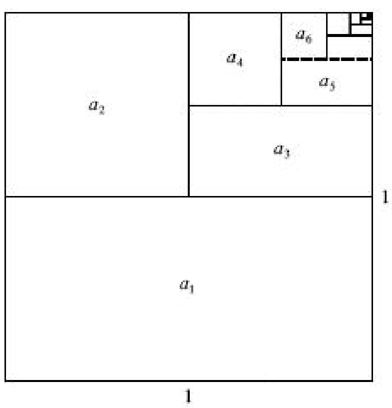
The geometric series diverges for all other values of r

Back to our example:

$$2+\frac{2}{3}+\frac{2}{9}+\frac{2}{27}+\cdots+\frac{2}{3^{n-1}}+\cdots \;=2\left(1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\cdots+\frac{1}{3^{n-1}}+\cdots\right)$$

$a=$ the first term

$r=$ the ratio b/w the terms



Area of square $=1$

sum of the series should also be 1

Find $\sum_{n=1}^{\infty} a_n$.

$$\sum_{n=1}^{\infty} a_n=\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots+\frac{1}{2^n}+\cdots \;=1$$

Represent $2.\overline{15}$ as an improper fraction by using a geometric series.

$$2.\overline{15} = 2.151515\ldots$$

$$= 2 +$$

a geometric series with $a =$ ____ and $r =$ ____

$$s = \frac{a}{1-r} =$$

A **telescoping series** is one in which the middle terms cancel and the sum collapses into just a few terms.

Example:

$$\sum_{n=1}^{\infty} \frac{3}{n(n+3)}$$

$S_n =$

If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then ______________.

Converse:

If $\lim_{n\to\infty} a_n = 0$, then the series $\sum_{n=1}^{\infty} a_n$ is convergent. This is ________

$\left(\text{Just because } \lim_{n\to\infty} a_n = 0, \text{ you } \_\_\_\_\_\_\_ \text{ conclude that the series } \sum_{n=1}^{\infty} a_n \text{ is convergent.}\right)$

Contrapositive:

Test for Divergence:

$$\sum_{n=1}^{\infty} \frac{3n^2}{n(n+3)}$$

If $\sum a_n$ and $\sum b_n$ are convergent series,

then so are the series $\sum ca_n$ (where c is a constant),

$\sum (a_n + b_n)$, and $\sum (a_n - b_n)$,

i) $\sum ca_n =$

ii) $\sum (a_n + b_n) =$

iii) $\sum (a_n - b_n) =$