12.3 The Integral Test

Math 104 – Rimmer
12.3 Integral Test

If $f(x)$ is: $a)$ continuous, on the interval $[k,\infty)$ (constant $k>0$)

$b)$ positive,

$c)$ and decreasing

,then the series $\sum_{n=k}^{\infty} a_n$ $\left(\text{with } a_n = f(n)\right)$

$i)$ is convergent when $\int_k^{\infty} f(x)\,dx$ is convergent.

$ii)$ is divergent when $\int_k^{\infty} f(x)\,dx$ is divergent.

Note:

the function does not necessarily have to be decreasing for all $x \in [k,\infty)$

as long as the function is decreasing "eventually"

(there is some number N so that f is decreasing for all $x > N$)

The next two slides give you a feeling of **how** the integral test works.

Math 104 – Rimmer
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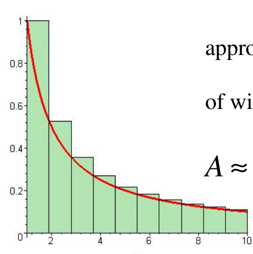
$f(x) = \dfrac{1}{x}$

on $[1,\infty)$

$a)$ continuous,

$b)$ positive,

$c)$ and decreasing



approximate the area $\int_1^{\infty} \frac{1}{x}\,dx$ with rectangles

of width 1 using the left endpoint

$$A \approx 1(1) + 1\left(\tfrac{1}{2}\right) + 1\left(\tfrac{1}{3}\right) + \cdots = 1 + \tfrac{1}{2} + \tfrac{1}{3} + \cdots$$

$$A \approx \sum_{n=1}^{\infty} \frac{1}{n}$$ but this is an **overestimate**

$$\Rightarrow \int_1^{\infty} \frac{1}{x}\,dx < \sum_{n=1}^{\infty} \frac{1}{n}$$

$$\text{But } \int_1^{\infty} \frac{1}{x}\,dx = \lim_{b\to\infty} \int_1^{b} \frac{1}{x}\,dx = \lim_{b\to\infty} \ln x \Big|_1^b = \lim_{b\to\infty} \ln b = \infty$$

The integral $\int_1^{\infty} \frac{1}{x}\,dx$ diverges and $\int_1^{\infty} \frac{1}{x}\,dx < \sum_{n=1}^{\infty} \frac{1}{n}$

$\Rightarrow$ The series $\sum_{n=1}^{\infty} \frac{1}{n}$ must also diverge ← the **harmonic series**

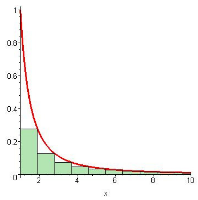
$f(x)=\dfrac{1}{x^2}$

on $[1,\infty)$

$a)$ continuous,

$b)$ positive,

$c)$ and decreasing



approximate the area $\int_1^\infty \frac{1}{x^2}dx$ with rectangles of width 1 using the right endpoint

$$A\approx 1\left(\tfrac{1}{4}\right)+1\left(\tfrac{1}{9}\right)+1\left(\tfrac{1}{16}\right)+\cdots=\tfrac{1}{4}+\tfrac{1}{9}+\tfrac{1}{16}+\cdots$$

$$A\approx\sum_{n=2}^{\infty}\frac{1}{n^2}$$ but this is an **underestimate**

$$\Rightarrow \sum_{n=2}^{\infty}\frac{1}{n^2}<\int_1^\infty\frac{1}{x^2}dx \quad \Rightarrow 1+\sum_{n=2}^{\infty}\frac{1}{n^2}<1+\int_1^\infty\frac{1}{x^2}dx$$

$$\Rightarrow \sum_{n=1}^{\infty}\frac{1}{n^2}<1+\int_1^\infty\frac{1}{x^2}dx$$

But $\displaystyle\int_1^\infty\frac{1}{x^2}dx=\lim_{b\to\infty}\int_1^b x^{-2}dx=\lim_{b\to\infty}\left.\frac{-1}{x}\right|_1^b=\lim_{b\to\infty}\frac{-1}{b}+1=1$

The integral $\displaystyle\int_1^\infty\frac{1}{x^2}dx$ converges and $\Rightarrow \displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}<2$

(The sequence of partial sums S_n is a bounded increasing sequence $\Rightarrow$ this sequence converges)

$\Rightarrow$ The series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$ also converges

$f(x)=\dfrac{1}{x^p}$

on $[1,\infty)$

$a)$ continuous,

$b)$ positive,

$c)$ and decreasing

For what values of p does the integral converge?

$$\int_1^\infty\frac{1}{x^p}dx=\lim_{b\to\infty}\int_1^b x^{-p}dx=\lim_{b\to\infty}\left.\frac{x^{-p+1}}{-p+1}\right|_1^b$$

need $-p+1$ to be negative so that we can get convergence by moving the $x-$term to the denominator

$$-p+1<0\Rightarrow \boxed{p>1}$$

corresponding to this function is the series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^p}$

this is called a $p-$series

$i)$ $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^p}$ **converges** when $p>1$

$ii)$ $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^p}$ **diverges** when $p\le 1$

Which of these converge?

$a)\ \sum_{n=1}^{\infty}\frac{1}{n^{5/2}}\qquad b)\ \sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}\qquad c)\ \sum_{n=1}^{\infty}\frac{3}{2n^3}\qquad d)\ \sum_{n=1}^{\infty}n^{-e}$

$a)\ \sum_{n=1}^{\infty}\frac{1}{n^{5/2}}$ **converges** $p-$series with $p=2.5$

$b)\ \sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ **diverges** $p-$series with $p=\frac{1}{2}$

$c)\ \sum_{n=1}^{\infty}\frac{3}{2n^3}=\frac{3}{2}\sum_{n=1}^{\infty}\frac{1}{n^3}$ **converges** $p-$series with $p=3$

$d)\ \sum_{n=1}^{\infty}n^{-e}=\sum_{n=1}^{\infty}\frac{1}{n^e}$ **converges** $p-$series with $p=e$

Which of these converge?

$a)\ \sum_{n=1}^{\infty}\frac{1}{n^2+4}\qquad b)\ \sum_{n=2}^{\infty}\frac{1}{n\ln n}\qquad c)\ \sum_{n=2}^{\infty}\frac{\ln n}{n^2}$

$a)\ \sum_{n=1}^{\infty}\frac{1}{n^2+4}$ $\quad f(x)=\frac{1}{x^2+4}$ continuous,positive, and decreasing on $[1,\infty)$

$$\int_1^{\infty}\frac{1}{x^2+4}\,dx=\lim_{b\to\infty}\int_1^{b}\frac{1}{x^2+4}\,dx=\lim_{b\to\infty}\frac{1}{2}\arctan\left(\frac{x}{2}\right)\Big|_1^b$$

$$=\lim_{b\to\infty}\frac{1}{2}\arctan\left(\frac{b}{2}\right)-\arctan\left(\frac{1}{2}\right)\qquad\left(\arctan x\to\frac{\pi}{2}\text{ as }x\to\infty\right)$$

$$=\frac{\pi}{4}-\frac{1}{2}\arctan\left(\frac{1}{2}\right)\quad\Rightarrow\ \text{the integral converges}$$

so, $\sum_{n=1}^{\infty}\frac{1}{n^2+4}$ **converges** by the integral test.

Which of these converge?

$a)\ \sum_{n=1}^{\infty}\frac{1}{n^2+4}$ $b)\ \sum_{n=2}^{\infty}\frac{1}{n\ln n}$ $c)\ \sum_{n=2}^{\infty}\frac{\ln n}{n^2}$

$b)\ \sum_{n=2}^{\infty}\frac{1}{n\ln n}$ $f(x)=\frac{1}{x\ln x}$ continuous,positive, and decreasing on $[2,\infty)$

$\left(f'(x)=\frac{-[\ln(x)+1]}{(x\ln x)^2}\text{ always negative on }[2,\infty)\right)$

$$\int_2^{\infty}\frac{1}{x\ln x}dx=\lim_{b\to\infty}\int_2^{b}\frac{1}{x\ln x}dx=\lim_{b\to\infty}\ln(\ln x)\Big|_2^b=\lim_{b\to\infty}\ln(\ln b)-\ln(\ln 2)$$

$u=\ln x$
$du=\frac{1}{x}dx \quad \int\frac{1}{u}du=\ln|u|+C$

$=\infty$ $(\ln x\to\infty \text{ as } x\to\infty)$

$\Rightarrow$ the integral diverges

so, $\sum_{n=2}^{\infty}\frac{1}{n\ln n}$ **diverges** by the integral test.

Which of these converge?

Math 104 – Rimmer
12.3 Integral Test

$a)\ \sum_{n=1}^{\infty}\frac{1}{n^2+4}$ $b)\ \sum_{n=2}^{\infty}\frac{1}{n\ln n}$ $c)\ \sum_{n=2}^{\infty}\frac{\ln n}{n^2}$

$f'(x)=\frac{x-2x\ln(x)}{x^4}=\frac{1-2\ln(x)}{x^3}$
$1-2\ln x<0\Rightarrow\ln x>\frac{1}{2}\Rightarrow x>e^{1/2}$
f is decreasing for $x>\sqrt{e}\sim 1.65$

$c)\ \sum_{n=2}^{\infty}\frac{\ln n}{n^2}$ $f(x)=\frac{\ln x}{x^2}$ continuous,positive, and decreasing on $[2,\infty)$

$$\int_2^{\infty}\frac{\ln x}{x^2}dx=\lim_{b\to\infty}\int_2^{b}\frac{\ln x}{x^2}dx=\lim_{b\to\infty}\frac{-(1+\ln x)}{x}\Bigg|_2^b=\lim_{b\to\infty}\frac{-(1+\ln b)}{b}+\frac{1+\ln 2}{2}=\frac{1+\ln 2}{2}$$

$u=\ln x \quad dv=\frac{1}{x^2}dx$
$du=\frac{1}{x}dx \quad v=\frac{-1}{x}$

$$uv-\int v\,du=\frac{-1}{x}\ln x+\int\frac{1}{x^2}dx=\frac{-1}{x}\ln x+\frac{-1}{x}=\frac{-(1+\ln x)}{x}$$

since, $\lim_{b\to\infty}\frac{-(\ln b+1)}{b}=\frac{-\infty}{\infty}\overset{L'H}{=}\lim_{b\to\infty}\frac{-\frac{1}{b}}{1}=0$

$\Rightarrow$ the integral converges

so, $\sum_{n=1}^{\infty}\frac{\ln n}{n^2}$ **converges** by the integral test.

Remainder Estimate for the Integral Test

If $f(x)$ is: $a)$ continuous, on the interval $\underset{\text{constant } k>0}{[k,\infty)}$
$b)$ positive,
$c)$ and decreasing $\left(\text{with } a_n = f(n)\right)$

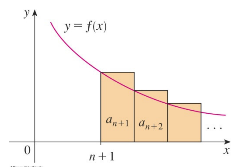
and the series $\sum_{n=k}^{\infty} a_n$ is convergent.

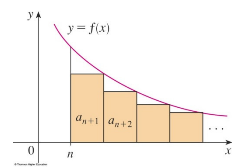
, then the remainder $R_n = s_n - s$ can be bounded above by $\int_n^{\infty} f(x)\,dx$.

$$s = \underbrace{a_1 + a_2 + a_3 + \cdots + a_n}_{s_n} + \underbrace{a_{n+1} + a_{n+2} + a_{n+3} + \cdots}_{R_n}$$

$\left(\text{and below by } \int_{n+1}^{\infty} f(x)\,dx.\right)$

$$\Rightarrow R_n = a_{n+1} + a_{n+2} + a_{n+3} + \cdots$$





$$\int_{n+1}^{\infty} f(x)\,dx \le R_n \le \int_n^{\infty} f(x)\,dx$$

Exercise # 34

Find the sum of the series $\sum_{n=1}^{\infty} \frac{1}{n^5}$ correct to 3 decimal places.

First find the value of n so that s_n is within 0.0005.

$$\int_n^{\infty} \frac{1}{x^5}\,dx = \lim_{b\to\infty}\left(\frac{-1}{4x^4}\right)_n^b = \underbrace{\lim_{b\to\infty}\left(\frac{-1}{4b^4}\right)}_{0} - \left(\frac{-1}{4n^4}\right) = \frac{1}{4n^4}$$

$$R_n \le \int_n^{\infty} f(x)\,dx = \frac{1}{4n^4} \overset{set}{=} \frac{5}{10{,}000} = \frac{1}{2{,}000}$$

$$\frac{1}{4n^4} = \frac{1}{2{,}000} \Rightarrow 4n^4 = 2{,}000 \Rightarrow n^4 = 500 \Rightarrow n = \sqrt[4]{500} \approx 4.73$$

So we need to go out to **5 terms**.

$$\sum_{n=1}^{\infty} \frac{1}{n^5} \approx 1 + \frac{1}{32} + \frac{1}{243} + \frac{1}{1024} + \frac{1}{3125} \approx 1.036661789$$

The actual value is $\zeta(5) \approx 1.036927755$ (look up the Riemann Zeta Function)