12.4 Comparison Tests

The Comparison Test:

Given the series $\sum_{n=1}^{\infty} a_n$, $(a_n \geq 0)$

(i) if the terms a_n are **smaller** than the terms b_n of a known **convergent** series $\sum_{n=1}^{\infty} b_n$ $(b_n \geq 0)$, then our series $\sum_{n=1}^{\infty} a_n$ is also **convergent**.

(ii) if the terms a_n are **larger** than the terms b_n of a known **divergent** series $\sum_{n=1}^{\infty} b_n$ $(b_n \geq 0)$, then our series $\sum_{n=1}^{\infty} a_n$ is also **divergent**.

For the series $\sum_{n=1}^{\infty} b_n$, it must be known whether it converges or diverges, so it is usually chosen to be a $p-$series or a geometric series.

search for the **dominating terms** in both the numerator and the denominator of a_n, choose your b_n to be the ratio of these dominating terms

Consider the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1} \cdot 4^n}$

Choose $b_n = \frac{1}{4^n}$ since as n gets large 4^n is much larger than $\sqrt{n+1}$.

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{4^n} \text{ is a convergent geometric series}$$

$\underbrace{\frac{1}{\sqrt{n+1} \cdot 4^n}}_{a_n} < \underbrace{\frac{1}{4^n}}_{b_n}$ for $n > 1$ since you are making the denominator smaller by taking away $\sqrt{n+1}$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1} \cdot 4^n} \text{ also converges by the Comparison Test}$$

Consider the series $\sum_{n=9}^{\infty} \frac{\sqrt{n}}{n-8}$

Choose $b_n = \frac{\sqrt{n}}{n} = \frac{1}{\sqrt{n}}$ since as n gets large n is much larger than 8.

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \text{ is a divergent } p\text{-series}$$

$$\underbrace{\frac{1}{\sqrt{n}}}_{b_n} < \underbrace{\frac{\sqrt{n}}{n-8}}_{a_n} \text{ since } n-8 < n$$

$$\Rightarrow \sum_{n=9}^{\infty} \frac{\sqrt{n}}{n-8} \text{ also diverges by the Comparison Test}$$

The inequality $a_n \le b_n$ or $b_n \le a_n$ doesn't need to be satisfied for all values of n.
If it doesn't hold for the first few terms but it holds for all $n > N$ for some N, then the comparison test will still work.

Consider the series $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ Choose $b_n = \frac{1}{n}$ $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n}$ is the harmonic series, so it is divergent

Since $\sum_{n=1}^{\infty} b_n$ is divergent, the inequality should be $\frac{1}{n} \le \frac{\ln n}{n}$

$$\Rightarrow n \le n \ln n \quad \Rightarrow \frac{n}{n} \le \frac{n \ln n}{n} \quad \Rightarrow 1 \le \ln n \quad \Rightarrow e^1 \le e^{\ln n} \quad \Rightarrow n > e$$

The inequality doesn't hold for $n = 1$ or $n = 2$ but it holds for all $n \ge 3$

The convergence or divergence of the series does not depend on the first two terms. These terms can be subtracted off and we can look at both series starting at $n = 3$.

$$\sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges } \Rightarrow \sum_{n=1}^{\infty} \frac{\ln n}{n} \text{ also diverges by the Comparison Test}$$

Consider the series $\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^3+1}}$

Choose $b_n = \frac{n}{n^{3/2}} = \frac{1}{\sqrt{n}}$ since as n gets large n the "+8" won't matter.

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \text{ is a divergent } p-\text{series}$$

but the inequality is going the wrong way!

$$\underbrace{\frac{n}{\sqrt{n^3+1}}}_{a_n} < \underbrace{\frac{1}{\sqrt{n}}}_{b_n} \text{ since } n^{3/2} < \sqrt{n^3+1}$$

The Comparison Test does not apply. $\Rightarrow$ We must use another test.

The Limit Comparison Test:

Given the series $\sum_{n=1}^{\infty} a_n,\ (a_n > 0)$ and a known

convergent or divergent series $\sum_{n=1}^{\infty} b_n,\ (b_n > 0)$

If the $\lim_{n\to\infty} \frac{a_n}{b_n} = c$ where c is a finite positive number, then

the series will behave alike, i.e. either both converge or both diverge.

Back to the series $\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^3+1}}$. Choose the same $b_n = \frac{n}{n^{3/2}} = \frac{1}{\sqrt{n}}$.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{n}{\sqrt{n^3+1}}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} \frac{n^{3/2}}{\sqrt{n^3+1}} = \lim_{n\to\infty} \frac{n^{3/2}}{\sqrt{n^3+1}} \cdot \frac{\frac{1}{n^{3/2}}}{\frac{1}{n^{3/2}}}$$

$$= \lim_{n\to\infty} \frac{1}{\sqrt{\frac{n^3}{n^3} + \frac{1}{n^3}}} = 1$$

$\Rightarrow$ the series will behave alike by the Limit Comparison Test

$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges $\Rightarrow \sum_{n=1}^{\infty} \frac{n}{\sqrt{n^3+1}}$ also diverges

Consider the series $\sum_{n=1}^{\infty} \frac{1+3^n}{4+2^n}$ Choose $b_n = \frac{3^n}{2^n}$

$\sum_{n=1}^{\infty} \left(\frac{3}{2}\right)^n$ is a divergent geometric series w/ $r = \frac{3}{2}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1+3^n}{4+2^n}}{\frac{3^n}{2^n}} = \lim_{n\to\infty} \frac{1+3^n}{4+2^n} \cdot \frac{2^n}{3^n} = \lim_{n\to\infty} \frac{2^n+6^n}{4\cdot 3^n + 6^n} = \lim_{n\to\infty} \frac{\frac{2^n}{6^n} + \frac{6^n}{6^n}}{\frac{4\cdot 3^n}{6^n} + \frac{6^n}{6^n}}$$

$$= \lim_{n\to\infty} \frac{\left(\frac{1}{3}\right)^n + 1}{4\cdot\left(\frac{1}{2}\right)^n + 1} = \lim_{n\to\infty} \frac{\overbrace{\left(\frac{1}{3}\right)^n}^{\to 0} + 1}{4\cdot \overbrace{\left(\frac{1}{2}\right)^n}^{\to 0} + 1} = 1$$

$\Rightarrow$ the series will behave alike by the Limit Comparison Test

$\sum_{n=1}^{\infty} \left(\frac{3}{2}\right)^n$ diverges $\Rightarrow \sum_{n=1}^{\infty} \frac{1+3^n}{4+2^n}$ also diverges

Consider the series $\sum_{n=1}^{\infty} \frac{3n+4}{(2n+1)^3}$ Choose $b_n = \frac{n}{n^3} = \frac{1}{n^2}$

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent $p-$series w/ $p=2$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{3n+4}{(2n+1)^3}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{3n+4}{(2n+1)^3} \cdot \frac{n^2}{1} = \lim_{n\to\infty} \frac{3n^3+4n^2}{(2n+1)^3} = \frac{3}{8}$$

$\deg(num.) = \deg.(denom.)$

$$\lim_{x\to\infty} \frac{a_m x^m + a_{m-1}x^{m-1} + \cdots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0} = \frac{a_m}{b_m}$$

$\Rightarrow$ the series will behave alike by the Limit Comparison Test

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges $\Rightarrow \sum_{n=1}^{\infty} \frac{3n+4}{(2n+1)^3}$ also converges