# 12.5 Alternating Series Test



An **alternating series** is of the form $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (-1)^{n-1} b_n$ or $\sum_{n=1}^{\infty} (-1)^n b_n$, (where $b_n > 0$)

(it has successive terms of opposite signs) $b_n = |a_n|$

Example: $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} - \cdots$

Example: $\sum_{n=1}^{\infty} \frac{(-1)^n n^2}{n+5} = -\frac{1}{6} + \frac{4}{7} - \frac{9}{8} + \frac{16}{9} - \cdots$

Forms for the term that makes the series alternate in sign:

$(-1)^{n-1}$ $(-1)^n$ $(-1)^{n+1}$

$\cos(n\pi)$ $\sin\left((2n-1)\frac{\pi}{2}\right)$



# The Alternating Series Test

If the alternating series $\sum_{n=1}^{\infty} (-1)^{n-1} b_n$ (where $b_n > 0$) satisfies:

$i)$ $\lim_{n\to\infty} b_n = 0$

$ii)$ $\{b_n\}$ is a decreasing sequence, and

,then the series is **convergent**.

Note:

$a)$ This test is for convergence only. It says nothing about divergence.

$b)$ Like the function in the Integral Test, the sequence $\{b_n\}$ needs to be decreasing "eventually" i.e., for all $n > N$ for some $N$

Example 1:

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n} \qquad b_n=\frac{1}{n} \qquad \lim_{n\to\infty}\frac{1}{n}=0$$

$$\text{consider } f(x)=\frac{1}{x}$$

$$f'(x)=\frac{-1}{x^2} \qquad f'(x)<0 \text{ for all positive } x \quad \Rightarrow \{b_n\} \text{ is decreasing}$$

$$\underbrace{\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}}$$ is **convergent** by the Alternating Series Test

The Alternating Harmonic Series converges.

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Example 2:

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}n^2}{n^2+5} \qquad b_n=\frac{n^2}{n^2+5} \qquad \lim_{n\to\infty}\frac{n^2}{n^2+5}=1$$

the Alternating Series Test Does not apply

$$\lim_{n\to\infty}a_n=\lim_{n\to\infty}\frac{(-1)^{n+1}n^2}{n^2+5}=\lim_{n\to\infty}(-1)^{n+1}\cdot\lim_{n\to\infty}\frac{n^2}{n^2+5}=\lim_{n\to\infty}(-1)^{n+1}\cdot 1 \Rightarrow \text{The limit does not exist.}$$

The series **diverges** by the Test For Divergence, since  does not exist.



Example 3:

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\ln n}{n} \qquad b_n=\frac{\ln n}{n}$$

$$\lim_{n\to\infty}b_n=\lim_{n\to\infty}\frac{\ln n}{n} = \text{"}\frac{\infty}{\infty}\text{"} \quad \text{Indeterminate form} \Rightarrow \text{Use L'Hopistals Rule}$$

$$\overset{L'H}{=}\lim_{n\to\infty}\frac{\frac{1}{n}}{1}=0$$

$$\text{consider } f(x)=\frac{\ln x}{x}$$

$$f'(x)=\frac{x\cdot\frac{1}{x}-\ln x}{x^2}=\frac{1-\ln x}{x^2}$$

$$f'(x) \text{ will be negative when } 1-\ln x<0 \quad \Rightarrow \ln x>1$$

$$e^{\ln x}>e^1$$

$$x>e$$

$$\{b_n\} \text{ is decreasing for } n>2$$

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\ln n}{n}$$ is **convergent** by the Alternating Series Test

# Alternating Series Estimation Theorem



If the alternating series $\sum_{n=1}^{\infty}(-1)^{n-1}b_n$ $(\text{where } b_n > 0)$ satisfies:

$i)$ $\lim_{n\to\infty} b_n = 0$

$ii)$ $\{b_n\}$ is a decreasing sequence

then $|R_n| = |s - s_n| \le b_{n+1}$

The size of the error is at most the size of the first omitted term.

The actual sum is between $s_n - b_{n+1}$ and $s_n + b_{n+1}$.

The error has the same sign as the first omitted term.



$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2} = \underbrace{\underbrace{1-\frac{1}{4}+\frac{1}{9}-\frac{1}{16}+\frac{1}{25}-\frac{1}{36}+\frac{1}{49}-\frac{1}{64}+\frac{1}{81}}_{s_9}\underbrace{-\frac{1}{100}+\frac{1}{121}-\frac{1}{144}\cdots}_{R_9}}_{s}$$

The error committed in using the 9th partial sum to approximate the total sum is $R_9$

The size of this error is at most the size of the first omitted term.

$$|R_9| = |s - s_9| \le \frac{1}{100} \quad \Rightarrow \frac{-1}{100} \le s - s_9 \le \frac{1}{100}$$

$$s_9 - \frac{1}{100} \le s \le s_9 + \frac{1}{100}$$

The actual sum is between $s_n - b_{n+1}$ and $s_n + b_{n+1}$.

The sign of the error is the sign of the first omitted term.

$R_9 = s - s_9 < 0$ $\Rightarrow s_9 > s$ $s_9$ is an overestimate

since $a_{10} = -\frac{1}{100}$