12.6 Absolute Convergence

An infinite series

$\sum_{n=1}^{\infty} a_n$ is called **absolutely convergent** if the positive series $\sum_{n=1}^{\infty} |a_n|$ converges.

Absolute convergence implies converges.

(If the series of absolute value converges, then the original series also converges)

If the series of absolute value **diverges**, it is still possible for the original series to converge.

Use the Alternating Series Test on the original series.
If the Alternating Series Test gives convergence, then this is a special type of convergence.

An infinite series

$\sum_{n=1}^{\infty} a_n$ is called **conditionally convergent** if it converges but $\sum_{n=1}^{\infty} |a_n|$ diverges.

$$\sum_{n=1}^{\infty} |a_n|$$

convergent

$\sum_{n=1}^{\infty} a_n$ is **absolutely convergent**

$\sum_{n=1}^{\infty} a_n$ is **convergent**

divergent

On $\sum_{n=1}^{\infty} a_n$ try :

a) the Alternating Series Test, or

b) the Test for Divergence

If the Alternating Series Test gives convergence,

$\sum_{n=1}^{\infty} a_n$ is **conditionally convergent**

If the Test for Divergence gives divergence,

$\sum_{n=1}^{\infty} a_n$ is **divergent**

A major difference between absolutely convergent and conditionally convergent comes in the rearrangement of the terms.

If $\sum_{n=1}^{\infty} a_n$ is absolutely convergent with sum s,
then any rearrangement of the sum $\sum_{n=1}^{\infty} a_n$ will have the same sum s.

If $\sum_{n=1}^{\infty} a_n$ is conditionally convergent and r is any real number,
then there is a rearrangement of the sum $\sum_{n=1}^{\infty} a_n$ that has the sum r.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots = \ln 2 \quad \text{(We will show this later)}$$

$$\frac{1}{2}\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots\right) = \frac{1}{2}\ln 2$$

$$\frac{1}{2} - \frac{1}{4} + \frac{1}{6} + \cdots = \frac{1}{2}\ln 2$$

$$0 + \frac{1}{2} + 0 - \frac{1}{4} + 0 + \frac{1}{6} + \cdots = \frac{1}{2}\ln 2$$

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \cdots = \ln 2$$

$$+ \quad 0 + \frac{1}{2} + 0 - \frac{1}{4} + 0 + \frac{1}{6} + 0 + -\frac{1}{8} + \cdots = \frac{1}{2}\ln 2$$

$$1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \cdots = \frac{3}{2}\ln 2$$

different sums

same terms

Determine whether the series is absolutely convergent, conditionally convergent, or divergent.

$i)\ \sum_{n=1}^{\infty} \frac{\cos(n\pi)}{\sqrt{n^3}}$

$$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3}} = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$$

convergent $p-$series

$\Rightarrow \sum_{n=1}^{\infty} \frac{\cos(n\pi)}{\sqrt{n^3}}$ is absolutely convergent

$ii)\ \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$

$$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$$

divergent $p-$series

$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$ use $A.S.T.$:

$b_n = \frac{1}{\sqrt{n}}$ is decreasing,

and $\lim_{n\to\infty} b_n = 0$

convergent by $A.S.T.$

$\Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$ is conditionally convergent

$iii)\ \sum_{n=1}^{\infty} \frac{(-4)^{n+1}}{3^n}$

$$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{4^{n+1}}{3^n} = 4\sum_{n=1}^{\infty} \left(\frac{4}{3}\right)^n$$

divergent geom. series

$\sum_{n=1}^{\infty} \frac{(-4)^{n+1}}{3^n}$ use $T.F.D.$:

$\lim_{n\to\infty} \frac{(-4)^{n+1}}{3^n}$ does not exist

$\lim_{n\to\infty} \frac{(-4)^{n+1}}{3^n} = -\infty$, for n even

$\lim_{n\to\infty} \frac{(-4)^{n+1}}{3^n} = \infty$, for n odd

$\Rightarrow \sum_{n=1}^{\infty} \frac{(-4)^{n+1}}{3^n}$ is divergent

12.6 The Ratio Test

Let $\{a_n\}$ be a sequence and assume that the following limit exists: $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L$

$i)$ If $L<1,$ then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

$ii)$ If $L>1$ or if the limit is infinite, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

$iii)$ If $L=1,$ the Ratio Test is inconclusive.

(the series could be absolutely convergent, conditionally convergent, or divergent)

12.6 The Root Test

Let $\{a_n\}$ be a sequence and assume that the following limit exists: $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L$

$i)$ If $L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

$ii)$ If $L > 1$ or if the limit is infinite, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

$iii)$ If $L = 1$, the Root Test is inconclusive.

(the series could be absolutely convergent, conditionally convergent, or divergent)

Determine whether the series is convergent or divergent.

$i)\ \sum_{n=1}^{\infty} \frac{n^3}{4^n}$ $\qquad ii)\ \sum_{n=1}^{\infty} \frac{n!}{4^n}$

$$i)\ \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(n+1)^3}{4^{n+1}}}{\frac{n^3}{4^n}}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^3}{n^3}\cdot\frac{4^n}{4^{n+1}}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^3}{n^3}\ \boxed{\nearrow 1}\cdot\frac{4^n}{4\cdot 4^n}\right| = \frac{1}{4}$$

$$\boxed{\sum_{n=1}^{\infty}\frac{n^3}{4^n} \text{ is convergent}}$$

$$ii)\ \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(n+1)!}{4^{n+1}}}{\frac{n!}{4^n}}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!}{n!}\cdot\frac{4^n}{4^{n+1}}\right| = \lim_{n\to\infty}\left|\frac{(n+1)\,n!}{n!}\cdot\frac{4^n}{4\cdot 4^n}\right| = \lim_{n\to\infty}\frac{n+1}{4} = \infty$$

$$\boxed{\sum_{n=1}^{\infty}\frac{n!}{4^n} \text{ is divergent}}$$

Determine whether the series is convergent or divergent.

$iii)\ \sum_{n=1}^{\infty}\frac{(2n+1)^n}{n^{2n}}$ $\qquad$ $iv)\ \sum_{n=1}^{\infty}\frac{n^2}{(2n+1)!}$

$$iii)\ \lim_{n\to\infty}\sqrt[n]{|a_n|}=\lim_{n\to\infty}\sqrt[n]{\left|\frac{(2n+1)^n}{(n^2)^n}\right|}=\lim_{n\to\infty}\left[\left(\frac{2n+1}{n^2}\right)^n\right]^{1/n}=\lim_{n\to\infty}\frac{2n+1}{n^2}=0$$

$\deg(num.)<\deg.(denom.)$

$$\sum_{n=1}^{\infty}\frac{(2n+1)^n}{n^{2n}}\text{ is convergent}$$

$$iv)\ \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{\dfrac{(n+1)^2}{(2[n+1]+1)!}}{\dfrac{n^2}{(2n+1)!}}\right|=\lim_{n\to\infty}\left|\frac{(n+1)^2}{n^2}\cdot\frac{(2n+1)!}{(2n+3)!}\right|$$

$$=\lim_{n\to\infty}\left|\frac{(n+1)^2}{n^2}\nearrow 1\cdot\frac{(2n+1)!}{(2n+3)\cdot(2n+2)(2n+1)!}\right|=\lim_{n\to\infty}\frac{1}{(2n+3)\cdot(2n+2)}=0$$

$$\sum_{n=1}^{\infty}\frac{n^2}{(2n+1)!}\text{ is convergent}$$